ELASTIC WAVES IN TWO SOLIDS AS PROPAGATION OF SINGULARITIES PHENOMENON

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In this paper we shall study elastic waves in two isotropic media with different densities and Lamé's constants. In seismology elastic waves are studied when the border of two media is a hyperplane, however there are no results on elastic waves in the non-flat border case. First in Section 1 we shall show an existence theorem of the solutions of an initial boundary value problem which is satisfied by the displacements of two media. Next we shall discuss about propagation of singularities of the solutions, for Hörmander and Lax-Nirenberg showed that an appearance of propagation of singularities is similar to one of propagation of waves.

As first part on singularities in Section 3 we shall show existence of Stoneley waves as propagation of singularities, which is explained as follows: if the boundary values of the initial data of the solutions have singularities, then there exist singularities of the solutions which start from the singularities of the initial data and propagate in the elliptic region of the border of two media according to the passage of time.

Second part on singularities is to study relations between incident waves and reflected and refracted waves, which are stated in Sections 4, 5. Under various conditions on the incident angle of singularities corresponding to the fast waves or the slow waves we have interesting refractive phenomena. For example if we assume that densities and Lamé's constants satisfy some conditions, and that the incident angle of singularities corresponding to the fast waves is not sharp, then the solutions have only refracted singularities corresponding to the slow waves (see Theorem 4.1). If the singularities corresponding to the slow waves make incidence, the solutions have both refracted singularities corresponding to the fast and slow waves or they have only refracted singularities cor-
responding to the slow waves (see Theorems 5.1, 5.2).

§ 1. Existence of solutions

In this section we shall show existence of the solutions of initial boundary value problems in two solids. We suppose that there exist two simply connected open subsets \( O_i \) \((i = 1, 2)\) of \( \mathbb{R}^3 \) with smooth boundaries \( \partial O_i \) such that \( \partial O_i \) is compact, or \( \partial O_i \) is equal to a hyperplane in \( \{ x : |x| > R \} \) for some positive constant \( R \). Then domains \( \Omega_1, \Omega_2 \) of two solids are defined as follows; \( \Omega_i = O_i \cap O_i^c \) and \( \Omega_i = O_i \backslash \overline{O}_i \). The strain tensors \( \varepsilon_{jk}^i(u) \) \((j, k = 1, 2, 3)\) are \( \varepsilon_{jk}^i(u) = \frac{1}{2}(\partial u_j/\partial x_k + \partial u_k/\partial x_j)/2 \) and in the stress tensors \( \sigma_{jk}^i(u) \) are \( \lambda_i (\text{div} \, u) \delta_{jk} + 2\mu_i \varepsilon_{jk}^i(u) \), where \( u = (u_1, u_2, u_3) \) and \( \lambda_i \) and \( \mu_i \) are Lamè constants such that \( \mu_i, 3\lambda_i + 2\mu_i \) and \( \lambda_i + \mu_i \) are positive. If the solids are isotropic, then the displacements \( u_i(x, t) = (u_{i1}, u_{i2}, u_{i3}) \) in \( \Omega_i \) satisfy the following boundary value problem:

\[
\begin{align*}
(1.1) & \quad \rho_i \frac{\partial^2 u_{ij}}{\partial t^2} - \sum_{k=1}^3 \frac{\partial (\sigma_{jk}^{i1}(u_i))}{\partial x_k} = 0 \quad \text{in } \Omega_i \times R, \\
(1.2) & \quad u_1 = u_2 \quad \text{on } \Gamma \times R, \\
(1.3) & \quad \sum_j n_j(x) \sigma_{jk}^{i1}(u_i) = \sum_j n_j(x) \sigma_{jk}^{i2}(u_2) \quad \text{on } \Gamma \times R, \\
(1.4) & \quad \sum_j n_j(x) \sigma_{jk}^{i1}(u_i) = 0 \quad \text{on } \Gamma_i \times R,
\end{align*}
\]

where \( \rho_i > 0 \) is the density of \( \Omega_i \), \( \Gamma = \partial O_1 \cap \partial O_2 \), \( \Gamma_i = \partial O_i \setminus \Gamma \), and \( n(x) = \hat{n}(n_1(x), n_2(x), n_3(x)) \) is the unit normal vector of \( \Gamma \) or \( \Gamma_i \) \((j = 1, 2)\).

We consider an initial boundary value problem (1.1) to (1.4) with data

\[
(1.5) \quad u_i(x, 0) = f_i(x), \quad (\partial u_i/\partial t)(x, 0) = g_i(x) \quad \text{in } \Omega_i.
\]

We introduce a Hilbert space \( \mathcal{H} \) whose elements are equal to \((L^2(\Omega_i))^3\), with the inner product \( (f, g)_\mathcal{H} = \rho_i(f, g)_{L^2(\Omega_i)} + \rho_2(f, g)_{L^2(\Omega_2)} \), and a subspace \( D \) of \( \mathcal{H} \) such that \( f \) belongs to \( D \) if \( f \in (H^1(\Omega_1))^3 \) and the distributions \( L_i(f) = (\sum_k \partial (\sigma_{jk}^{i1}(f))/\partial x_k)_{\Omega_i} \) belong to \((L^2(\Omega_i))^3\). The boundary conditions (1.3) and (1.4) are represented as follows:

\[
(1.6) \quad \int_{\Omega_i} \left( L_i(f) \cdot v + \sum_{j,k} \sigma_{jk}^{i1}(f) \partial v_k/\partial x_j \right) dx + \int_{\partial_1} \left( L_i(f) \cdot v + \sum_{j,k} \sigma_{jk}^{i2}(f) \partial v_k/\partial x_j \right) dx = 0
\]

for any \( v = (v_1, v_2, v_3) \in (H^1(\Omega_i))^3 \). The operator \( A \) on \( D_N = \{ f \in D : f \) satisfies (1.6)\} is defined by \( Af|_{\Omega_i} = -L_i(f)/\rho_i \). We have the following
THEOREM 1.1. A is a self-adjoint operator on $\mathcal{H}$.

Proof. Since a set of $C_0^\infty(\Omega_i)$ functions which vanish in a neighbourhooed of $\partial\Omega_1 \cup \partial\Omega_2$ is a dense subset of $\mathcal{H}$, $D_N$ is clearly a dense subset of $\mathcal{H}$. If $f$ belongs to $D_N$, from (1.6) we have that

\begin{equation}
(\mathcal{A}f, g)_\mathcal{H} = \sum_{i=1}^3 \int_{\Omega_i} \left( \sum_{j,k} (\lambda_j \delta_{jk} + 2\mu_k) \varepsilon_{jk}(\varepsilon f) \varepsilon_{jk}(\varepsilon g) \right) dx
\end{equation}

for any $g \in (H'((O_1))^3$. From the above equality and the assumption $3\lambda_i + 2\mu_i > 0$ it follows that $A \subset A^*$ and $A \geq 0$. Thus in order to prove self-adjointness of $A$ we have to show that the range of $I + A$ is equal to $\mathcal{H}$. Let $\mathcal{H}(O_1) = (H'(O_1))^3$ be a Hilbert space with the inner product $(f, g)_\mathcal{H}$ defined by $(f, g)_\mathcal{H} = \langle f, g \rangle + \text{right hand side of (1.7)}$. By Korn’s inequality on $O_1$ (see p. 110 of [1]) it follows that $\|f\|^2_{H'(O_1)} \leq C(f, f)$. This fact and Riesz’s theorem imply that for any $g \in \mathcal{H}$ there exists $f \in \mathcal{H}'(O_1)$ such that $(g, v)_\mathcal{H} = (f, v)$, for any $v \in H'(O_1)$. From (1.6) this implies that $f \in D_N$ and $(I + A)f = g$. The proof is completed.

Since the domain $D(A)$ of $A$ is a dense subset of $\mathcal{H}$, we have the following

PROPOSITION 1.2. The domain of $A^{1/2}$ is equal to $\mathcal{H}'(O_1)$ and

\begin{equation}
\|A^{1/2} f\|^2 = \sum_{i=1}^3 \left( \sum_{j,k} (\lambda_j \delta_{jk} + 2\mu_k) \|\varepsilon_{jk}(f)\|_{L^2(\Omega_i)}^2 \right).
\end{equation}

In (1.5) we assume that $f = (f_1, f_2) \in D_N$ and $g = (g_1, g_2) \in \mathcal{H}'(O_1)$, then $u^{(1)}(x, t) = u(x, t)\mid_{\partial \Omega}$, where $u(x, t) = (\cos tA^{1/2})f + A^{-1/2}(\sin tA^{1/2})g$, satisfies (1.1) and (1.5) in the distribution sense. On (1.2) (1.3) and (1.4) we have the following

PROPOSITION 1.3. We put $\Gamma_0 = \Gamma \setminus (\Gamma_1 \cup \Gamma_2)$. Then $u_i(x, t)\mid_{\Gamma_0 \times \mathbb{R}} \in C(R_i; H^{1/2}_{\text{loc}}(\Gamma_0))$, $\sum_{j} n_j(x)\sigma_j^0(u_i)\mid_{\Gamma_0 \times \mathbb{R}} \in C(R_i; H^{-1/2}_{\text{loc}}(\Gamma_0))$ where $h = 0, i (i = 1, 2)$, and these satisfy (1.2), (1.3) and (1.4).

