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RESEARCH ARTICLE

On local Galois deformation rings: generalised tori

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Abstract
We study deformation theory of mod p Galois representations of p-adic fields with values in generalised tori, such as L-groups of (possibly non-split) tori. We show that the corresponding deformation rings are formally smooth over a group algebra of a finite abelian p-group. We compute their dimension and the set of irreducible components.

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1. Introduction

Let p denote any prime number, let F be a finite extension of $\mathbb{Q}_p$ and let Γ_F denote its absolute Galois group. Let L be another finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $k = \mathcal{O}/\varpi$. Let G be a smooth affine group scheme over $\mathcal{O}$, such that its neutral component G^0 is a torus and the component group G/G^0 is a finite étale group scheme over $\mathcal{O}$. We call such group schemes *generalised tori*. We do not make any assumptions regarding p and the component group of G. After replacing L by a finite unramified extension, one may assume that G^0 is a split torus and G/G^0 is a constant group scheme. We will assume this for the rest of the introduction.

An important example of a generalised torus that we have in mind is the L-group of a torus defined over F: let H be a torus over F, which splits over a finite Galois extension E over F. Then the L-group of H is ${}^L H = \hat{H} \rtimes \underline{\mathrm{Gal}(E/F)}$, where $\hat{H}$ is the split $\mathcal{O}$-torus, such that the character lattice of $\hat{H}$ is equal to the cocharacter lattice of H_E and $\underline{\mathrm{Gal}(E/F)}$ is the constant group scheme associated to $\mathrm{Gal}(E/F)$. In this example, the surjection $G \twoheadrightarrow G/G^0$ has a section of group schemes. We do not assume this in general.

We fix a continuous representation $\overline{\rho} : \Gamma_F \to G(k)$ and denote by $D^{\square}_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ the functor from the category $\mathfrak{A}_{\mathcal{O}}$ of local artinian $\mathcal{O}$-algebras with residue field k to the category of sets, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D^{\square}_{\overline{\rho}}(A)$ is the set of continuous representations $\rho_A : \Gamma_F \to G(A)$, such that

$$\rho_A(\gamma) \equiv \overline{\rho}(\gamma) \pmod{\mathfrak{m}_A}, \quad \forall \gamma \in \Gamma_F.$$

The functor $D^{\square}_{\overline{\rho}}$ of framed deformations of $\overline{\rho}$ is pro-represented by a complete local noetherian $\mathcal{O}$-algebra $R^{\square}_{\overline{\rho}}$ with residue field k.

Theorem 1.1. *There is a finite extension L' of L with ring of integers $\mathcal{O}'$ and a continuous representation $\rho : \Gamma_F \to G(\mathcal{O}')$ lifting $\overline{\rho}$.*

Remark 1.2. If $G \cong G^0 \rtimes (G/G^0)$, then a lift as in Theorem 1.1 can be constructed over $\mathcal{O}$ using the Teichmüller lift to define a section of $G^0(\mathcal{O}) \twoheadrightarrow G^0(k)$, which then induces a section $\sigma : G(k) \to G(\mathcal{O})$. Then $\sigma \circ \overline{\rho}$ is the required lift. In particular, in this case, Theorem 1.1 holds if we replace Γ_F with any profinite group. However, it seems nontrivial to prove Theorem 1.1 in general, and our argument uses that the Euler–Poincaré characteristic formula holds for Γ_F.

Let Γ_E be the kernel of the composition $\Gamma_F \xrightarrow{\overline{\rho}} G(k) \to (G/G^0)(k)$ and let Δ be the image of this map. We identify $\Delta = \mathrm{Gal}(E/F)$. Let M be the character lattice of G^0. The action of G on G^0 by conjugation induces an action of Δ on M. Let $\Gamma_E^{\mathrm{ab},p}$ be the maximal abelian pro-p quotient of Γ_E. Below, we will consider the diagonal action of Δ on $\Gamma_E^{\mathrm{ab},p} \otimes M$ and $\mu_{p^\infty}(E) \otimes M$, where $\mu_{p^\infty}(E)$ is the subgroup of p-power roots of unity in E.

Theorem 1.3. *Assume that Theorem 1.1 holds with $\mathcal{O} = \mathcal{O}'$. Then*

$$R^{\square}_{\overline{\rho}} \cong \mathcal{O}[(\mu_{p^\infty}(E) \otimes M)^{\Delta}][\![x_1, \dots, x_m]\!],$$

where $m = \mathrm{rank}_{\mathbb{Z}} M \cdot ([F : \mathbb{Q}_p] + 1)$.

