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A NOTE ON THE MULTIPLICITY OF COHEN-MACAULAY ALGEBRAS WITH PURE RESOLUTIONS

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1. Introduction, main theorem, and examples. Let $R = k[X_1, \dots, X_n]$ with k a field, and let $I \subset R$ be a homogeneous ideal. The algebra R/I is said to have a *pure resolution* if its homogeneous minimal resolution has the form

$$0 \to R^{b_p}(-d_p) \to \dots \to R^{b_2}(-d_2) \to R^{b_1}(-d_1) \to R.$$

Some of the known examples of pure resolutions include the coordinate rings of: the tangent cone of a minimally elliptic singularity or a rational surface singularity [**15**], a variety defined by generic maximal Pfaffians [**2**], a variety defined by maximal minors of a generic matrix [**3**], a variety defined by the submaximal minors of a generic square matrix [**6**], and certain of the Segre-Veronese varieties [**1**].

If I is in addition Cohen-Macaulay, then Herzog and Kühl have shown that the betti numbers b_i are completely determined by the twists d_i. The main purpose of this note is to give a closed formula for the multiplicity of R/I in the case that R/I is Cohen-Macaulay with a pure resolution. We denote the multiplicity $e(R/I)$. Our main result is that

$$e(R/I) = \left(\prod_{i=1}^{p} d_i\right)/p!.$$

This formula imposes strong restrictions on the d_i (as $e(R/I)$ must be an integer), and suggests the question: when does the multiplicity completely determine the resolution type $(d_1, \dots, d_p)$ of R/I? This question gives rise to a Diophantine equation. If R/I is Gorenstein of codimension four, then the multiplicity $e = e(R/I)$ has the form $x^4y^2(y^2 - 1)/12$, where the integers x and y determine the resolution type. If the equation

$$x^4y^2(y^2 - 1) = z^4u^2(u^2 - 1) \tag{1.1}$$

has only the resolutions $x = z$, $y = u$ in $\mathbf{N}$, then it follows that e determines the resolution type of R/I. We have been unable to show

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uniqueness, but (1.1) has only the trivial solutions for $e \leqq 10^{12}$.

In the second section we investigate numerical constraints on pure resolutions. For instance if $e \leqq 2^{-3/5}p^{29/20}$ and e has no prime divisors greater than p, then the last twist d_p is at most $p + p^{11/20} + 1$, provided that p is sufficiently large. Indeed, there is a delicate interplay between d_j for j close to 1 or close to p that allows us to conclude that certain algebras can not have pure resolutions. By computing the socle degree of a suitable 0-dimensional quotient, we can apply our techniques to show that the cone of the s-uple embedding of $\mathbf{P}^2$ into $\mathbf{P}^{N-1}$, for $N = \binom{s+2}{2}$, has a pure resolution if and only if $s \leqq 3$, a result which extends those in **[1]**.

We conclude by investigating which Hilbert functions are possible for 0-dimensional algebras R/I with pure resolutions and $(\mathfrak{m}_R)^4 = 0$. It would be very interesting to completely determine which Hilbert functions can give rise to pure resolutions.

In **[12]** Peskine and Szpiro have given a multiplicity formula for a graded module with a graded free resolution (not necessarily pure). In our setting their formula becomes

$$e(R/I) = \frac{1}{p!} \sum_{i=0}^{p} (-1)^{i+p} b_i d_i^p$$

which is similar, but not identical, to the formula of Herzog and Kühl **[9]** that we use. To obtain a closed expression one must still use the result of Herzog and Kühl that expresses the b_i in terms of the d_i, but it is interesting to note that our proof shows that d_i^p can be replaced by an arbitrary monic polynomial of degree p in d_i.

THEOREM 1.2. *Let $R = k[X_1, \dots, X_n]$ with k a field, and let $I \subset R$ be a homogeneous ideal. Suppose that R/I is Cohen-Macaulay and has a pure resolution of type $(d_1, \dots, d_p)$. Then*

$$e(R/I) = \left(\prod_{i=1}^{p} d_i\right)/p!.$$

Proof. It follows from elementary considerations (as in **[9]**) that

$$e(R/I) = \sum_{i=1}^{p} (-1)^{i+p} b_i \binom{d_i}{p}$$

where the binomial coefficient denotes $d_i(d_i - 1) \dots (d_i - p + 1)/p!$. By using the formula

$$(1.3) \quad b_i = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{d_j - d_i}$$

of **[9]**, we conclude that

$$e(R/I) = \sum_{i=1}^{p} \left(\left(\prod_{j \neq i} d_j \right) \Big/ \prod_{j \neq i} (d_i - d_j) \right) \binom{d_i}{p}$$

$$= \frac{\prod_{i=1}^{p} d_i}{p!} \sum_{i=1}^{p} \prod_{j=1}^{p-1} (d_i - j) \Big/ \prod_{j \neq i} (d_i - d_j).$$