Proof. Since $u_i(x, t)$ belongs to $C(R_i; H'(\Omega_i))$, it follows that $u_i(x, t)\mid_{\Gamma_0 \times \mathbb{R}} \in C(R_i; H^{1/2}_{\text{loc}}(\Gamma_0))$ and $u_i(x, t) = u_i(x, t)$ on $\Gamma_0 \times \mathbb{R}$. Let $x_0$ be a point of $\Gamma_0$ and $U$ be an open neighbourhood of $x_0$ such that $U \subset O_i$ and that there exists a diffeomorphism $\kappa$ from $U$ to $y \in \mathbb{R}^3: |y| < \delta$ which maps $U \cap \Omega_i$ to $\{ y \in \mathbb{R}^3: |y| < \delta, y_3 > 0 \}$. For any $\phi(x) \in C_c^\infty(U)$ we put $\phi_i(y, t) = (\phi\kappa^{-1}(\kappa^{-1}(y)), t)$. Then from Theorem 4.3.1 of [2] we see that $\phi_i(y, t) \in C(R_i; H_{(x, y)}^{1/2}(\mathbb{R}^3))$, where $H_{(x, y)}^{1/2}(\mathbb{R}^3)$ is a function space denoted in Definition 2.5.1 of [2]. By
the trace theorem (see Theorem 2.5.6) in [2] it follows that the trace of
\( \sum_i n_i(x)\sigma^{(i)}_{jk}(u_i) \) on \( \Gamma_0 \times \mathbb{R} \) belongs to \( C(R_+: H^{-1/2}_{\text{loc}}(\Gamma_0)) \). Put \( v_\epsilon(y, t) = \epsilon^{-1}\int \rho((y' - z')/\epsilon)v(y', y, t)dz' \), where a non-negative function \( \rho(y') \) belongs to \( C_\mathbb{T}(\{y' \in \mathbb{R}^2: |y'| < 1\}) \) and \( \int \rho(y')dy' = 1 \). Then using Theorem 2.5.4 of [2], we can easily prove that \( v_\epsilon(y, t) \in C(R_+: H_{\omega, 0}(\mathbb{R}^1_+)) \), supp \( v_\epsilon \subset \{y \in \mathbb{R}^1: |y| < \delta\} \) if \( \epsilon \) is sufficiently small, and \( v_\epsilon(y, t) \) converges to \( v(y, t) \) in the topology of \( C(R_+: H_{\omega, -1}(\mathbb{R}^1_+)) \). These facts imply that the divergence theorem is valid for \( (\sum_j (\sigma^{(j)}_{\nu}(u_j)\nu_j))_{b=1,2,3} \), where \( v = (v_1, v_2, v_3) \in (H^1(O))^3 \). Similarly we can prove the same fact for \( \phi u_2 \). By (1.6) it follows that
\[
\int_{\Gamma_0} \sum_{j,k} n_j(x)(\sigma^{(j)}_{\nu}(u_j) - \sigma^{(j)}_{\nu}(u_2))v_\epsilon(x)dx = 0
\]
for any \( v(x) \in (C^\infty_\mathbb{T}(U))^3 \). The (1.3) is valid. Similarly we can prove (1.4). The proof is completed.

Remark 1.4. The arguments used in this section are easily extended for finite number of media which are not isotropic, whose displacements satisfy the similar boundary value problem to (1.1) to (1.5).

§ 2. Reduction to first order systems and definition of rays

In this section in order to study propagation of singularities to the solutions of (1.1) to (1.3) we shall reduce the considered boundary value problem to the first order system. After that we shall define an incident ray, a reflected ray, a transferred reflected ray, a refracted ray and a transferred refracted ray, which are half null bicharacteristics of \( \tau^\epsilon - \sigma^\epsilon_j [\xi^\epsilon]^j \) or \( \tau^\epsilon - \beta^\epsilon_j - \beta^\epsilon_j [\xi^\epsilon]^j \).

Let us consider a solution \( u_i(x, t) \) of (1.1). Hereafter we assume that \( u_i \) is an extensible distribution, that is, there exists a distribution \( U_i(x, t) \) on \( R^i \) such that \( U_i = u_i \) on \( \Omega_i \times \mathbb{R} \). Thus by Theorem 4.3.1 of [2] the traces of \( u_i|_{\Gamma_0 \times \mathbb{R}} \) and \( \sigma^{(j)}_{\nu}(u_i)|_{\Gamma_0 \times \mathbb{R}} \) are distributions on \( \Gamma_0 \times \mathbb{R} \), and we can suppose these distributions satisfy the conditions (1.2) and (1.3). From now on we assume that \( n(x) \) appearing in (1.3) is the unit outer normal vector of \( \Omega_i \) at \( \Gamma_0 \). Since the boundary value problem (1.1) to (1.3) is rotation free, we may assume that the origin of \( R^i \) belongs to \( \Gamma_0 \) and \( n(0) = \epsilon'(0, 0, -1) \). In a neighbourhood \( U_0 \) of \( 0 \) \( \Gamma_0 \) is defined by \( x_i = g(x') \), where \( x' = (x_1, x_2) \). Making use of the coordinate transform \( \kappa; y' = x' \), \( y_3 = x_3 - g(x') \) such that \( \Omega_i \cap U_0 \) is tranformed into \( \{y: y_3 > 0\} \) and putting
$U_t(y, t) = A^\phi^{y, t} u_n, D_y^2 u_t)$, where $A$ is a pseudo-differential operator with the symbol $A(\gamma y, \tau) = (|G|^2 + \tau^2 + 1)^{1/2}$, the problem (1.1) to (1.3) is reduced to the following boundary value problem (1.1) to (1.3) is reduced to the following boundary value problem (see section 1.1 of [9]):

$$
\begin{cases}
    D_y U_t = M_i(y', D_y', D_t)U_i & \text{in } (-1)^{i+1} y_s > 0, \\
    (I_0, 0)U_t = (I_0, 0)U_s & \text{on } y_s = 0, \\
    B_i(y', D_y', D_t)U_i = B_s(y', D_y', D_t)U_s & \text{on } y_s = 0,
\end{cases}
$$

where $I_i$ is the $3 \times 3$ identity matrix and the principal symbol $(B_i, B_s)$ of $B_i = (B_{ii}, B_{is})(y', D_y', D_t)$ is

$$
\begin{align*}
    (B_{ii}(y', \gamma'), \tau) &= (\lambda_i G'^2 + \mu \tilde{\gamma}^2 G + \mu_i G^2) A_i^{-1}, \\
    (B_{is}(y', \gamma'), \tau) &= (\lambda_i + \mu_i) G'^2 G + \mu_i G^2 I_i,
\end{align*}
$$

with $G = \langle -F g(y') \rangle$ and $\tilde{\gamma} = \langle \gamma_i, \tilde{\gamma}_s, 0 \rangle$. Here the principal symbol $M_i(y', \gamma', \tau)$ of $M_i(y', D_y', D_t)$ satisfies that $\det(\gamma_i I_s - M_i) = (\gamma_i - \alpha)^2 + p_i((\gamma_i - \alpha)^2 + s_i)^2$, where $a(y', \gamma') = \gamma' \cdot F g(y')/|G|^2$, $s_i(y', \gamma', \tau) = (|\gamma'|^2 - \tau^2/\alpha_i^2) - (\gamma' \cdot F g)|G|^2/|G|^2$ and $p_i(y', \gamma', \tau) = (|\gamma'|^2 - \tau^2/\beta_i^2) - (\gamma' \cdot F g)|G|^2/|G|^2$ with $\alpha_i = \mu_i/\rho_i$ and $\beta_i = (\lambda_i + 2\mu_i)/\rho_i$.

We shall use notions of wave front set $WF(G)$ for $G(y', t) \in \mathcal{D}'(R_{y', t})$ defined in [3] and micro-local smoothness of $F \in C^\infty([0, \delta] \cap \mathcal{D}'(R_{y', t})) \cup C^\infty([-\delta, 0] \cap \mathcal{D}'(R_{y', t}))$ at $\rho \in T^*(R_{y', t}) \setminus \{0\}$, which means that there exists a properly supported pseudo-differential operator $A(y', t, D_y', D_t)$ such that $A$ is elliptic at $\rho$ and $(AF)(y', t) \in C^\infty([0, \pm \varepsilon] \times R_{y', t})$ for some $\varepsilon > 0$.

Let us consider a point $(0, \gamma_0, \tau_0)$ such that $\gamma_0/\tau_0 = (p_0, s_0, \delta_0)(0, \gamma_0, \tau_0) \neq 0$. Put $\bar{a}_i^\pm(y', \gamma', \tau) = a \pm (-1)^{i+1}(-s_i)^{1/2}$ if $s_i(0, \gamma_0, \tau_0) > 0$ and put $\bar{a}_i^\pm(y', \gamma', \tau) = a \pm (-1)^{i+1}(-s_i)^{1/2}$ if $s_i(0, \gamma_0, \tau_0) < 0$ and $\varepsilon = \text{sgn} \tau_0$, where the branch of $(-1)^{1/2}$ is the imaginary unit. Similarly making use of $p_i$ instead of $s_i$, we define $\bar{b}_i^\pm(y', \gamma', \tau)$. Then by Lemma 1.1 of [9] $\bar{a}_i^\pm$ and $\bar{b}_i^\pm$ are eigen values of $M_i(y', \gamma', \tau)$ and there exist eigen vectors $s_{ik}^\pm$ and $s_{ik}^\pm$ of $\bar{a}_i^\pm$ and $\bar{b}_i^\pm$, respectively, which are linearly independent. By the argument of Section 2 of [8] we can reduce the boundary value problem (1.2) as follows: There exists an elliptic pseudo-differential operator $S_i(y', D_y', D_t)$ of order 0 defined in a conic neighbourhood of $\rho_0 = (0, t, \gamma_0, \tau_0) \in T^*(R_{y', t}) \setminus \{0\}$ with the principal symbol $(s_{ik}^\pm, s_{ik}^\pm, s_{ik}^\pm, s_{ik}^\pm, s_{ik}^\pm)$ such that the boundary value problem (2.1) is micro-locally reduced to the following.
\[
\begin{aligned}
&V_i = S_i^{-1} U_i, \quad C_i(y', D_y, D_t) = \begin{pmatrix} I & 0 \\ B_{1i} & B_{2i} \end{pmatrix} S_i, \quad \rho_0 \in WF(G) \quad \text{and} \quad F_i \quad \text{is} \quad \text{smooth at} \quad \rho. \quad \text{Moreover the principal symbol of} \quad A_i(y', D_y, D_t) \quad \text{is the diagonal matrix} \quad \tilde{a}_i(y', \gamma', \tau)I_3 \quad \text{and the principal symbol of} \quad b_i(y', D_y, D_t) \quad \text{is} \quad \tilde{b}_i(y', \gamma', \tau). \quad \text{At} \quad y = 0 \quad \text{the principal symbols of} \quad C_1 \quad \text{and} \quad C_2 \quad \text{are simplified as follows:}
\end{aligned}
\]