Let $\overline{\Theta}$ be the G-pseudocharacter associated to $\overline{\rho}$ and let $D^{\mathrm{ps}}_{\overline{\Theta}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ be the deformation functor such that $D^{\mathrm{ps}}_{\overline{\Theta}}(A)$ is the set of continuous A-valued G-pseudocharacters deforming $\overline{\Theta}$. These notions are reviewed in Section 6. The functor $D^{\mathrm{ps}}_{\overline{\Theta}}$ is pro-represented by a complete local noetherian $\mathcal{O}$-algebra $R^{\mathrm{ps}}_{\overline{\Theta}}$ with residue field k, and we denote the universal deformation by Θ^u. Sending a deformation of $\overline{\rho}$ to its G-pseudocharacter induces a natural transformation $D^{\square}_{\overline{\rho}} \to D^{\mathrm{ps}}_{\overline{\Theta}}$, and hence a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\square}_{\overline{\rho}}$. If G is a torus or, more generally, when G is commutative, then representations and G-pseudocharacters coincide, and this map is an isomorphism.

Theorem 1.4. *The map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\square}_{\overline{\rho}}$ is formally smooth. Moreover, if Theorem 1.1 holds with $\mathcal{O}' = \mathcal{O}$, then*

$$R^{\mathrm{ps}}_{\overline{\Theta}} \cong \mathcal{O}[\![(\Gamma_E^{\mathrm{ab},p} \otimes M)^{\Delta}]\!] \cong \mathcal{O}[(\mu_{p^\infty}(E) \otimes M)^{\Delta}][\![x_1, \dots, x_r]\!],$$

where $r = \mathrm{rank}_{\mathbb{Z}} M \cdot [F : \mathbb{Q}_p] + \mathrm{rank}_{\mathbb{Z}} M_{\Delta}$.

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We let $X^{\mathrm{gen}}_{G,\overline{\Theta}} : R^{\mathrm{ps}}_{\overline{\Theta}}\text{-alg} \to \mathrm{Set}$ be the functor such that $X^{\mathrm{gen}}_{G,\overline{\Theta}}(A)$ is the set of representations $\rho : \Gamma_F \to G(A)$ such that its G-pseudocharacter Θ_ρ satisfies $\Theta_\rho = \Theta^u \otimes_{R^{\mathrm{ps}}_{\overline{\Theta}}} A$. This functor is representable by a finite type $R^{\mathrm{ps}}_{\overline{\Theta}}$-algebra $A^{\mathrm{gen}}_{G,\overline{\Theta}}$.

Theorem 1.5. *The map $R^{\mathrm{ps}}_{\overline{\Theta}} \to A^{\mathrm{gen}}_{G,\overline{\Theta}}$ is smooth. Moreover, if Theorem 1.1 holds with $\mathcal{O}' = \mathcal{O}$, then*

$$A^{\mathrm{gen}}_{G,\overline{\Theta}} \cong R^{\mathrm{ps}}_{\overline{\Theta}}[t_1^{\pm 1}, \dots, t_s^{\pm 1}],$$

where $s = \mathrm{rank}_{\mathbb{Z}} M - \mathrm{rank}_{\mathbb{Z}} M_\Delta$.

We note that $\mu_{p^\infty}(E)$ is a finite p-group. This implies that $\mu := (\mu_{p^\infty}(E) \otimes M)^\Delta$ is also a finite p-group, and we denote its order by p^m. We assume further that $\mathcal{O}$ contains all p^m-th roots of 1 and let $\mathrm{X}(\mu)$ be the group of characters $\chi : \mu \to \mathcal{O}^\times$. It then follows from Theorems 1.3, 1.4, 1.5 that the sets of irreducible components of $R^\square_{\overline{\rho}}$, $R^{\mathrm{ps}}_{\overline{\Theta}}$ and $A^{\mathrm{gen}}_{G,\overline{\Theta}}$, respectively, are in $\mathrm{X}(\mu)$-equivariant bijection with $\mathrm{X}(\mu)$. Moreover, the irreducible components and their special fibres are regular. The identification of the set of components with $\mathrm{X}(\mu)$ is non-canonical in general, as one has to distinguish one component, which corresponds to the trivial character. However, we explain in Section 7.4 that there is a canonical action of $\mathrm{X}(\mu)$ on the set of irreducible components, which is faithful and transitive.