To show that the large summation equals 1, consider the rational function

$$f(z) = \prod_{j=1}^{p-1} (z - j) \Big/ \prod_{j=1}^{p} (z - d_j).$$

This function has simple poles contained in the set $\{d_1, \ldots, d_p\}$ since $d_i \neq d_j$ for $i \neq j$; and since d_i is at worst a simple pole,

$$\mathrm{Res}_{d_i} f(z) = \prod_{j=1}^{p-1} (d_i - j) \Big/ \prod_{j \neq i} (d_i - d_j).$$

Hence we obtain the formula

$$(1.4) \qquad e(R/I) = \frac{\prod_{i=1}^{p} d_i}{p!} \sum_{i=1}^{p} \mathrm{Res}_{d_i} f(z).$$

The sum of all the residues of a rational function at all poles (including ∞) is zero [**11**, p. 233], so

$$\sum_{i=1}^{p} \mathrm{Res}_{d_i} f(z) = -\mathrm{Res}_{\infty} f(z).$$

By integrating around a circle of sufficiently large radius and making a change of variables $z \to 1/z$, we see that

$$\mathrm{Res}_{\infty} f(z) = -\mathrm{Res}_0 f\left(\frac{1}{z}\right) / z^2,$$

In our situation

$$f\left(\frac{1}{z}\right)/z^2 = \prod_{j=1}^{p-1} \left(\frac{1}{z} - j\right) \Big/ z^2 \prod_{j=1}^{p} \left(\frac{1}{z} - d_j\right)$$

$$= \prod_{j=1}^{p-1} (1 - jz) \Big/ z \prod_{j=1}^{p} (1 - d_j z).$$

Clearly $f(1/z)/z^2$ has a simple pole with residue 1 at 0; thus by (1.4) we have the desired formula.

We give a few examples to which Theorem 1.2 applies. Throughout we assume that R is a polynomial ring over a field, I is homogeneous, and R/I is Cohen-Macaulay.

Example 1.5. Suppose $\mathrm{ht}(I) = 2$. Then $I = I_n(X)$ is generated by the $n \times n$ minors of an $n \times (n + 1)$ matrix X and R/I has resolution

$$0 \to R^n(-nd - d) \to R^{n+1}(-nd) \to R$$

if the entries of X are forms of degree d. Hence

$$e(R/I) = (1/2)d^2n(n + 1) = d^2\binom{n+1}{2}$$

by the theorem.

Example 1.6. Suppose $\mathrm{ht}(I) = 3$ and R/I is Gorenstein, so that by Buchsbaum and Eisenbud [**2**] $I = Pf_{2n}(X)$, where X is a $(2n + 1) \times (2n + 1)$ skew-symmetric matrix. If the entries of X are forms of degree d, then R/I has resolution

$$0 \to R(-2dn - d) \to R^{2n+1}(-dn - d) \to R^{2n+1}(-dn) \to R,$$

and so $e(R/I) = d^3n(n + 1)(2n + 1)/6$.

Example 1.7. Suppose X is an $(n + 1) \times (n + 1)$ matrix of forms of degree d such that $\mathrm{ht}(I_n(X)) = 4$. Then $R/I_n(X)$ is Gorenstein and by Gulliksen and Negard [**6**] it has resolution

$$0 \to R(-2dn - 2d) \to R^{\mu}(-dn - 2d) \to R^{2\mu-2}(-dn - d) \to R^{\mu}(-dn) \to R$$

with $\mu = (n + 1)^2$. Hence $e(R/I) = d^4n(n + 1)^2(n + 2)/12$.