**Lemma 2.1.** If \( \tau_1, \gamma'_1((p_1, p_2, s_1, s_2)(0, \gamma'_0, \tau_0) = 0 \), then we may assume that at \( y' = 0 \) the principal symbol \( C_i = (C_i^+, C_i^-) \) of \( C_i(y', D_y, D_t) \) is given by

\[
C_i^+ = \begin{pmatrix} a_i^+ |\gamma'|^2 A_i^{-2} & 0 & \gamma|\gamma'| A_i^{-3} \\ 0 & |\gamma'|^2 A_i^{-2} & 0 \\ -|\gamma'|^2 A_i^{-2} & 0 & b_i^+ A_i^{-1} \end{pmatrix},
\]

\[
C_i^- = \begin{pmatrix} (\rho, \tau^2 - 2\mu_1 |\gamma'|^2) |\gamma'| A_i^{-2} & 0 & 2\mu_1 b_i^+ |\gamma'| A_i^{-3} \\ 0 & \mu_1 a_i^+ |\gamma'| A_i^{-2} & 0 \\ -2\mu_1 a_i^+ |\gamma'| A_i^{-2} & 0 & (\rho, \tau^2 - 2\mu_1 |\gamma'|^2) A_i^{-2} \end{pmatrix}
\]

with \( a_i^+(\gamma', \tau) = \tilde{a}_i(0, \gamma', \tau) \) and \( b_i^+(\gamma, \tau) = \tilde{b}_i(0, \gamma', \tau) \).

**Proof.** In order to simplify the principal symbol of \( C_i(y', D_y, D_t) \) we use an elliptic pseudo-differential operator \( D_2(D_y, D_t) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \) of order 0, where the principal symbol \( d_{ij}(\gamma', \tau) \) of the components of \( 3 \times 3 \) matrix \( D(D_y, D_t) \) are \( d_{11} = d_{22} = \gamma_1 A_1(\gamma', \tau), \quad d_{12} = -d_{21} = \gamma_2 A_1(\gamma', \tau), \quad d_{33} = 1 \) and \( d_{13} = d_{23} = d_{31} = d_{32} = 0 \). By (1.8) of [9] we can take the principal symbol of \( C_i \) as follows; \( s_{ii} = \begin{pmatrix} (\gamma_{1i} A_1)^{-1} w_{1i} \\ \gamma_{1i} A_1^{-1} w_{2i} \\ \gamma_{1i} A_1^{-1} w_{3i} \end{pmatrix} \) with \( w_{1i} = (\tilde{a}_i(\gamma' - \tilde{a}_i V g), -|\gamma' - \tilde{a}_i V g|^2) A_i^{-1}, \quad s_{i2} = \begin{pmatrix} (\gamma_{2i} A_1 A_1^{-1} w_{2i}) \\ -a_{i2} A_1^{-1} w_{3i} \end{pmatrix} \) with \( w_{2i} = (-(\gamma_1 - a_{i1} \delta g/\delta y_1), \gamma_2 - a_{i2} \delta g/\delta y_1, 0) A_1^{-1} \) and \( s_{i3} = \begin{pmatrix} (\gamma_{3i} A_1^{-1} w_{3i}) \\ -a_{i3} A_1^{-1} w_{2i} \end{pmatrix} \) with \( w_{3i} = (\gamma' - b_{ij} G, b_{ij}) A_1^{-1}. \) Since \( G = 0 \) at \( y' = 0 \), making use of (2.2), we can compute the principal symbol \( D_2 C_i(y', D_y, D_t) \), which is given by (2.4) at \( y' = 0 \). The proof is completed.

Let us consider the incident plane wave \( P_i(\omega) \) in \( \Omega_i \), hitting on \( (0, t_0) \) with a direction \( \omega = (\omega_1, \omega_2, \omega_3) \in S^2 \) such that \( 0 < n(0)\cdot \omega < 1 \), which is the half connected null bicharacteristic \( \{(-\beta_0 t, t_0 - t, -\varepsilon_0, \varepsilon_0^2) \in T^*(\Omega \times R) : t > 0 \} \) of \( \tau^2 - \rho_0^2 |\xi|^2 \) passing through \( \rho_0 = (0, t_0, -\varepsilon_0, \varepsilon_0^2) \), where \( \varepsilon_0^2 = 1. \) Since
the outer unit normal vector of $\Omega$, at 0 is $(0, 0, -1)$, the reflected $P$ ray $P_\omega(x)$ of $P_\omega(x)$ is given by the connected ray $(\beta(t)x, t + t_\omega, \omega_\beta) \in T^*(\Omega \times R)$, where $\omega = (\omega, \omega_\beta, -\omega)$. Hereafter we say that a ray $r(t)$ in $\Omega$ parametrized by time $t$ is outgoing (incoming) if $(-1)(dx/dt)(0) < 0 (>0)$, where $x(t)$ is the $x$ component of $r(t)$. The half connected outgoing null bicharacteristic $S_\omega(x)$ of $\tau^2 - \alpha^2_2|\xi|^2$ in $T^*(\Omega \times R)$ passing through $\rho$ with $\pi(\rho) = \pi(\rho_0)$ is called the transferred reflected ray of $P_\omega(x)$, where $\pi$ is the projection from $T*(\Omega \times R)$ to $T*(\Omega \times R)$. If there exists the half connected outgoing null bicharacteristic $P_\omega(x)$ of $\tau^2 - \beta^2_2|\xi|^2$ in $T^*(\Omega \times R)$ passing through $p$ with $\pi(p) = \pi(p_0)$, we call it the refracted $P$ ray (the transferred refracted $S$ ray) of $P_\omega(x)$. Similarly for the incident $S$ ray $S_\omega(x)$ passing through $p = (0, t_\omega, -\omega_\beta, \epsilon_j\beta)$, the reflected $S$ ray $S_\omega(x)$, the transferred reflected $P$ ray $P_\omega(x)$, the refracted $S$ ray $S_\omega(x)$ and the transferred refracted $P$ ray $P_\omega(x)$ are defined, if these rays exist. The rays are concretely denoted as follows:

**Lemma 2.2.** i) Put \( \omega = (\omega', \pm (\beta^2_2/\alpha^2_2 - 1 + (n(0)\omega)^2)^{1/2}), \) then $S_\omega(x) = \{(\alpha^2_2\omega^2_\beta/\beta, t + t_\omega, -\omega_\beta, \epsilon_j\beta) \in T^*(\Omega \times R) : t > 0 \}$. A similar statement is valid for $P_\omega(x)$, if $1 - \alpha^2_2/\beta^2_2 < (n(0)\omega)^2$. 

ii) Put \( \omega'' = (\omega', \pm (\beta^2_2/\beta^2_2 - 1 + (n(0)\omega)^2)^{1/2}), \) if $1 - \beta^2_2/\beta^2_2 < (n(0)\omega)^2$; then $\tilde{P}_\omega(x) = \{(\beta^2_2\omega''^2/\beta, t + t_\omega, -\omega_\beta, \epsilon_j\beta) \in T^*(\Omega \times R) : t > 0 \}$. Similarly, $\tilde{S}_\omega(x)$ is defined, if $1 - \alpha^2_2/\beta^2_2 < (n(0)\omega)^2$. 

iii) Put \( \omega''' = (\omega', \pm (\beta^2_2/\alpha^2_2 - 1 + (n(0)\omega)^2)^{1/2}), \) if $1 - \beta^2_2/\alpha^2_2 < (n(0)\omega)^2$, then $\tilde{S}_\omega(x) = \{(\alpha^2_2\omega'''^2/\beta, t + t_\omega, -\omega_\beta, \epsilon_j\beta) \in T^*(\Omega \times R) : t > 0 \}$. Similarly, $\tilde{P}_\omega(x)$ is defined, if $1 - \alpha^2_2/\beta^2_2 < (n(0)\omega)^2$.

The proof of Lemma 2.2 is easily derived from the definitions of the rays. For the incident $P$ ray $P_\omega(x)$ we also denote by $S_\omega(x), \tilde{P}_\omega(x)$ and $\tilde{S}_\omega(x)$ half connected incoming null bicharacteristics of $\tau^2 - \alpha^2_2|\xi|^2$ and $\tau^2 - \beta^2_2|\xi|^2$ passing through $(0, t_\omega, -\omega_\beta, \epsilon_j\beta), (0, t_\omega, -\omega_\beta, \epsilon_j\beta)$ and $(0, t_\omega, -\omega_\beta, \epsilon_j\beta)$, respectively, if these rays exist. Similarly for the incident $S$ ray $S_\omega(x)P_\omega(x), \tilde{P}_\omega(x)$ and $\tilde{S}_\omega(x)$ are defined.