Remark 1.6. Let us point out that if $G \cong G^0 \rtimes G/G^0$, then the lift constructed in Remark 1.2 is canonical, as it is minimally ramified. This distinguishes the irreducible component that it lies on. Thus, in this case, there is a canonical $\mathrm{X}(\mu)$-equivariant bijection between $\mathrm{X}(\mu)$ and the set of irreducible components.

The above theorems are used in an essential way in a companion paper [12], where we study deformations of Galois representations with values in generalised reductive $\mathcal{O}$-group schemes G, which means that the neutral component G^0 is reductive and the component group G/G^0 is finite étale. If we let G' be the derived subgroup scheme of G^0, then G^0/G' is a torus and G/G' is a generalised torus as considered in this paper. If $\overline{\rho} : \Gamma_F \to G(k)$ is a continuous representation, then we show in [12, Proposition 4.20] that its deformation ring $R^\square_{\overline{\rho}}$ can be presented over the deformation ring of $\varphi \circ \overline{\rho} : \Gamma_F \to (G/G')(k)$, where $\varphi : G \to G/G'$ is the quotient map. This allows us to split up the arguments into 'torus part', carried out in this paper, and 'semisimple part', carried out in [12].

If μ is any finite abelian p-group, then $\mu \cong \prod_{i=1}^{l} \mathbb{Z}/p^{e_i}$, and hence,

$$\mathcal{O}[\mu] \cong \bigotimes_{i=1}^{l} \frac{\mathcal{O}[z_i]}{((1+z_i)^{p^{e_i}} - 1)}$$

is complete intersection. It then follows from Theorems 1.3, 1.4, 1.5 that the rings $R^\square_{\overline{\rho}}$, $R^{\mathrm{ps}}_{\overline{\Theta}}$ and $A^{\mathrm{gen}}_{G,\overline{\Theta}}$ are locally complete intersections. To relate their dimensions to the dimensions of the deformation rings appearing in [12], we note that $\mathrm{rank}_{\mathbb{Z}} M = \dim G_k$, and if $\Delta = (G/G^0)(\bar{k})$ (which we may assume without changing the functors $D^\square_{\overline{\rho}}$, $D^{\mathrm{ps}}_{\overline{\Theta}}$ and $X^{\mathrm{gen}}_{G,\overline{\Theta}}$), then $\mathrm{rank}_{\mathbb{Z}} M_\Delta = \dim Z(G)_k$. The scheme $X^{\mathrm{gen}}_{G,\overline{\Theta}}$ coincides with the scheme denoted by $X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}}$ in [12], when G is a generalised torus. In [12], in the definition of $X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}}$, an additional condition is imposed on representations, called $R^{\mathrm{ps}}_{\overline{\Theta}}$-condensed, and we show in Lemma 9.6 that when G is a generalised torus, this condition holds automatically.

1.1. Arithmetic setting

Let us first recall how one proves the result when $G = \mathbb{G}_m$. The first step is to observe that Theorem 1.1 holds with $\mathcal{O}' = \mathcal{O}$ – for example, using the Teichmüller lift $\sigma : k^\times \to \mathcal{O}^\times$ and letting $\rho_0 = \sigma \circ \overline{\rho}$. The second step is to observe that the map $\rho \mapsto \rho\rho_0^{-1}$ induces a bijection between $D^\square_{\overline{\rho}} \to D^\square_{\mathbf{1}}$, where $\mathbf{1}$ is the trivial representation. One may identify $D^\square_{\mathbf{1}}(A)$ with the set of continuous group homomorphisms

$\Gamma_F \to 1 + \mathfrak{m}_A$, and since the target is an abelian p-group, the functor is representable by $\mathcal{O}[\![\Gamma_F^{\mathrm{ab},p}]\!]$. An interesting feature of this argument is that $D_{\mathbf{1}}^{\square}$ takes values in Ab as opposed to Set, and the map

$$D_{\mathbf{1}}^{\square}(A) \times D_{\overline{\rho}}^{\square}(A) \to D_{\overline{\rho}}^{\square}(A), \quad (\Phi, \rho) \mapsto \Phi\rho$$

defines an action of an abelian group $D_{\mathbf{1}}^{\square}(A)$ on the set $D_{\overline{\rho}}^{\square}(A)$. Moreover, if the set is nonempty, then this action is faithful and transitive.