Example 1.8. If R/I is Cohen-Macaulay of minimal multiplicity, i.e., $e(R/I) = \mathrm{ht}(I) + 1$, then by Sally [**13**] R/I has a pure resolution of form

$$0 \to R^{b_p}(-p - 1) \to \dots \to R^{b_2}(-3) \to R^{b_1}(-2) \to R.$$

Conversely, if R/I has such a resolution, then

$$e(R/I) = (p + 1)! = \mathrm{ht}(I) + 1$$

and R/I has minimal multiplicity.

More generally if R/I has a d-linear resolution (i.e., $d_1 = d$, $d_i = d + i - 1$ for $2 \leqq i \leqq p$) then Schenzel [**14**] and Eisenbud and Goto [**4**, Proposition 1.7] have shown that

$$e(R/I) = \binom{p+d-1}{p},$$

in agreement with Theorem 1.2. Schenzel has also proved an analogous result for R/I Gorenstein.

Example 1.9. Let X be an $r \times s$ matrix with $r < s$ whose entries are forms of degree d. Suppose

$$\mathrm{ht}(I_r(X)) = p = s - r + 1,$$

the maximum possible. Then Eagon and Northcott [**3**] have shown that $R/I_r(X)$ has a resolution of form

$$0 \to R^{b_p}(-ds) \to \ldots \to R^{b_2}(-d(r + 1)) \to R^{b_1}(-dr) \to R.$$

Therefore

$$e(R/I_r(X)) = d^{s-r+1}\binom{s}{s - r + 1}.$$

Example 1.10. Assume char $k \neq 2$. Let X be an $n \times n$ symmetric matrix and $I = I_{n-1}(X)$. If $\mathrm{ht}(I) = 3$ and the entries of X are forms of degree d, then by Goto and Tachibana [**5**] (or, without the characteristic assumption, by Józefiak [**10**], whose article appeared slightly later) the resolution R/I has the form

$$0 \to R^{\binom{n}{2}}(-d(n + 1)) \to R^{n^2-1}(-dn)$$
$$\to R^{\binom{n+1}{2}}(-d(n - 1)) \to R.$$

Thus $e(R/I) = d^3n(n^2 - 1)/6$.

Next we turn to the question: given a pure resolution of a codimension p Cohen-Macaulay algebra, to what extent does $e(R/I)$ determine the resolution type $(d_1, \ldots, d_p)$? If R/I is codimension four Gorenstein, then in fact the type is uniquely determined if the multiplicity is at most a trillion!

LEMMA 1.11. *Suppose $R = k[X_1, \ldots, X_n]$ and R/I is a graded Gorenstein algebra of codimension four with a pure resolution. Then there are positive integers x and y such that*

$$e(R/I) = x^4y(y + 1)^2(y + 2)/12$$

and the resolution of R/I has the form

$$0 \to R(-2x(y + 1)) \to R^{\mu}(-x(y + 2))$$
$$\to R^{2\mu-2}(-x(y + 1)) \to R^{\mu}(-xy) \to R$$

where $\mu = (y + 1)^2$.

Proof. Since R/I is Gorenstein, the shifts in the resolution are dual and the resolution must have the form

$$0 \to R(-2d_2) \to R^{b_1}(-2d_2 + d_1) \to R^{b_2}(-d_2) \to R^{b_1}(-d_1) \to R.$$

Set $x = d_2 - d_1$. By Herzog and Kühl [**9**], or equation (1.3) above,

$$b_1 = \frac{d_2^2}{(d_2 - d_1)^2} = \left(\frac{x + d_1}{x}\right)^2.$$

Hence x must divide d_1. If we set $d_1 = xy$ then we obtain the desired resolution type, and the multiplicity formula follows by Theorem 1.2.

All the above resolution types are actually attainable as Example 1.7 shows.

PROPOSITION 1.12. *Let $R = k[X_1, \dots, X_n]$ and suppose R/I is a Gorenstein algebra of codimension four with a pure resolution. If $e = e(R/I) \leqq 10^{12}$, then the resolution type of R/I is uniquely determined by e.*