§ 3. Singularities corresponding to Stonly waves

In this section we analyze singularities to a solution of (1.1) to (1.3) near an elliptic point $(0, t_\omega, \gamma_\omega, \tau_\omega)$, that is, $s_i(0, \gamma_\omega, \tau_\omega) > 0$ for $i = 1, 2$. In [7] he proved that if $\Gamma_0$ is a hyperplane of $R^2$, there are surface waves satisfying (1.1) to (1.3) and propagating on the boundary $\Gamma_0 \times R$. In this
section without assuming the flatness of $\Gamma$, we shall show that there exist rays belonging to the wave front set of a solution of (1.1) to (1.3) which propagates in the elliptic region of the boundary $\Gamma_0 \times R$.

We shall consider the Lopatinski matrix of the boundary value problem (2.3) in the elliptic region $\{(y', t, \eta', \tau): s_i(y', \eta', \tau) > 0 \text{ for } i = 1, 2\}$, where $p_i(y', \eta', \tau) > 0 \text{ for } i = 1, 2$. We remark that if $y' = 0$, then the elliptic region is $\{(0, t, \eta', \tau): \min(\alpha^i, \alpha^j)|\eta'|^2 > \tau^2\}$.

**Lemma 3.1.** We assume that $\tau_0 \neq 0$ and $\min(\alpha^i, \alpha^j)|\eta'|^2 > \tau_0^2$. Then the necessary and sufficient condition that the determinant $(C^i, C^j)$ is zero is given by $F(\tau^2/|\eta'|^2) = 0$, where

$$F(s) = (\rho_i a^i(s) - \rho_j a^j(s))(\rho_i b^i(s) - \rho_j b^j(s)) + (\rho_i - \rho_j)s^2$$

$$- 4(\mu_i - \mu_j)(\rho_i a^i b^i(s) - \rho_j a^j b^j(s) + \rho_i - \rho_j)s$$

$$+ 4(\mu_i - \mu_j)^2((a^i b^i a^j b^j)(s) + (a^i b^j a^j b^i)(s) + (a^i b^i a^j b^j)(s) + 1)$$

with $a^i(s) = (-1)^{i+1}i(1 - s/\alpha^i)^{1/2}$ and $b^i(s) = (-1)^{i+1}i(1 - s/\beta^i)^{1/2}$.

**Proof.** By (2.4) the second and fifth column vectors of $(C^i, C^j)$ are linearly independent to the other column vectors, and these two vectors are linearly independent. Thus the condition $\det(C^i, C^j) = 0$ is equivalent to the condition $\det M_i = 0$, where $M_i$ is the $4 \times 4$ square matrix generated by eliminating the second and fifth column and line vectors of $(C^i, -C^j)$. By simple calculations we can show that $\det M_i = C|\eta'|^6F(\tau^2/|\eta'|^2)$, where $C$ is a non zero constant. The proof is completed.

If we assume the Wiechert condition in seismology, that is, $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$, then we can get informations on the roots of $F(s) = 0$.

**Lemma 3.2.** We assume $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$, and put $\alpha = \alpha_1$ and $\beta = \beta_1$. Then we have the following statements.

i) The roots of $F(s) = 0$ in $(0, \alpha^2)$ are also the roots of $f(x) = 0$ in $(0, 1)$ where $x = s/\alpha^2$ and $f(x) = x^4 - (\gamma + 1 + 8\gamma M)x^3 + [1 + 24\gamma M + 8M + 8\gamma M^2 - M^2 - 16\gamma^2 M^2]x^2 - 8\gamma M^3 - 4\gamma^2 M^2 + 3M + 2\gamma M^2 + 16(M + \gamma^2 M)$ with $\gamma = \alpha/\beta$ and $M = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)^2$.

ii) $f(x)$ has at least one root in $(0, 1)$.

iii) If all roots $f(x) = 0$ in $(0, 1)$ are simple, then the set $(y', t, \eta', \tau): \det(C^i, C^j)(y', \eta', \tau) = 0$ is locally given by $\tau = h(y', \eta')$, where $h(y', \eta')$ is a positively homogeneous function of degree 1.

**Proof.** In $F(s)$ put $s/\alpha^2 = x$, then $F(s)$ is equal to $\alpha^4[(\rho_1 - \rho_2)^2(x^4 +
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4(1 - x)(2 - r x) - ((ρ_1 + ρ_2)x^2 + 4(ρ_1 - ρ_2)(2 - x))(1 - r x)^{1/2}(1 - x)^{1/2} = α'[g_1(x) - g_2(α)], where g_1(x) is defined by the last equality. Since g_1(x) and g_2(α) are positive if x ∈ (0, 1), the condition F(s) = 0 is equivalent to g_1(x) - g_2(α) = (ρ_1 + ρ_2)x^2f(x) = 0. The statement ii) is clear, for f(0) = 16M(1 - r^2) > 0 and f(1) = (ρ_1 + ρ_2)^{-1}(g_1(1) - g_2(1)) = -M^2 < 0. The statement iii) is a consequence of the implicit function theorem. The proof is completed.

We shall check conditions that the polynomial f(x) has only simple roots in (0, 1).

Remark 3.3. i) One of the equivalent conditions that a polynomial f(x) has a double roots is that the discriminant of f(x) is zero. In this case the discriminant of (ρ_1 + ρ_2)x^2f(x) is a polynomial with respect to ρ_1, ρ_2, α and β. Thus for almost everywhere (ρ_1, ρ_2, α, β) the equation f(x) = 0 has only simple roots.

ii) A simple condition of simplicity of the roots in (0, 1) of f(x) = 0 is given as follows: If f''(1) ≥ 0 and (r + 1 + 8M)/4r ≥ 1, where x = (r + 1 + 8M)/4r is the symmetric axis of f''(x) = 0, then f''(x) > 0 in [0, 1]. This condition implies that f(x) = 0 has only one root in (0, 1). Thus if 3r + 8M(rM + 1) ≥ (16r^2 + 1)M^2 + 2 and 8rM + 1 ≥ 3r, then f(x) = 0 has only one simple root in (0, 1).

Let us consider the Lopatinski determinant of the boundary value problem (2.3) in a conic neighbourhood of ρ_0 = (0, t_0, γ_0, 0).

Lemma 3.4. If τ_0 = 0, then the Lopatinski determinant of (2.3), that is, the determinant of the principal symbol (C^*_x, C^*_z) is not zero at ρ_0.

Proof. In the case τ_0 = 0, in Section 1.2 of [9] S_k(0, γ_0, 0) (k = 1, 2) is given as follows: Put S_k(0, γ_0, 0) = (s^x_k, s^y_k, s^z_k, s^x_+^k, s^x_-^k), then

\[ \langle s^x_k \rangle = (\pm(1)\gamma_0^x, \pm(1)\gamma_0^y, \pm(1)^x\gamma_0^x, \pm(1)^x\gamma_0^y, \pm(1)^x\gamma_0^z, \pm(1)^x\gamma_0^w) \],

\[ \langle s^y_k \rangle = (w_k, w_k, w_k, w_k, w_k, w_k) \] with \[ w_k = (-\gamma_0^x, \gamma_0^y, 0)A_1^{-1}, \]

\[ \langle s^z_k \rangle = (w_k, w_k, w_k, w_k, w_k, w_k) \] with \[ w_k = (\gamma_0^x, \gamma_0^y, 0)A_1^{-1}. \]

Using S_k(0, γ_0, 0) and (2.2) we can easily compute det (C^*_x, C^*_z)(0, γ_0, 0), which is equal to C(μ_1 + μ_2) × \{μ_1(λ_1 + 2μ_1) + μ_2(λ_1 + 3μ_1)\}μ_2(λ_2 + 2μ_2) + μ_1(λ_2 + 3μ_2)\}ρ_1ρ_2 with a non zero constant C. That is not zero. The proof is completed.

For the solution u_i of (1.1) we define WF_{s}(u_i) ⊂ (T^*(Ω_i × R)\0) ∪ (T^*(Γ_i × R)\0) as follows: i) ρ ∈ T^*(Ω_i × R) belongs to WF_{s}(u_i), if ρ belongs
to $WF(u_{1}|_{a_{2}})$, ii) $\rho \in T^{*}(\Gamma _{0}\times R)$ does not belong to $WF_{b}(u_{i})$, if $u_{i}(\kappa ^{-1}(y), t)$ has the property of micro-local smoothness at $\pi(\kappa ^{*}(\rho ))$, where $\pi$ is the projection from $T^{*}(R_{x_{i}}, t)$ to $T^{*}(\{y_{5} = 0\} \times R)$ and $\kappa ^{*}$ is the diffeomorphism from $T^{*}(U_{0} \times R)$ to $T^{*}(R_{x_{i}}, t)$ induced from $\kappa$, that is, $\kappa ^{*}(x, t, \xi, \tau ) = (x', x_{3} - g(x'), t, \xi' + (fg)(x')\xi_{3}, \xi_{a}, \tau )$.