If G is connected, then $G = \mathfrak{D}(M)$, where M is the character lattice, and so $G(A) = \mathrm{Hom}(M, A^{\times})$. The same argument goes through, and we only need to observe that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Group}}(\Gamma_F, G(A)) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Group}}(\Gamma_F^{\mathrm{ab}} \otimes M, A^{\times}),$$

which implies that $D_{\overline{\rho}}^{\square}$ is representable by $\mathcal{O}[\![\Gamma_F^{\mathrm{ab},p} \otimes M]\!]$.

Let us consider the general case now. The first issue, alluded to in Remark 1.2, is that it is not clear anymore that such a lift exists. However, let us postpone the explanation how we deal with this problem until Section 1.3 and let us give ourselves a lift $\rho_0 : \Gamma_F \to G(\mathcal{O})$ of $\overline{\rho}$. Then the map $\rho \mapsto [\gamma \mapsto \rho(\gamma)\rho_0(\gamma)^{-1}]$ considered above induces a bijection between $D_{\overline{\rho}}^{\square}(A)$ and the set $\widehat{Z}^1(A)$ of continuous 1-cocycles $\Phi : \Gamma_F \to \mathrm{Hom}(M, 1 + \mathfrak{m}_A)$. Further, one may interpret $\widehat{Z}^1$ as $D_{\overline{\rho}}^{\square}$ for the group $G = \mathfrak{D}(M) \rtimes \underline{\Delta}$ and $\overline{\rho} : \Gamma_F \to G(k)$ given by $\overline{\rho}(\gamma) = (1, \pi(\gamma))$, where $\pi : \Gamma_F \to \Delta$ is the quotient map. The functor $\widehat{Z}^1$ is the analogue of $D_{\mathbf{1}}^{\square}$, and we have to find the ring pro-representing the functor $\widehat{Z}^1$.

Before we describe our solution, let us point out one dead end. If $\Phi \in \widehat{Z}^1(A)$, then its restriction to Γ_E is just a continuous homomorphism

$$\Phi|_{\Gamma_E} : \Gamma_E \to \mathrm{Hom}(M, 1 + \mathfrak{m}_A),$$

as Γ_E acts trivially on M. This induces a continuous homomorphism

$$\varphi : \Gamma_E^{\mathrm{ab},p} \otimes M \to 1 + \mathfrak{m}_A,$$

and the fact that the homomorphism is obtained by a restriction of a cocycle to Γ_E implies that φ factors through $(\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta \to 1 + \mathfrak{m}_A$. However, in Theorem 1.3 we have Δ-invariants and not Δ-coinvariants appearing. Tate group cohomology gives us an exact sequence

$$0 \to \widehat{H}^{-1}(\Delta, \Gamma_E^{\mathrm{ab},p} \otimes M) \to (\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta \to (\Gamma_E^{\mathrm{ab},p} \otimes M)^\Delta \to \widehat{H}^0(\Delta, \Gamma_E^{\mathrm{ab},p} \otimes M) \to 0.$$

One may show that the Tate group cohomology groups $\widehat{H}^i(\Delta, \Gamma_E^{\mathrm{ab},p} \otimes M)$ are finite for all $i \in \mathbb{Z}$, but there is no reason why they should vanish, unless p does not divide $|\Delta|$. We conclude that the restriction to Γ_E will lose information in general.

Our solution is motivated by Langlands' paper [11] on his correspondence for tori, where he shows that the universal coefficient theorem induces an isomorphism

$$H^1(W_{E/F}, \mathrm{Hom}(M, \mathbb{C}^{\times})) \cong \mathrm{Hom}(H_1(W_{E/F}, M), \mathbb{C}^{\times}), \tag{1.1}$$

and there are natural isomorphisms

$$H_1(W_{E/F}, M) \cong H_1(E^{\times}, M)^{\Delta} \cong (E^{\times} \otimes M)^{\Delta}, \tag{1.2}$$

where the Weil group $W_{E/F}$ fits into a short exact sequence

$$0 \to E^{\times} \to W_{E/F} \to \Delta \to 0, \tag{1.3}$$

corresponding to the fundamental class $[u_{E/F}] \in H^2(\Delta, E^\times)$. Since every continuous representation $\Gamma_F \to G(A)$ with image in $(G/G^0)(A)$ equal to Δ factors through the profinite completion of $W_{E/F}$, this motivates us to study the functor

$$\mathrm{CRing} \to \mathrm{Ab}, \quad A \mapsto Z^1(W_{E/F}, G(A))$$

without imposing any continuity conditions on the cocycles and allowing any commutative ring A. We determine the ring representing this functor and show that the ring representing $\widehat{Z}^1$ arises as a completion of it with respect to the maximal ideal corresponding to the trivial cocycle.