Proof. By Lemma 1.11 with y replacing $y + 1$, the statement of the proposition is equivalent to saying that the equation

$$(1.13) \quad 12e = x^4y^2(y^2 - 1) = z^4u^2(u^2 - 1)$$

has only the solutions $x = z$, $y = u$ in $\mathbf{N}$ if

$$x^4y^2(y^2 - 1) = 12e \leqq 12 \cdot 10^{12}.$$

Since $f(t) = (1 - t)^{1/4}$ is concave, one readily verifies that

$$0 < xy - (x^4y^2(y^2 - 1))^{1/4} < x/(3.6y)$$

if $y \geqq 2$ (which is implicit anyway if $\mathrm{ht}(I) = 4$ or $e > 0$). In particular, if $x \leqq 3.6y$, then

$$[(12e)^{1/4} + 1] = xy,$$

where $[r]$ denotes the greatest integer in r. Suppose there is a solution to (1.13) with both $x \leqq 3.6y$ and $z \leqq 3.6u$. Then

$$xy = [(12e)^{1/4} + 1] = zu.$$

This equality together with (1.13) forces $x^2 = z^2$ and hence $x = z$, $y = u$. Therefore, if (1.13) has a solution with $x \neq z$, we may assume that $x > 3.6y$. It follows that

$$(3.6y)^4y^2(y^2 - 1) < x^4y^2(y^2 - 1) \leqq 12 \cdot 10^{12},$$

which implies $y \leqq 22$. A check of a table of prime factorizations of integers up to 10^4 shows that (1.13) has no solutions with $y \leqq 22$ and $u \leqq 10^4$. (Even though x and z have not been determined, we only need to check that the exponents of the primes in the factorizations of $(y - 1)y^2(y + 1)$ and $(u - 1)u^2(u + 1)$ are never congruent modulo 4 if $y \leqq 22$ and $u \leqq 10^4$.) Thus $x = z$ and $y = u$ if $u \leqq 10^4$. However $u > 10^4$ is impossible for $e \leqq 10^{12}$ since

$$u^2(u^2 - 1) > 10^{15} > 12e.$$

2. Numerical constraints for pure resolutions. The equation

$$e \cdot p! = \prod_{i=1}^{p} d_i,$$

together with the equations (1.3) of Herzog and Kühl, imposes strong restrictions on the possible resolution types for given codimension p and multiplicity e, assuming that R/I has a pure resolution. In some cases one can show that R/I can not have a pure resolution because no resolution type is possible.

Throughout this section we let $R = k[X_1, \dots, X_n]$ with k a field, I be a homogeneous ideal of height $p \geqq 3$, and we assume that R/I is Cohen-Macaulay with a pure resolution of type $(d_1, \dots, d_p)$. We shall have the following standing hypotheses: $d_1 \geqq 2$ (and hence $d_i \geqq i + 1$ for all i); if q is a prime divisor of the multiplicity $e = e(R/I)$, then $q \leqq p$.

LEMMA 2.1. *With the hypotheses listed above*

$$d_p \leqq \min\{e, d_1 p\}.$$

If m is a positive integer such that $e < (p + 2)(p + 1)/(m + 1)$, then $d_i = i + 1$ for $1 \leqq i \leqq m$.

Proof. Since $d_i \geqq i + 1$ we have

$$p!e = \prod_{i=1}^{p} d_i \geqq d_p \cdot p!$$

by Theorem 1.2. Hence $e \geqq d_p$. To see that $d_p \leqq d_1 p$ let us choose a regular sequence of minimal generators of I, say $y_1, \dots, y_p$. Since all the y_i have degree d_1 the last free module in the Koszul resolution $\mathbf{K}_\bullet(\mathbf{y})$ is $R(-d_1 p)$. The natural map

$$\alpha_0 : R/(\mathbf{y}) \to R/I$$

extends to a map of homogeneous free resolutions

$$\alpha_0 : \mathbf{K}_\bullet(\mathbf{y}) \to \mathbf{F}_\bullet,$$

and in particular there is a map

$$\alpha_p : R(-d_1 p) \to R^{b_p}(-d_p)$$

of non-negative degree. Hence $d_p \leqq d_1 p$.

If the last assertion fails then $d_j > j + 1$ for some $j \leqq m$. If j is the least

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such index, then $d_i = i + 1$ for $i < j$ and $d_i > i + 1$ for $i \geqq j$. It follows that

$$p!e = \prod_{i=1}^{p} d_i \geqq (p + 2)!/(j + 1)$$

and hence

$$e \geqq (p + 2)(p + 1)/(m + 1),$$

which contradicts the hypothesis.

The lemma enables us to draw strong conclusions about the resolution type if e is small relative to p^2. Once we have $d_1 = 2$, then

$$d_p \leqq \min\{e, 2p\}.$$

By formula (1.3) $d_p - d_1$ must divide $\prod_{i=1}^{p} d_i = p!e$. If $d_p - d_1$ is a prime larger than p, then this division is impossible in view of our standing hypothesis on e. The next result formalizes this argument.