This definition of $WF_{b}(u_{i})$ is invariant by the diffeomorphism $\kappa$ (see Proposition 1.2 in [6]). Let $\Sigma '_{\xi}$ be an elliptic region on the boundary, i.e., $\Sigma '_{\xi } = \{(y', t, \eta ', \tau ) \in T^{*}(R_{x_{i}}, t): s_{i}(y', \eta ', \tau ) > 0, i = 1, 2\}$, and $\Sigma _{\xi }^{0}$ be a subset of $\Sigma '_{\xi }$ such that $\det (C_{1}^{*}, C_{2}^{*})(y', \eta ', \tau ) = 0$. Put $\Sigma _{\xi } = (\pi \circ \kappa ^{*})^{-1}(\Sigma _{\xi }^{0})$ and $\Sigma _{\xi }^{0} = (\pi \circ \kappa ^{*})^{-1}(\Sigma _{\xi }^{0})$; Then we have the following

**THEOREM 3.5.** We assume that $\alpha _{i} = \alpha _{2}$ and $\beta _{1} = \beta _{2}$ and that $f(x)$ of Lemma 3.2 has only simple roots in $(0, 1)$. Then $WF_{b}(u_{i}) \cap \Sigma _{\xi } \subset \Sigma _{\xi }^{0}$, where $\Sigma _{\xi }$ is locally given by $\tau = h(x, \xi )$ with $C^{\infty }$ homogeneous function $h(x, \xi )$ on $T^{*}(\Gamma _{0}) 0$ of order 1, and $WF_{b}(u_{i}) \cup WF_{b}(u_{2})$ is invariant under the Hamilton vector field $H_{\tau _{\text{h}}}$ on $T^{*}(\Gamma _{0} \times R)$ 0.

**Proof.** In (2.3) we denote $V_{\xi } = ('V_{\xi }^{*}, 'V_{\xi }^{*})$. Then since $V_{\xi }^{*}$ and $V_{\xi }^{*}$ satisfy backward parabolic equations, $\rho _{0}$ does not belong to $WF(V_{\xi _{1}}^{*}|_{y_{5_1} = 0}) \cup WF(V_{\xi _{2}}^{*}|_{y_{5_2} = 0})$. It follows that $WF(C_{1}^{*}V_{\xi _{1}}^{*}|_{y_{5_1} = 0} - C_{2}^{*}V_{\xi _{2}}^{*}|_{y_{5_2} = 0})$ does not contain $\rho _{0}$. By Hörmander’s theorem on propagation of singularities (see Theorem 6.1.1 of [4] we have the desired statement.

**Remark 3.6.** i) Let $\{f_{i}, g_{i}\}$ be the initial data of the solution $u(x, t)$ of (1.1) to (1.5). Assume that $(x_{0}, \xi _{0}) \in WF(f_{i}|_{\tau _{0}}) \cup WF(g_{i}|_{\tau _{0}})$, Then by Theorem 2.5.11V of [3] there exists $\tau _{0}$ such that an element $\rho _{0} = (x_{0}, 0, \xi _{0}, \tau _{0})$ of $T^{*}(\Gamma _{0} \times R)$ belongs to $WF(u_{i}|_{\tau _{0} = 0})$. If $\rho _{0}$ is an elliptic point, then by Theorem 3.5 there exists a ray belonging to $WF_{b}(u_{i}) \cup WF_{b}(u_{2})$, which starts at $\rho _{0}$ and propagates on the border $\Gamma _{0} \times R$.

ii) From the form of $F(s)$, the null points of $F(s) = 0$ are roots of some polynomial of degree 22 whose coefficients are polynomial of $(\rho _{1}, \lambda _{1}, \mu _{1}, \rho _{2}, \lambda _{2}, \mu _{2})$. Thus for almost all $(\rho _{1}, \lambda _{1}, \mu _{1}, \rho _{2}, \lambda _{2}, \mu _{2})$ with $F(s)|_{s = 0}$ $F(\min (\alpha _{1}^{p}, \alpha _{2}^{p})) < 0$ Theorem 3.5 holds.

§ 4. Incident $P$ singularities

In the case that $\alpha _{2} < \beta _{1} < \beta _{2}$ and $\alpha _{1} \neq \alpha _{2}$ there exist interesting reflective and refractive phenomena. Thus in this section we assume the above condition. We shall consider incident $P$ singularities and show
the following theorems on reflective and refractive phenomena of singularities.

**Theorem 4.1.** i) We assume $\beta^i_1 < \beta^j_1 (1 - (n(0) \cdot \omega)^2)$. Then $\tilde{P}_{in}(\omega)$ and $\tilde{P}_{r}(\omega)$ do not exist. Furthermore we suppose that $S_{in}(\omega) \cap WF(u_c) = \tilde{S}_{in}(\omega) \cap WF(u_c) = \phi$ and $P_{r}(\omega) \subset WF(u_c)$. Then $P_{r}(\omega) \cup S_{in}(\omega) \subset WF(u_c)$ and $\tilde{S}_{in}(\omega) \subset WF(u_c)$.

ii) We assume $\beta^i_1 > \beta^j_1 (1 - (n(0) \cdot \omega)^2)$. Then there exists a function $G_s(\omega)$ whose null points are at most 30 such that if $G_s(\omega) \neq 0$, $S_{in}(\omega) \cap WF(u_c) = \tilde{P}_{in}(\omega) \cup \tilde{S}_{in}(\omega) \cap WF(u_c) = \phi$ and $P_{r}(\omega) \subset WF(u_c)$, then $P_{r}(\omega) \cup S_{tr}(\omega) \subset WF(u_c)$ and $\tilde{S}_{tr}(\omega) \subset WF(u_c)$.

The concrete form of $G_s(\omega)$ is given in Lemma 4.3. The idea of proving the above theorem is as follows: First we shall look for an elliptic pseudodifferential operator $A$ such that some components of $AC_i (i = 1, 2)$ vanish. After that making use of the assumptions of $WF(u_c)$, we shall check the conditions to the wave front sets of the components of $V_{c}(\mid x = 0)$ ($i = 1, 2$), which derive the statements of the theorem.

Let $\rho_0$ be $(0, t_0, -\omega, \varepsilon \beta_j)$, where $\varepsilon^2 = 1, 0 < n(0) \cdot \omega < 1$. Then the projected point $\rho_i$ of $\rho_0$ to $T^* (\partial \Omega \times R)$ is $(0, t_0, -\omega', \varepsilon \beta_j)$. In a conic neighbourhood $\Gamma_1$ of $\rho_1$ in $T^* (R^{d,i}) \backslash 0$ we may assume that at $y = 0$ the principal symbols of $A^i_1$, $b^i_1$ in (2.3) are $a^i_1(y', \tau) = \pm (\varepsilon^2 |\tau|^{1/2}) I_d$, $a^1_1(y', \tau) = \pm (\varepsilon^2 |\tau|^{1/2}) I_d$, and $b^1_1(y', \tau) = \pm (\varepsilon^2 |\tau|^{1/2}) I_d$, and $b^1_1(y', \tau)$ is $\mp i(y^2 - \tau^2/\beta^2_1)$ if $\beta^2_1 < \beta^2_1 (1 - (n(0) \cdot \omega)^2)$ and is $\pm (\varepsilon^2 |\tau|^{1/2}) I_d$ if $\beta^2_1 > \beta^2_1 (1 - (n(0) \cdot \omega)^2)$. The boundary operator $C_0(y', D_y, D_t)$ of (2.3) is also defined by using these notations. We say that a pseudodifferential operator $P(y', t, D_y, D_t) \in L^{-\infty}(\Gamma)$, where $\Gamma$ is a conic open set of $T^* (R^{d,i})$, if the symbol of $P$ is rapidly decreasing with respect to $(\gamma', \tau)$ in $\Gamma$. We have the following

**Lemma 4.2.** There exists a pseudo-differential operator $A(y', D_y, D_t)$ of order 0 defined in a conic neighbourhood $\Gamma_1$ of $\rho$, such that the principal symbol $A_0(y', \gamma', \tau)$ of $A$ is the identity matrix $I_d$ at $y' = 0$, and that the (1, 2), (2, 1), (2, 3), (3, 1), (4, 2), (5, 1), (5, 3) and (6, 2) components of $(AC_1)(y', D_y, D_t)$ (i = 1, 2) are 0 modulo $L^{-\infty}(\Gamma_1)$.

**Proof.** Put $C_1 (y', D_y, D_t) = (c_1, c_2, c_3)(y', D_y, D_t)$, $C_2 (y', D_y, D_t) = (c_4, c_5, c_6)(y', D_y, D_t)$ and denote by $a_j(y', D_y, D_t)$ the $j$-th line vector of $A(y', D_y, D_t)$. Then the required conditions are
(4.1) \[ a_j \cdot c_2 = a_j \cdot c_3 = 0 \mod L^{-\infty}(\Gamma_j) \quad (j = 1, 3, 4, 6) \]
(4.2) \[ a_j \cdot c_1 = a_j \cdot c_3 = a_j \cdot c_4 = a_j \cdot c_6 = 0 \mod L^{-\infty}(\Gamma_j) \quad (j = 2, 5). \]

We denote the symbol of \( a_j \) by \( \sum_{i=0}^{\infty} a_{j,i}(y', \gamma', \tau) \), where \( a_{j,k} \) is of order \(-k\), and the principal symbol of \( c_j \) by \( c_{j,0}(y', \gamma', \tau) \). Put \( a_{10} = f_1 + x_{12}c_{20} + x_{15}c_{50} \), where \( f_1 = (1, 0, \cdots, 0) \in R^n \). Then the condition \( \langle a_{10}, c_{20} \rangle = \langle a_{10}, c_{50} \rangle = 0 \) is equivalent to

\[
\begin{bmatrix}
\langle c_{20}, c_{20} \rangle & \langle c_{20}, c_{50} \rangle \\
\langle c_{25}, c_{50} \rangle & \langle c_{50}, c_{50} \rangle
\end{bmatrix} \begin{bmatrix} x_{12} \\ x_{15} \end{bmatrix} = - \begin{bmatrix} \langle f_1, c_{20} \rangle \\ \langle f_1, c_{50} \rangle \end{bmatrix} .
\]