1.2. Abstract setting

We carry out this study in Section 4 in the following abstract setting: let Γ_1 be an abstract group, Γ_2 a normal subgroup of Γ_1 of finite index and let $\Delta := \Gamma_1/\Gamma_2$. The exact sequence

$$0 \to \Gamma_2^{\mathrm{ab}} \to \Gamma_1/[\Gamma_2, \Gamma_2] \to \Delta \to 0 \tag{1.4}$$

defines a class in $H^2(\Delta, \Gamma_2^{\mathrm{ab}})$. Its image in $\mathrm{Ext}^1_{\mathbb{Z}[\Delta]}(I_\Delta, \Gamma_2^{\mathrm{ab}})$, where I_Δ is the augmentation ideal in the group ring $\mathbb{Z}[\Delta]$, defines an extension of $\mathbb{Z}[\Delta]$-modules

$$0 \to \Gamma_2^{\mathrm{ab}} \to \mathcal{E} \to I_\Delta \to 0. \tag{1.5}$$

Theorem 1.7. *The functor*

$$\mathrm{CRing} \to \mathrm{Ab}, \quad A \mapsto Z^1(\Gamma_1, \mathrm{Hom}(M, A^\times))$$

is represented by the group algebra $\mathbb{Z}[(\mathcal{E} \otimes M)_\Delta]$. *The fpqc sheafification of the functor*

$$\mathrm{CRing} \to \mathrm{Ab}, \quad A \mapsto H^1(\Gamma_1, \mathrm{Hom}(M, A^\times))$$

is represented by the group algebra $\mathbb{Z}[H_1(\Gamma_1, M)]$.

We prove Theorem 1.7 by first replacing M with $\mathbb{Z}[\Delta] \otimes M$. In this case,

$$\mathrm{Hom}(\mathbb{Z}[\Delta] \otimes M, A^\times) \cong \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, A^\times)$$

and we can use Shapiro's lemma to identify H^1 with

$$\mathrm{Hom}(\Gamma_2, \mathrm{Hom}(M, A^\times)) \cong \mathrm{Hom}(\Gamma_2^{\mathrm{ab}} \otimes M, A^\times).$$

Further, after choosing a representative $\bar{c}$ for each coset $c \in \Gamma_1/\Gamma_2$, we construct in Section 3 a homomorphism of abelian groups,

$$H^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V) \to Z^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V),$$

for any Γ_2-module V, which is functorial in V and becomes the identity, when composed with the natural quotient map. These ingredients allow us to show that the functor

$$\mathrm{Ab} \to \mathrm{Ab}, \quad V \mapsto Z^1(\Gamma_1, \mathrm{Hom}(\mathbb{Z}[\Delta] \otimes M, V)) \tag{1.6}$$

is representable by $(\Gamma_2^{\mathrm{ab}} \oplus I_\Delta) \otimes M$. We then define an action of Δ on $\mathbb{Z}[\Delta] \otimes M$, which commutes with the action of Γ_1, such that

$$Z^1(\Gamma_1, \mathrm{Hom}(M, V)) \cong Z^1(\Gamma_1, \mathrm{Hom}(\mathbb{Z}[\Delta] \otimes M, V))^\Delta.$$

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To prove Theorem 1.7, we verify that the action of Δ on $(\Gamma_2^{\mathrm{ab}} \oplus I_\Delta) \otimes M$ induced by its action on the functor in (1.6) is isomorphic to $\mathcal{E} \otimes M$. This calculation is carried out in Section 4. The last part of Theorem 1.7 follows by observing that (1.1) remains an isomorphism if $\mathbb{C}^\times$ is replaced by any divisible group, and for any A, there is a faithfully flat map $A \to B$ such that $B^\times$ is divisible.

Let G be a generalised torus over some base ring $\mathcal{O}$. Let $\mathrm{PC}_G^{\Gamma_1} : \mathcal{O}\text{-alg} \to \text{ Set}$ be the functor such that $\mathrm{PC}_G^{\Gamma_1}(A)$ is the set of A-valued G-pseudocharacters of Γ_1. Let $\mathrm{Rep}_G^{\Gamma_1} : \mathcal{O}\text{-alg} \to \text{ Set}$ be the functor such that $\mathrm{Rep}_G^{\Gamma_1}(A)$ is the set of all representations $\rho : \Gamma_1 \to G(A)$.