PROPOSITION 2.2. *Retain the hypotheses listed prior to Lemma* 2.1 *and also assume* $d_1 = 2$. *Let* $S = \{\pi_1, \dots, \pi_N\}$ *be the set of primes in the interval* $p < x \leqq \nu$, *where* $\nu = \min\{e, 2p\}$. *Assume* $\pi_1 < \pi_2 < \dots < \pi_N$ *and set*

$$K = \max\{\pi_j - \pi_{j-1}, \nu - \pi_N | 2 \leqq j \leqq N\}.$$

If $S = \emptyset$, *set* $K = \infty$. *Then*

$$e < (p + 2)(p + 1)/(K + 1)$$

implies that $d_p \leqq \pi_1 + 1$.

Proof. Since the proposition is vacuous for $K = \infty$ we may assume that K is finite. Note that $K = 0$ if and only if $N = 1$ and $\nu = \pi_1$ is prime. In this case $\nu = e$, and by the lemma $e \geqq d_p \geqq p + 1$, in contradiction to our assumption that no prime divisor of e is larger than p. Hence $K \geqq 1$ and we conclude from Lemma 2.1 that $d_i = i + 1$ for $1 \leqq i \leqq K$.

Now suppose $d_p > \pi_1 + 1$. Then $\pi_j < d_p \leqq \pi_{j+1}$ for some j, and hence

$$1 \leqq d_p - \pi_j \leqq \pi_{j+1} - \pi_j \leqq K.$$

(If $j = N$, then $1 \leqq d_p - \pi_N \leqq \nu - \pi_N \leqq K$.) Set $m = d_p - \pi_j$. If $m \geqq 2$, then by (2.1) $d_{m-1} = m$. Therefore

$$\pi_j = d_p - d_{m-1}.$$

By formula (1.3), π_j divides $\prod_{i=1}^{p} d_i = p!e$. But $\pi_j > p$, whereas all the prime divisors of $p!$ and e are $\leqq p$. Thus we must have $m = 1$. In this case $j \geqq 2$ and

$$2 \leqq d_p - \pi_{j-1} = 1 + \pi_j - \pi_{j-1} \leqq K + 1;$$

consequently $d_p - \pi_{j-1} = d_i$ for some $i \leqq K$, and a contradiction follows just as above. Therefore $d_p \leqq \pi_1 + 1$.

Although the proposition no doubt sounds strange, if not completely bizarre, it is quite useful. Before giving some examples, we offer a corollary, which is easy to state and which eliminates the possibility that $S = \emptyset$.

COROLLARY 2.3. *Fix the notation as in the proposition. Assume that the prime factors of e are at most p and that $e \leqq 2^{-3/5}p^{29/20}$. Then for p sufficiently large,*

$$d_p \leqq p + p^{11/20} + 1.$$

Proof. For $x \gg 0$ by [**8**] there is a prime between x and $x + x^{11/20}$. In particular $K \leqq (2p)^{11/20}$. For $e \leqq 2^{-3/5}p^{29/20}$ and $p \gg 0$ it is easy to see that

$$e < (p + 1)(p + 2)/(K + 1);$$

Proposition 2.2 immediately yields the conclusion.

In the next series of examples we shall see that the method of proof of (2.2) can be used to constrain d_j for $j < p$, as well as d_p itself, and thus rule out the possibility of a pure resolution quite efficiently. In these examples e is simply a power of a small prime, so our standing hypothesis applies.

In [**1**] Bǎrcǎnescu and Manolache studied Segre-Veronese varieties and showed that certain ones have pure resolutions. We shall demonstrate that many do not. Let $R_{r,s}$ denote the coordinate ring of the affine cone of the s-uple embedding of $\mathbf{P}^r$ in $\mathbf{P}^{N-1}$, where $N = \binom{s+r}{r}$. Of course $R_{r,s}$ is just the subring of $k[X_0, \ldots, X_r]$ generated by all the monomials of degree s. The algebra $R_{r,s}$ is Cohen-Macaulay of embedding dimension N, codimension $N - r - 1$, and multiplicity s^r (see [**1**] or [**7**, p. 54]).