Since from (2.4) \( c_{20} \) and \( c_{50} \) are linearly independent, we can solve the equation (4.3) and \( x_{12} \) and \( x_{15} \) are zero at \( y' = 0 \). Similarly if we put \( a_{18} = x_{12}c_{20} + x_{15}c_{50} \), where \( x_{12}(y', \gamma', \tau) \) and \( x_{15}(y', \gamma', \tau) \) are of order \(-k\), we can decide the required \( a_{18} \). Similarly in order to construct \( a_j(y', D_v', D_v) \) which satisfies (4.2) we only check that \( c_{10}, c_{30}, c_{40} \) and \( c_{60} \) are linearly independent. This condition is equivalent to the condition \( \det M(y', \tau) \neq 0 \), where \( M(y', \tau) \) is the \( 4 \times 4 \) square matrix which is generated by eliminating the second and fourth column and line vectors of \((C^1, C^2)(0, y', \tau)\). We have

\[
\det M(y', \tau) = \left\{ \begin{array}{l}
\left[ p_1 a_1^1 b_1^1 + p_2 a_1^2 b_2^2 - \rho_1 p_2 a_1^1 b_2^2 + a_1^1 b_1^1 \right] + (\rho_1 - \rho_2)^2 y' \tau^4 \\
+ 4(\mu_1 - \mu_2)(x_1 a_1^1 b_1^1 - \rho_1 a_2^1 b_2^1 - (\rho_1 - \rho_2)y' \tau^2) \\
+ 4(\mu_1 - \mu_2)^2(a_1^2 b_2^2 + (a_2^1 b_1^1 + a_2^1 b_1^2)^2 + (a_2^1 b_1^1)^2) \right\} \tau^4 .
\end{array} \right.
\]

If \( b_1^1(y', \tau) \) is real valued, then \( |y'|^{-A} A_1^0 \times (4.4) \) is equal to \( (\rho_1 - \rho_2)^2 - 2(\mu_1 - \mu_2)|y'|^2 + (\rho_1^2 + 2(\mu_1 - \mu_2)|y'|^2 a_1^1 b_1^1 + (\rho_1^2 - 2(\mu_1 - \mu_2)|y'|^2 a_2^1 b_2^1 - \rho_1 \rho_2 (a_1^1 b_2^1 + a_1^1 b_2^1)^2 + 4(\mu_1 - \mu_2)^2 a_1^1 b_1^1 a_2^1 b_2^1 |y'|^2) \), which is positive because \( a_2^1 b_2^1 \) and \( a_2^1 b_2^1 \) are positive and \( a_1^1 b_2^1 \) and \( a_2^1 b_2^1 \) are negative. If \( b_1^1(y', \tau) \) is pure imaginary, then \( \Re(\det M(y', \tau)) \) is also positive. Thus \( c_{10}, c_{30}, c_{40} \) and \( c_{60} \) are linearly independent. The proof is completed.

Next we shall compute the principal symbol of \( (C^-)^{-1} C_1^+ \), where \( C^- = (C_1^-, C_2^-) \), which is elliptic from the proof of Lemma 4.2. Define \( a_i(s) = (s/\alpha_i^2 - 1)^{1/2} \) and \( b_i(s) = (s/\beta_i^2 - 1)^{1/2} \) \((i = 1, 2)\) and put

\[
g_1(s) = (\rho_1 s + 2(\mu_1 - \mu_2))((\rho_1^2 - \rho_2) s + 2(\mu_1 - \mu_2)) \\
- 2(\mu_1 - \mu_2)(\rho_1 s - 2(\mu_1 - \mu_2)) a_1(s) b_2(s),
\]
\[
g_2(s) = \rho_1 (a^2_1 b_2 - b_2 a_1)(s)^2 + (\rho_1 s + 2(\mu_1 - \mu_2))^2 (a_1 b_2)(s) \\
- (\rho_1 s - 2(\mu_1 - \mu_2))^2 (a_1 b_2)(s) + 4(\mu_1 - \mu_2)^2 (a_1 b_2)(s) \\
- ((\rho_1 - \rho_2) s - 2(\mu_1 - \mu_2))^2 .
\]
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\[ g_i(s) = \{ (\rho_i - \rho_j)^2 - 4(\mu_i - \mu_j)^2(\alpha_i^2 \beta_j^2) \} s \]

\[ - 4(\mu_i - \mu_j)((\rho_i - \rho_j) - (\mu_i - \mu_j)(\alpha_i^2 + \beta_j^2)), \]

\[ g_s(s) = (\rho_i^2 \alpha_i^2 - \beta_j^2(\alpha_j^2)s^2 + (\rho_j + \rho_j)((\rho_i - \rho_j) + 4(\mu_i - \mu_j)(\alpha_i^2 + \beta_j^2))s \]

\[ + 4(\mu_i - \mu_j)((\mu_i - \mu_j)(\alpha_i^2 - \beta_j^2) - (\rho_i + \rho_j)). \]

Then we have

**Lemma 4.3.** i) If \( \rho_1 \neq \rho_2, \mu_1 \neq \mu_2 \) and \( \beta_i^2 < \beta^2(1 - (n(0) \cdot \omega)^2) \), then the (1.3), (3.3) and (4.3) component of \( (C^\gamma)^{-1} C^\gamma_i \) are elliptic at \( \rho_1 \), where \( C^\gamma_i = (C^\gamma_i, C^\gamma_i) \).

ii) If \( \beta_i^2 > \beta^2(1 - (n(0) \cdot \omega)^2) \) and \( g_s(\beta^2_i(1 - (n(0) \cdot \omega)^2)) = 0 \), then the \( j, 3 \) component of \( (C^\gamma)^{-1} C^\gamma_j \) is elliptic at \( \rho_1 \), where \( j = 1, 3, 4, 6 \).

iii) If \( \alpha_i \neq \alpha_j \), then \( G(s) = (g_1(s), g_3(s)) \) has at most 30 null points in \((\beta^2_i, \infty)\).

**Proof.** Let us denote by \( M_i(y', y, \tau) = (c_1, c_2, c_3, c_4) \) the 4 \times 4 square matrix which is generated by eliminating the second and fourth column and line vectors of the principal symbol of \( C^\gamma_i(y', D_y, D_t) \) and put \( c(y', y, \tau) \) to be the column vector which is generated by eliminating the second and fourth components of the principal symbol of the third column vector of \( C^\gamma_i(y', D_y, D_t) \). Then from Lemma 4.2 and Cramer’s formula we may check that the determinant of \( (c, c_2, c_3, c_4) \), where \( 1 \leq i, j, k \leq 4 \) and \( i \neq k \), is not zero at \( \rho_1 \). From (2.4) it follows that

\[ \text{det}(c, c_2, c_3, c_4)(0, y', \tau) = A_1[(\rho_i \tau^2 + 2(\mu_i - \mu_j)(\eta' \tau^2)] \]

\[ \times ((\rho_i - \rho_j)\tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2) + 2(\mu_i - \mu_j)(\rho_i \tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2)a_i b_j, \]

\[ \text{det}(c, c_2, c_3, c_4)(0, y', \tau) = A_2[(\rho_i \rho_j(a_i b_j + a_j b_i) + (\rho_i \tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2)a_i b_j \]

\[ - (\rho_i \rho_j(a_i b_j + a_j b_i) - (\rho_i \tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2)a_i b_j \]

\[ (\rho_i - \rho_j)\tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2), \]

\[ \text{det}(c, c_2, c_3, c_4)(0, y', \tau) = A_3[(\rho_i - \rho_j)\tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2 a_i b_j) \]

\[ (\rho_i \tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2)a_i b_j), \]

\[ \text{det}(c, c_2, c_3, c_4)(0, y', \tau) = A_4[(\rho_i \tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2)a_i b_j \]

\[ - (\rho_i \tau^2 - 2(\mu_i - \mu_j)(\eta' \tau^2)a_i b_j), \]

where \( A_j(\eta', \tau) \) is not zero, if \( \eta' \neq 0 \) and \( \tau \neq 0 \). If \( \rho_1 \neq \rho_2, \mu_1 \neq \mu_2 \) and \( b_j \) is pure imaginary, then the real part and the imaginary part of (4.5) do not vanish at the same point and the imaginary parts of (4.6) and (4.7) are not zero. It follows that the statement i) holds. (4.5) and (4.6) clearly implies that the statement ii) holds for \( j = 1, 3, g(s) \) and \( g(s) \) are equal.
to $[((\rho_1 - \rho_2)s - 2(\mu_1 - \mu_2))^2 - 4(\mu_1 - \mu_2)(a_1^2 b_2^2)]/s$ and $[((\rho_2 s + 2(\mu_1 - \mu_2)a_1^2) - 2(\mu_1 - \mu_2)a_2^2)]/s$. Thus when $\rho_1 \neq \rho_2$ and $\mu_1 \neq \mu_2$, $g_1(s)$, $g_2(s)$ and $g_6(s)$ are not identically zero and one of the equivalent conditions that $g_1(s) = 0$ is $[((\rho_2 s + 2(\mu_1 - \mu_2)((\rho_1 - \rho_2)s + 2(\mu_1 - \mu_2)^2 - 4(\mu_1 - \mu_2)(a_2^2 b_1(s)])^2)]/s = 0$. It follows that the null points of $(g_1, g_2, g_6)(s)$ are at most 6. By the form of $g_1(s)$ the null points of $g_5(s)$ are roots of some polynomial of degree 24. These show that the null points of $G_i(s)$ in $(\tilde{\beta}_i^2, \infty)$ are at most 30. The proof is completed.