Theorem 1.8. *Mapping a representation to its G-pseudocharacter induces an isomorphism of schemes*

$$\mathrm{Rep}_G^{\Gamma_1} /\!\!/ G^0 \xrightarrow{\cong} \mathrm{PC}_G^{\Gamma_1}.$$

The proof of Theorem 1.8 follows the arguments of Emerson–Morel [8], where they prove an analogous result, when G is a connected reductive group over a field of characteristic 0. Since in our case G^0 is a torus, it is linearly reductive over $\mathcal{O}$, and their arguments carry over integrally.

Let us assume that G^0 is split over $\mathcal{O}$ and G/G^0 is a constant group scheme. Let $\rho_0 : \Gamma_1 \to G(\mathcal{O})$ be a representation and let $\mathrm{Rep}_{G,\pi}^{\Gamma_1}$ be the subfunctor of $\mathrm{Rep}_G^{\Gamma_1}$ consisting of representations ρ such that $\Pi \circ \rho = \Pi \circ \rho_0$, where $\Pi : G \to G/G^0$ is the quotient map. One may similarly define $\mathrm{PC}_{G,\pi}^{\Gamma_1}$. We show that the map

$$Z^1(\Gamma_1, G^0(A)) \to \mathrm{Rep}_{G,\pi}^{\Gamma_1}(A), \quad \Phi \mapsto \Phi\rho_0$$

is bijective and use Theorem 1.7 to prove:

Theorem 1.9. *The functors* $\mathrm{Rep}_{G,\pi}^{\Gamma_1}$ *and* $\mathrm{PC}_{G,\pi}^{\Gamma_1}$ *are represented by the group algebras* $\mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$ *and* $\mathcal{O}[H_1(\Gamma_1, M)]$*, respectively.*

If $\Gamma_1 = W_{E/F}$, then $H_1(\Gamma_1, M) \cong (E^\times \otimes M)^\Delta$ by (1.2). To prove Theorem 1.4, we relate the completions of the local rings of $\mathrm{Rep}_{G,\pi}^{\Gamma_1}$ and $\mathrm{PC}_{G,\pi}^{\Gamma_1}$ to deformation rings parameterising continuous deformations and G-pseudocharacters of the profinite completion $\widehat{\Gamma}_1$ of Γ_1. These arguments are carried out in Section 7. We expect that the abstract setting will also be useful studying deformations of global Galois groups.

1.3. Producing a lift

Let us go back to the problem of exhibiting a lift of $\overline{\rho} : \Gamma_F \to G(k)$ to characteristic zero as claimed in Theorem 1.1. Using $\overline{\rho}$ instead of ρ_0, we may identify the restriction of $D_{\overline{\rho}}^{\square}$ to $\mathfrak{A}_k$ with the restriction of $\widehat{Z}^1$ to $\mathfrak{A}_k$. This allows us to conclude that

$$R_{\overline{\rho}}^{\square}/\varpi \cong k[\![((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}]\!],$$

where $\mathcal{E}$ is defined by (1.5), where (1.4) is equal to (1.3), so that $\Gamma_1 = W_{E/F}$ and $\Gamma_2 = E^\times$ and $(-)^{\wedge,p}$ denotes the pro-p completion.

In Section 8, we compute

$$\mathrm{rank}_{\mathbb{Z}_p}((\mathcal{E} \otimes M)_\Delta)^{\wedge,p} = \mathrm{rank}_{\mathbb{Z}} M \cdot ([F : \mathbb{Q}_p] + 1),$$

which is then also equal to $\dim R_{\overline{\rho}}^{\square}/\varpi$. Moreover, using Mazur's obstruction theory and the Euler–Poincaré characteristic formula, we show that

$$R_{\overline{\rho}}^{\square} \cong \frac{\mathcal{O}[\![x_1, \ldots, x_r]\!]}{(f_1, \ldots, f_s)},$$

where $r - s = \dim_k \operatorname{Lie} G_k \cdot ([F : \mathbb{Q}_p] + 1)$. Since $\operatorname{rank}_{\mathbb{Z}} M = \dim_k \operatorname{Lie} G_k$, a little commutative algebra implies that $\varpi, f_1, \dots, f_s$ is a regular sequence in $\mathcal{O}[\![x_1, \dots, x_r]\!]$, and hence, $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free. A closed point of the generic fibre gives us the required lift.