Example 2.4. $R_{3,3}$ does not have a pure resolution. Here $p = 16$ and $e = 27 = 3^3$. In the notation of the proposition $\nu = 27$, $S = \{17, 19, 23\}$, and $K = 4$. Since $27 < 17 \cdot 18/5$,

$$d_{16} \leqq \pi_1 + 1 = 18.$$

However, $d_1 \ldots d_p = 16! \cdot 27$ is clearly not possible because 3^9 divides the right hand side, but at best 3^7 divides the left.

Example 2.5. [**1**] $R_{7,2}$ does not have a pure resolution. Here $p = 28$ and $e = 128$. In this case $\nu = 56$, $S = \{29, 31, 37, 41, 43, 47, 53\}$, and $K = 6$. Since 128 is not less than $29 \cdot 30/7$, we can not use the proposition itself, but we can conclude from the lemma that $d_i = i + 1$ for $1 \leqq i \leqq 5$ and $d_{28} \leqq 56$. By the proof of the proposition we can eliminate all integers of form $\pi + i$ with $\pi \in S$ and $2 \leqq i \leqq 6$ as possibilities for d_{28} (since $d_{28} - d_{i-1} = \pi$ can not divide $28! \cdot 2^7$). Clearly $d_1 \ldots d_p = 28! \cdot 2^7$ does not allow $d_{28} \in S$. This leaves the possibilities $d_{28} = 30$, $d_{28} = 38$, or $d_{28} = 54$. The first is easy to discard. Since 2^{24} divides $28!$, some $d_j = 32$ in order for $d_1 \ldots d_{28} = 28! \cdot 2^7$ with $d_{28} = 38$. Then

$d_j - d_2 = 32 - 3 = 29$ divides $\prod_{i=1}^{p} d_i$, and this is impossible. If $d_{28} =$

54, we again compare powers of 2. Since $d_2 = 3$ and $d_3 = 5$ we quickly see that the only even integers allowable as d_j's are 54, 38, and those $\leqq 30$.

This supplies us with at most 2^{27} in $\prod_{i=1}^{p} d_i$, whereas 2^{31} divides $28! \cdot 2^7$.

PROPOSITION 2.6. *The algebra $R_{2,s}$ has a pure resolution if and only if $s \leqq 3$.*

Proof. For $s = 1$ the assertion is obvious. Bǎrcǎnescu and Manolache [**1**] showed that $R_{2,s}$ has a pure resolution for $s = 2, 3$. We may therefore assume that $s > 3$.

In general, if $S = R/I$ is a graded Cohen-Macaulay algebra of codimension p, and the last module in its minimal homogeneous resolution is $\bigoplus_{i=1}^{b_p} R(-d_{pi})$, then the integers $d_{pi} - p$ are the degrees of the socle generators of $\bar{S}$, where $\bar{S}$ is the reduction of S modulo a maximal regular sequence consisting of forms of degree one. In particular if $R_{2,s}$ has a pure resolution then $d_p = p + \delta$ where δ is the degree of any element of socle($\bar{S}$). Now

$$R_{2,s} = k[M_\sigma | M_\sigma \text{ is a monomial of degree } s \text{ in } X_0, X_1, X_2].$$

The elements X_0^s, X_1^s, X_2^s form a maximal regular sequence of linear forms in $R_{2,s}$, and we set

$$\bar{R}_{2,s} = R_{2,s}/(X_0^s, X_1^s, X_2^s).$$

We claim that every monomial $\bar{M}$ of degree $2s$ is either 0 or in socle($\bar{R}_{2,s}$), and that not all such are 0. Suppose

$$M_1 = X_0^{i_0} X_1^{i_1} X_2^{i_2} \quad \text{with } i_0 + i_1 + i_2 = 2s \text{ and}$$

$$M_2 = X_0^{j_0} X_1^{j_1} X_2^{j_2} \quad \text{with } j_0 + j_1 + j_2 = s.$$

Then $\bar{M}_1 \bar{M}_2 = 0$ since

$$\sum_{k=0}^{2} (i_k + j_k) = 3s$$

implies $i_k + j_k \geqq s$ for some k. Hence $\bar{M}_1$ is in socle$(\bar{R}_{2,s})$. Not all such can be zero, as for instance

$$\bar{X}_0^{s-1}\bar{X}_1^{s-1}\bar{X}_2^2 \neq 0.$$

We conclude that $\delta = 2$ and $d_p = p + 2$.