The proof of Theorem 4.1. The proof of the statement i) is similar to that of the statement ii). Thus we only prove the statement ii). In (2.3) we put $V_t = \text{'}(V_t^\dagger, V_t^\ddagger)$ with $V_t^\dagger = (\nu_t^\dagger, \nu_t^\ddagger, \nu_t^\lambda)$. The assumption implies that $\rho_1 \in WF(v_1^\lambda) \cup WF(v_2^\lambda) \cup WF(v_3^\lambda)$ and $\rho_1 \in WF(v_1^\lambda \cup v_2^\lambda)$, where $\rho_1 = (0, t_0, -\varepsilon_\omega, \varepsilon_\omega')$. Thus the boundary condition in (2.3) is reduced to the following

$$\text{'}(V_t^\dagger, V_t^\ddagger) = -((C^-)^{-1}v_t^\lambda) + \tilde{G},$$

where $c_t^\lambda(\gamma', D', D)$ is the third column vector of $C_t^\lambda(\gamma', D', D)$ and $\rho_1$ does not belong to $WF(\tilde{G})$. From Lemma 4.3 and (4.9) we see the $\rho_1 \in WF(v_1^\lambda) \cap WF(v_2^\lambda) \cap WF(v_3^\lambda)$. Using Theorem 2.5.11’ of [3], we have the desired conclusions. The proof is completed.

§ 5. Incident S singularities

Let us consider incident S singularities. So all functions and pseudo-differential operators are defined in $\Gamma$, where $\Gamma$ is a conic neighbourhood of $\rho_1 = (0, t_0, -\varepsilon_\omega, \varepsilon_\omega')$. Under the assumption that $\alpha_2 < \tilde{\beta}_1 < \tilde{\beta}_2$ and $\alpha_1 \neq \alpha_2$ we shall show the following

**Theorem 5.1.** We assume that $\tilde{\beta}_1 < \alpha_2^\dagger(1 - (n(0), \omega)^2) < \tilde{\beta}_2$. Then $\tilde{P}_\infty(\omega)$ and $\tilde{P}_\infty(\omega)$ do not exist and there exists a function $H_1(s)$ whose null points are at most 48 such that if $H_1(\alpha_2^\dagger(1 - (n(0), \omega)^2)) \neq 0$, $S_\infty(\omega) \subset WF(u_1)$ and $P_\infty(\omega) \cap WF(u_1) = \tilde{S}_\infty(\omega) \cap WF(u_1) = \phi$, then one of the following two cases occurs: a) $S_\infty(\omega) \subset WF(u_1)$ and $\tilde{S}_\infty(\omega) \subset WF(u_2)$, b) $S_\infty(\omega) \subset WF(u_1)$, $\tilde{S}_\infty(\omega) \subset WF(u_1)$ and $P_\infty(\omega) \cap WF(u_1) = \phi$.

**Theorem 5.2.** We assume that $\tilde{\beta}_2^\dagger < \alpha_2^\dagger(1 - (n(0), \omega)^2)$. There exists a function $H_2(s)$ whose null points are at most 49 such that if $H_2(\alpha_2^\dagger(1 - (n(0), \omega)^2)) \neq 0$, $P_\infty(\omega) \cap WF(u_1) = (\tilde{P}_\infty(\omega) \cup \tilde{S}_\infty(\omega)) \cap WF(u_1) = \phi$ and $S_\infty(\omega) \subset WF(u_1)$, then one of the following two cases occurs: a) $S_\infty(\omega) \subset WF(u_1)$ and
$\tilde{S}_r(\omega) \cup \tilde{P}_{ir}(\omega) \subset WF(u_2)$. b) $S_r(\omega) \subset WF(u_i)$, $\tilde{S}_r(\omega) \subset WF(u_i)$ and $\tilde{P}_{ir}(\omega) \cap WF(u_2) = \phi$. In the above statement is not complete in the following sense: If we suppose that the assumptions mentioned in the above hold and in a small neighbourhood of $0$ the border $\Gamma_2$ is equal to a hyperplane in $\mathbb{R}^n$, then we have one of the following two cases; a') $(S_r(\omega) \cup P_{ir}(\omega)) \subset WF(u_i)$ and $(\tilde{S}_r(\omega) \cup \tilde{P}_{ir}(\omega)) \subset WF(u_2)$, b') $S_r(\omega) \subset WF(u_i)$, $\tilde{S}_r(\omega) \subset WF(u_2)$ and $P_{ir}(\omega) \cap WF(u_2) = \tilde{P}_{ir}(\omega) \cap WF(u_2) = \phi$.

In order to prove the above theorems we need to change the components of $V_i$ in (2.3) corresponding to $S$ waves. Put

$$h_i(s) = ((\rho_i - \rho_2)s - 2(\mu_i - \mu_2)(\rho_2s + 2(\mu_i - \mu_2)) + 2(\mu_i - \mu_2)(\rho_2s - 2(\mu_i - \mu_2)))(a_ib_i)(s),$$

$$h_2(s) = ((\rho_i - \rho_2)^2 - 4(\mu_i - \mu_2)^2(\beta_0^2\alpha_2^2)s - 4(\mu_i - \mu_2)((\rho_i - \rho_2) - (\mu_i - \mu_2)(\beta_1^z + \alpha_1^z)),

where $a_i = (s/\alpha_1^2 - 1)^{1/2}$ and $b_i = (s/\beta_0^2 - 1)^{1/2}$, then we have the following

**Lemma 5.3.** i) If $\beta_0^2 < \alpha_1^2(1 - (n(0) \cdot \omega)^2) < \beta_0^2$, then there exists a pseudodifferential operator $a(y', D_{y'}, D_t)$ of order $0$ such that the principal symbol of $a$ is zero at $y' = 0$ and that the $(3, 2)$ component of $(C^{-})^{-1}(C_1)(I_3 + A)$ is zero modulo $L^{-\infty}(\Gamma_2)$, where $I_3$ is the $3 \times 3$ identity matrix and $A(y', D_{y'}, D_t)$ is a $3 \times 3$ square matrix whose $(1, 2)$ component is $a$ and other components are $0$.

ii) We assume that $\alpha_1^2(1 - (n(0) \cdot \omega)^2) > \beta_0^2$. Then there exists a pseudodifferential operator $a(y', D_{y'}, D_t)$ such that if $h_i(\alpha_1^2(1 - (n(0) \cdot \omega)^2)) \neq 0$, $(3, 2)$ component of $(C^{-})^{-1}(C_1)(I_3 + A)$ is zero modulo $L^{-\infty}(\Gamma_2)$, where $A$ is a similar pseudodifferential operator to that mentioned in i). The similar property on the $(6, 2)$ component of $(C^{-})^{-1}(C_2)(I_3 + A)$ holds, if $h_2(\alpha_2^2(1 - (n(0) \cdot \omega)^2)) \neq 0$.

**Proof.** Define $c_j$ ($j = 1, \ldots, 6$) and $c_i^\dagger$ ($j = 1, 2, 3$) to be the $j$-th line and column vector of the principal symbol of $(C^{-})^{-1}$ and $C_1$, respectively. If we can show that $X = 'c_a \cdot c_i^\dagger$ or $Y = 'c_a \cdot c_i^\dagger$ is not zero at $y' = 0$, we have the statements of Lemma 5.3, by using the calculus on symbols of pseudo-differential operators. Let $f_i$ be the first column vector of the principal symbol of $C_1$. Then $X = 'c_a \cdot (c_i^\dagger - f_i)$ is equal to $2a_i^2|\gamma|\tau A_2^{-}\tau(c_{21} - 2\mu_2c_{ab})$ at $y' = 0$ and $Y = 'c_a \cdot (c_i^\dagger - f_i)$ is equal to $2a_i^2|\gamma|\tau A_1^{-}\tau(c_{21} - 2\mu_2c_{ab})$ at $y' = 0$, where $c_{ij}$ is the $(i, j)$ component of the principal symbol of $(C^{-})^{-1}$.
Using (2.4), we can easily derive that
\[
\begin{align*}
\sigma_{11} &= |\gamma|^4 A_{11}^{-\frac{s}{2}}[(\rho_1 - \rho_2)^2 - 2(\mu_1 - \mu_2)](\rho_1 \tau^2 - 2\rho_1 |\gamma|^2) \\
&\quad - 2\rho_1 \alpha^1 b_1 \{((\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\gamma|^2 + \rho_1 b_1^1 (\alpha_1 - \alpha_2)^2\}, \\
\sigma_{30} &= -|\gamma|^4 A_{30}^{-\frac{s}{2}}[(\rho_1 - \rho_2)^2 - 2(\mu_1 - \mu_2)|\gamma|^2](\alpha_1^2 b_1 - \rho_1 b_1^1 (\alpha_1 - \alpha_2)^2\}, \\
\sigma_{10} &= -|\gamma|^4 A_{10}^{-\frac{s}{2}}[(\rho_1 - \rho_2)^2 - 2(\mu_1 - \mu_2)|\gamma|^2](\alpha_1^1 b_1 - \rho_1 b_1^1 (\alpha_1 - \alpha_2)^2\}, \\
\sigma_{01} &= -|\gamma|^4 A_{01}^{-\frac{s}{2}}[(\rho_1 - \rho_2)^2 - 2(\mu_1 - \mu_2)|\gamma|^2](\alpha_{11}^2 b_1 - \rho_1 b_1^1 (\alpha_1 - \alpha_2)^2\}. \\
\end{align*}
\]
Thus if \(b_1^2\) is pure imaginary, \(\sigma_{11} - 2\rho_1 \sigma_{30}\) does not vanish. When \(b_1^2\) is real, \(\sigma_{11} - 2\rho_1 \sigma_{30}\) is not zero at \(\rho_1\), if \(h_3(\alpha_{11}^1(1 - (n(0) \omega)^2)) \neq 0\), and \(\sigma_{11} - 2\rho_1 \sigma_{30}\) is not zero at \(\rho_1\), if \(h_3(\alpha_{11}^1(1 - (n(0) \omega)^2)) \neq 0\). The proof is completed.