1.4. Overview by section

In Section 2, we define what a generalised torus is. In Section 3, we establish an explicit version of Shapiro's lemma. In Section 4, we compute the ring representing the functor $A \mapsto Z^1(\Gamma_1, \operatorname{Hom}(M, A^\times))$. This section is the technical heart of the paper. In Section 5, we relate the functor $\operatorname{Rep}^{\Gamma_1}_{G,\pi}$ to the functor $Z^1(\Gamma_1, G^0(-))$. In Section 6, we introduce Lafforgue's G-pseudocharacters and study the functor $\operatorname{PC}^{\Gamma_1}_{G,\pi}$. In Section 7, we transfer the results about abstract representations and G-pseudocharacters of Γ_1 to continuous representations and continuous G-pseudocharacters of its profinite completion $\widehat{\Gamma}_1$ and prove versions of Theorems 1.3, 1.4, 1.5 with Γ_F replaced by $\widehat{\Gamma}_1$. In Section 8, we compute $\mathbb{Z}_p$-ranks of certain pro-p completions appearing naturally in the arithmetic setting. We show in Lemma 8.7 that the pro-p completion of $(E^\times \otimes M)^\Delta$ is isomorphic to $(\Gamma_E^{\mathrm{ab},p} \otimes M)^\Delta$. In Section 9, we use the results proved in the previous sections to prove Theorems 1.1, 1.3, 1.4, 1.5.

1.5. Notation

Let p denote any prime number. Let F be a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F}$ of F. Let $\Gamma_F := \operatorname{Gal}(\overline{F}/F)$ be the absolute Galois group of F. Let L be another finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field k. However, $\mathcal{O}$ is allowed to be an arbitrary commutative ring such that $\operatorname{Spec} \mathcal{O}$ is connected in Sections 5 and 6.

Let $\mathfrak{A}_{\mathcal{O}}$ be the category of local artinian $\mathcal{O}$-algebras with residue field k. We let $\widehat{\mathfrak{A}}_{\mathcal{O}}$ be the category of pro-objects of $\mathfrak{A}_{\mathcal{O}}$. Concretely, one may identify $\widehat{\mathfrak{A}}_{\mathcal{O}}$ with the category of pseudo-compact local $\mathcal{O}$-algebras with residue field k. Let $\mathfrak{A}_k$ be the full subcategory of $\mathfrak{A}_{\mathcal{O}}$ consisting of objects killed by ϖ. Then $\widehat{\mathfrak{A}}_k$ is the full subcategory of $\widehat{\mathfrak{A}}_{\mathcal{O}}$ consisting of objects killed by ϖ, which can be identified with the category of local pseudo-compact k-algebras with residue field k.

The groups denoted by Γ_1 and Γ_2 are abstract groups with no topology. We will denote by $\widehat{\Gamma}_1$ the profinite completion of Γ_1. If $\mathcal{A}$ is an abelian group, then we will denote its pro-p completion by $\mathcal{A}^{\wedge,p}$. If X is a scheme, then we denote the ring of its global sections by $\mathcal{O}(X)$. If Γ is a group, then we denote by $\mathcal{O}[\Gamma]$ its group algebra over a ring $\mathcal{O}$.

2. Generalised tori

Definition 2.1. Let S be a scheme. A *generalised torus* is a smooth affine S-group scheme G, such that the geometric fibres of G^0 are tori and $G/G^0 \to S$ is finite.

Remark 2.2. It follows from [6, Proposition 3.1.3] that for a generalised torus G, the quotient G/G^0 is étale. Sean Cotner has shown in [7] that if S is locally noetherian, then for a smooth affine S-group scheme G, the identity component G^0 has reductive geometric fibers and G/G^0 is finite étale if and only if G is geometrically reductive in the sense of [1, Definition 9.1.1]. In particular, this holds over $S = \operatorname{Spec} \mathcal{O}$.

Remark 2.3. When G is a generalised torus over $\mathcal{O}$, then G^0 is linearly reductive (i.e., taking G^0-invariants is an exact functor on the category of $\mathcal{O}(G^0)$-comodules). It follows that if A is a commutative $\mathcal{O}$-algebra with trivial G^0-action, M is an A-module with G^0-action and N is an A-module with trivial G^0-action, then

$$(M \otimes_A N)^{G^0} = M^{G^0} \otimes_A N.$$

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If M is an abelian group, then we denote by $\mathfrak{D}(M) := \operatorname{Spec}(\mathcal{O}[M])$ the diagonalisable group scheme. For any $\mathcal{O}$-algebra A, we have

$$\mathfrak{D}(M)(A) = \operatorname{Hom}(M, A^\times).$$

If M is a finitely generated free abelian group, then $\mathfrak{D}(M)$ is a split torus.