From the formulas for codimension and multiplicity we have

$$e = s^2 \quad \text{and} \quad p = (s + 4)(s - 1)/2.$$

Since $d_p = p + 2$ and $d_1 \geqq 2$, there is an index $j \leqq p$ such that $d_i = i + 1$ for $i < j$ and $d_i = i + 2$ for $i \geqq j$. Hence

$$p!e = \prod_{i=1}^{p} d_i = (p + 2)!/(j + 1) \quad \text{and}$$

$$e = (p + 2)(p + 1)/(j + 1).$$

Therefore $(p + 2)(p + 1)/e$ is an integer; we conclude the argument by showing this is possible only if $s \leqq 3$.

Observe that

$$(p + 2)(p + 1)/e = (s + 3)^2/4 - (s + 3)/2s.$$

If s is odd, then $(s + 3)^2/4$ is an integer, so $(s + 3)/2s$ must also be an integer, which is true only if $s = 1, 3$. If s is even, then

$$(p + 2)(p + 1)/e = (s^2 + 6s)/4 + (7s - 6)/4s,$$

and the first term is clearly an integer. It follows that $s|6$, from which we conclude $s = 2$.

To conclude, we investigate which Hilbert functions are possible for certain 0-dimensional algebras with pure resolutions.

Example 2.8. Suppose $p \geqq 4$ is even and $R = k[X_1, \dots, X_p]$. If R/I is 0-dimensional with Hilbert function $(1, p, p, 1)$ then R/I can not have a pure resolution. For $e(R/I) = 2p + 2$, and if the resolution is pure the socle of R/I must sit in degree $\delta = 3$ alone. Then socle(R/I) is 1-dimensional, so R/I is Gorenstein, and $d_{p-1} = d_p - d_1$. Using the formula $d_p = p + \delta$ discussed above, and observing that $d_1 = 2$ (since $p < \binom{p}{2}$ some, and hence all generators of I are in degree 2), we obtain

$$(2p + 2)p! = (p + 3)(p + 1)d_{p-2} \dots d_1.$$

Since the d_i are distinct and $2 \leqq d_i \leqq p$ for $1 \leqq i \leqq p - 2$, we conclude that

$$p + 3 = 2p! \Big/ \prod_{i=1}^{p-2} d_i$$

is an even integer, which is a contradiction.

As a specific example, the Grassmannian of 2-planes in affine 6-space $G(2, 6)$ has (after specializing by linear forms) Hilbert function (1, 6, 6, 1). It follows that the coordinate ring can not have a pure resolution.

Example 2.9. Let $R = k[X, Y, Z]$ and I be a homogeneous ideal such that

$$\dim R/I = 0 \quad \text{and} \quad (X, Y, Z)^4 \subseteq I \subseteq (X, Y, Z)^2.$$

If R/I has a pure resolution, then the Hilbert function of R/I is one of the following:

i) 1, 3 ii) 1, 3, 1 iii) 1, 3, 6 iv) 1, 3, 3, 1

v) 1, 3, 6, 2 vi) 1, 3, 6, 5 vii) 1, 3, 6, 10

Furthermore, these examples are attainable as follows:

i′) $(X, Y, Z)^2$

ii′) specialization of the maximal Pfaffians of the generic 5×5 skew-symmetric matrix

iii′) $(X, Y, Z)^3$

iv′) a regular sequence of 3 quadrics

v′) $(X^3, Y^3, Z^3):J$, where J is the ideal of vi′)

vi′) $J = (X^3, Y^3, Z^3, XYZ, X^2Y + X^2Z + Y^2X + Y^2Z + YZ^2 + XZ^2)$

vii′) $(X, Y, Z)^4$.

Proof. That i′)-iv′) and vii′) yield R/I with pure resolutions and corresponding Hilbert functions follows from known resolutions. The resolution of vi′) is

$$0 \rightarrow R^5(-6) \rightarrow R^9(-5) \rightarrow R^5(-3) \rightarrow R.$$

One obtains the resolution for case v′) by applying the mapping cone construction to the above resolution:

$$0 \rightarrow R^2(-6) \rightarrow R^9(-4) \rightarrow R^8(-3) \rightarrow R.$$

In these last two cases the Hilbert functions can be readily computed to be v) and vi) respectively.