Next we shall check the ellipticity of the components of \((C^-)^{-1}(C_1)^{-1}\).

Put
\[
\begin{align*}
h_3(s) &= (\rho_3 s - 2\rho_2) \rho_3(s - 2(\mu_1 - \mu_2)) b_2(s) + \rho_2 \rho_3(s + 2(\mu_1 - \mu_2)) b_2(s) \\
&\quad + 2\rho_2 \rho_3(s - 2(\mu_1 - \mu_2)) a_2(s) + 4\rho_2(\mu_1 - \mu_2)(\alpha_1 b_1)(s) b_2 - a_2(s),
\end{align*}
\]
where \(b_2(s)\) is equal to \(i \in (1 - s/\beta^{1/2})^{1/2}\), if \(\beta_1^2 < \alpha_{11}^2(1 - (n(0) \omega)^2) < \beta_1^2\) and is equal to \((s/\beta_1^2 - 1)^{1/2}\), if \(\beta_1^2 < \alpha_{11}^2(1 - (n(0) \omega)^2)\). Then we have the following

**Lemma 5.4.** Let \(d_{ij}(\gamma', D_{\nu}', D_\nu)\) be the \((i, j)\) component of \((C^-)^{-1}C_1^{-1}\).

i) When \(\beta_1^2 < \alpha_{11}^2(1 - (n(0) \omega)^2) < \beta_1^2\), \(d_{11}\) is elliptic at \(\rho_1 = (0, t_0, -\omega', \varepsilon\alpha')\) and \((d_{1j})_{i=5, j=1,2}\) is also elliptic at \(\rho_1\), if \(h_3(\alpha_{11}^1(1 - (n(0) \omega)^2)) \neq 0\).

ii) When \(\beta_1^2 < \alpha_{11}^2(1 - (n(0) \omega)^2)\), \(d_{11}\) is elliptic at \(\rho_1\), if \(h_3(\alpha_{11}^1(1 - (n(0) \omega)^2)) \neq 0\), \(d_{01}\) is elliptic at \(\rho_1\), if \(h_3(\alpha_{11}^1(1 - (n(0) \omega)^2)) \neq 0\), and \((d_{1j})_{i=5, j=1,2}\) is also elliptic at \(\rho_1\), if \(h_3(\alpha_{11}^1(1 - (n(0) \omega)^2)) \neq 0\).

**Proof.** The ellipticity of \(d_{01}\) and \(d_{31}\) at \(\rho_1\) proved in the proof of Lemma 5.3. From (2.4) it follows that the principal symbols of \(d_{41}\) and \(d_{31}\) are zero at \(\rho_1\) and the one of \(d_{31}\) is not zero at \(\rho_1\). Thus we may prove that the principal symbol of \(d_{41}\) is not zero. By the same way as in the proof of Lemma 5.3, one of equivalent conditions of the ellipticity of \(d_{41}\) at \(\rho_1\) is the principal symbol of \(f_{41} - 2\mu_1 f_{46}\) is not zero at \(\rho_1\), where \(f_{4j}\) is the \((i, j)\) component of the principal symbol of \((C^-)^{-1}\). Making use of (2.4), we can easily derive that \(f_{41} - 2\mu_1 f_{46}\) is equal to \(-|\gamma|^4 A_1^{-\frac{s}{2}} h_3(\tau^2)|\gamma|^2)\) at \(\gamma' = 0\). The proof is completed. To prove the theorems we need the following function.
where \( a_i(s) = (s/\alpha_i^2 - 1)^{1/2} \) (\( i = 1, 2 \)), \( b_i(s) = (s/\beta_i^2 - 1)^{1/2} \) and \( b_2(s) \) is \( i\varepsilon(1 - s/\beta_2^2)^{1/2} \) if \( \beta_1^2 < \alpha_1^2/(1 - n(0) \cdot \omega)^2 < \beta_2^2 \) and is \( (s/\beta_2^2 - 1)^{1/2} \) if \( \beta_2^2 < \alpha_2^2/(1 - (n(0) \cdot \omega)^2) \).

**Proof of Theorem 5.1.** We shall use the same notations appeared in the proof of Theorem 4.1 and \( H_i(s) = (h_i h_i)(s) \). By Lemma 5.3 there exists a pseudo-differential operator \( a(y', D_y, t) \) such that the \((3, 2)\) component of \((C^-)^{-1}C^+_i(I_s + A)\) is essentially zero. We shall put

\[
'(\vec{v}_{i1}, \vec{v}_{i2}) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1}(\vec{v}_{i1}, \vec{v}_{i2}).
\]

Then they satisfy a hyperbolic equation \( (D_{y3} - \vec{A}_i)'(\vec{v}_{i1}, \vec{v}_{i2}) = g \) in \( y_3 > 0 \), where \( g \) is smooth at \( \rho_1 \) and the principal symbol \( \vec{A}_i(y', D_y, D_t) \) is \( \vec{a}_i(y', \sigma', \tau) \). From the assumption the boundary condition in (2.3) is reduced to

\[
(5.1) \quad \begin{pmatrix} V_{i1}^- \\ V_{i2}^- \end{pmatrix} = F(y', D_{y'}, D_t)(\vec{v}_{i1}^+, \vec{v}_{i2}^+) + G \quad \text{on} \quad y_3 = 0,
\]

where \( \rho_i \in WF(G) \) and the first and second column vectors of the \( 6 \times 2 \) matrix \( F = (c_{ij}) \) are equal to these of \(- (C^-)^{-1}C^+_i(I_s + A)\). The assumption \( S_1(\omega) \subset WF(u_i) \) is equivalent to \( \rho_i \in WF(\vec{v}_{i1}^-|_{y_3=0}) \cup WF(\vec{v}_{i2}^-|_{y_3=0}) \). From Lemma 5.4 it follows that \( \rho_i \in WF(\vec{v}_{i1}^-|_{y_3=0}) \cup WF(\vec{v}_{i2}^-|_{y_3=0}) \), which means \( \vec{S}_i(\omega) \subset WF(u_i) \). If we assume that \( \rho_i \in WF(\vec{v}_{i1}^-|_{y_3=0}) \), then from the third component of the right hand side of (5.1) we see that \( \rho_i \in WF(\vec{v}_{i1}^-|_{y_3=0}) \), that is \( P_{i1}(\omega)WF(u_i) \). On the other hand if \( \rho_i \notin WF(\vec{v}_{i1}^-|_{y_3=0}) \), then by the same reason it follows that \( \rho_i \notin WF(v_{i3}|_{y_3=0}) \), that is \( P_{i3}(\omega) \cap WF(u_i) = \varnothing \). Finally we shall show that \( \rho_i \notin WF(v_{i3}|_{y_3=0}) \cup WF(v_{i2}|_{y_3=0}) \), if \( h_i(\alpha_i^2/(1 - (n(0) \cdot \omega)^2)) \neq 0 \). We assume \( \rho_i \notin WF(v_{i3}|_{y_3=0}) \cup WF(v_{i2}|_{y_3=0}) \), then from the assumptions it follows that does not belong to the wave front set of \( F_i(y', D_{y'}, D_t)'(\vec{v}_{i1}, \vec{v}_{i2}, v_{i3}, v_{i4}, V_{j3})|_{y_3=0} \), where the first and second column vectors of \( F_i \) are equal to these of \( C_i^+(I_s + A) \), the third column vector of \( F_i \) is equal to one of \( C_i^- \), and the fourth, fifth and sixth column vectors of \( F_i \) are equal to these of \(- C_2 \). If \( F_i \) is elliptic at \( \rho_i \), we have \( \rho_i \) does not belong to \( WF(\vec{v}_{i1}^-|_{y_3=0}) \cup WF(\vec{v}_{i2}^-|_{y_3=0}) \). This is a contradiction. From (2.4) if \( h_i(\alpha_i^2/(1 - (n(0) \cdot \omega)^2)) \) is not zero, \( F_i \) is elliptic at \( \rho_i \). The proof is completed. Next we shall consider the case \( \beta_2^2 < \alpha_2^2/(1 - (n(0) \cdot \omega)^2) \).
Proof of Theorem 5.2. We put $H(s) = (h_2 h_1 h_0)(s)$. Using Lemma 5.3, we can prove the first part by the same way as in the proof of Theorem 5.1. If we assume that $T_0$ is flat near 0, then the reduced boundary value problem (2.3) does not depend on $y'$. Thus the symbol of $C^0(D_{y'}, D_0)$ is given by (2.4). It follows that the $(3,2)$ and $(6,2)$ components of $(C^0)^{-1}C^0$ is both zero. Using this fact, we can prove the later part of Theorem 5.1.

In the statement of the first part of Theorem 5.2 we only consider refracted singularities, however on reflected singularities we have the following

Remark 5.5. In assumptions of Theorem 5.2 we assume $(h_2(h_1 h_0)(\alpha^2_1|1 - (n(0) \cdot \omega)) = 0$ instead of $(h_2 h_1 h_0)(\alpha^2_1|0 - (n(0) \cdot \omega)) = 0$. Then making use of the statement ii) of Lemma 5.3 we have the following two cases: a”) $S_r(\omega) \cup P_{u}(\omega) \subset WF(u_i)$ and $\tilde{S}_r(\omega) \subset WF(u_2)$. b”) $P_{u}(\omega) \subset WF(u_i)$, $\tilde{S}_r(\omega) \subset WF(u_2)$ and $P_{u}(\omega) \cap WF(u_i) = \phi$.

References


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