For R/I satisfying the given conditions the Hilbert function must have form 1, 3, n, m, with $n \leqq 6$, $m \leqq 10$. If R/I has resolution

$$0 \rightarrow R^{b_3}(-d_3) \rightarrow R^{b_2}(-d_2) \rightarrow R^{b_1}(-d_1) \rightarrow R$$

then by duality the socle of R/I lives in degree $d_3 - 3$, and of course the generators of I live in degree d_1.

If $n = 0$, then $m = 0$ and we have case i). If $m = 0$ and $n < 6$, then the socle of R/I and the generators of I both lie in degree 2, so $d_3 = 5$ and $d_1 = 2$. Thus $d_2 = 4$ or 3. If $d_2 = 3$ we obtain Hilbert function ii) by using (1.3) and $d_2 = 4$ is impossible since b_1 must be an integer. If $n = 6$, $m = 0$ we have case iii), and if $n = 6$, $m = 10$ we have case vii).

Next assume $n = 6$ and $0 < m < 10$. Then both the generators of I and socle(R/I) lie in degree 3, so that $d_3 = 6$, $d_1 = 3$. Hence $d_2 = 4$ or 5. If $d_2 = 4$, we have case v), while if $d_2 = 5$ we have case vi).

Finally, we assume $n < 6$ and $m \neq 0$. Then $d_3 = 6$ and $d_1 = 2$. Arguing as before, we find $d_2 = 4$, whence $b_1 = 3$ and I is a complete intersection, precisely case iv).

This analysis is very simple, but we offer it as an indication of the strength of the assumptions. It would be interesting to detemine exactly which Hilbert functions correspond to pure resolutions, even in codimension 3. Three more possibilities arise if we relax the hypothesis to $(X, Y, Z)^5 \subset I$, namely: (1, 3, 6, 3, 1), (1, 3, 6, 10, 15), and (1, 3, 6, 10, 8). The difficulty in continuing is to find examples for the sequences that are numerically allowable. For instance the Hilbert function (1, 3, 6, 6, 3, 1) gives rise to $d_1 = 3$, $d_2 = 5$, $d_3 = 8$, and hence $b_1 = 4$, $b_2 = 4$, $b_3 = 1$. These would be the betti numbers for a codimension 3 Gorenstein algebra R/I with $\mu(I) = 4$, contradicting the structure theorem of [**2**].

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References

1. S. Bǎrcǎnescu and N. Manolache, *Betti numbers of Segre-Veronese singularities*, Rev. Roum. Math. Pures et Appl. *26* (1981), 549-565.

2. D. Buchsbaum and D. Eisenbud, *Algebra structures for some finite free resolutions and some structure theorems for ideals of codimension* 3, Amer. J. Math. *99* (1977), 447-485.

3. J. E. Eagon and D. G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. (Ser. A) *269* (1962), 188-204.

4. D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, preprint (1982).

5. S. Goto and S. Tachibana, *A complex associated with a symmetric matrix*, J. Math. Kyoto Univ. *17* (1977), 51-54.

6. T. Gulliksen and O. Negárd, *Un complexe résolvant pour certain idéaux déterminantiels*, C.R. Acad Sc. Paris (Sér. A) *274* (1972), 16-18.

7. R. Hartshorne, *Algebraic geometry* (Graduate Texts in Math. 52, Springer-Verlag, NY, 1977).

8. D. R. Heath-Brown and H. Iwaniec, *On the difference between consecutive primes*, Invent. Math. *55* (1979) 49-69.

9. J. Herzog and M. Kühl, *On the betti numbers of finite pure and linear resolutions*, Comm Alg. *12* (1984), 1627-1646.

10. T. Józefiak, *Ideals generated by minors or a symmetric matrix*, Comment. Math. Helv. *53* (1978), 595-607.

11. N. Levinson and R. Redheffer, *Complex variables* (Holden-Day, San Francisco, 1970).

12. C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C.R. Acad. Sc. Paris (Sér. A) *278* (1974), 1421-1424.

13. J. Sally, *Cohen-Macaulay rings of maximal embedding dimension*, J. Alg. *56* (1979), 168-183.

14. P. Schenzel, *Extremale Cohen-Macaulay-Ringe*, J. Alg. *64* (1980), 93-101.

15. J. Wahl, *Equations defining rational singularities*, Ann. Sci. Ecole Norm. Sup. *10* (1977), 231-264.

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