3. Shapiro's lemma

We recall an explicit version of Shapiro's lemma. Let Γ_1 be a group and let Γ_2 be a subgroup of Γ_1 of finite index. Let V be an abelian group with a Γ_2-action. We let $\operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V$ be the set of functions $f : \Gamma_1 \to V$, such that $f(kg) = kf(g)$ for all $k \in \Gamma_2$ and $g \in \Gamma_1$. Then $\operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V$ is naturally an abelian group, isomorphic to a finite direct sum of copies of V, on which Γ_1 acts by right translations; that is, $[g \cdot f](h) := f(hg)$ for all $g, h \in \Gamma_1$. In this section, topology does not play a role, all cohomology is just abstract group cohomology, and there is no continuity condition on the cocycles.

Proposition 3.1. *Let $\Phi : \Gamma_1 \to \operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V$ be a 1-cocycle. Then the function $\varphi : \Gamma_2 \to V$, given by*

$$\varphi(k) = [\Phi(k)](1),$$

is a 1-cocycle. Moreover, the above map induces an isomorphism in cohomology

$$H^1(\Gamma_1, \operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V) \cong H^1(\Gamma_2, V).$$

Proof. [10, Theorem 3.7]. □

We will construct an explicit inverse of the map above following [10]. For each right coset c of Γ_2 in Γ_1, we fix a coset representative $\bar{c}$, so that the representative of the trivial coset is 1. In particular,

$$\Gamma_1 = \bigcup_c \Gamma_2 \bar{c} = \bigcup_c \bar{c}^{-1} \Gamma_2.$$

Since $cg = \Gamma_2 \overline{cg}$ for every $g \in \Gamma_1$, we have

$$\bar{c} g \overline{cg}^{-1} \in \Gamma_2, \quad \forall g \in \Gamma_1.$$

Let $\mathcal{F}_V$ be the subgroup of $\operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V$ of functions with support in Γ_2. By evaluating at 1, we obtain a canonical isomorphism of Γ_2-representations between $\mathcal{F}_V$ and V; the inverse homomorphism is obtained by mapping v to the function $g \mapsto gv$. If $\varphi \in Z^1(\Gamma_2, V)$, then the isomorphism gives us a cocycle $f_\varphi \in Z^1(\Gamma_2, \mathcal{F}_V)$, where if $h \in \Gamma_2$, then $f_\varphi(h) \in \mathcal{F}_V$ is the function given by

$$[f_\varphi(h)](k) = k\varphi(h), \quad \forall k \in \Gamma_2. \tag{3.1}$$

Lemma 3.2. *Let $\varphi : \Gamma_2 \to V$ be a 1-cocycle, and let $f_\varphi : \Gamma_2 \to \mathcal{F}_V$ be the 1-cocycle corresponding to φ, via the canonical isomorphism $\mathcal{F}_V \cong V$ and let $\Phi_\varphi : \Gamma_1 \to \operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V$ be the function given by*

$$\Phi_\varphi(g) = \sum_c \bar{c}^{-1} f_\varphi(\bar{c} g \overline{cg}^{-1}).$$

Then Φ_φ is a 1-cocycle, such that

$$[\Phi_\varphi(k)](1) = \varphi(k), \quad \forall k \in \Gamma_2.$$

Proof. We have to show that

$$\Phi_\varphi(gh) = \Phi_\varphi(g) + g\Phi_\varphi(h), \quad \forall g, h \in \Gamma_1.$$

It is enough to show that the equality holds once we evaluate both sides at $k\overline{c}$, where $\overline{c}$ is a representative of the coset c and $k \in \Gamma_2$. If $f \in \mathcal{F}_V$, then $\operatorname{Supp} \overline{c}^{-1} f \subseteq c$; hence,

$$[\Phi_\varphi(gh)](k\overline{c}) = [f_\varphi(\overline{c}gh\overline{cgh}^{-1})](k), \quad [\Phi_\varphi(g)](k\overline{c}) = [f_\varphi(\overline{c}g\overline{cg}^{-1})](k),$$

$$\begin{aligned}[g\Phi_\varphi(h)](k\overline{c}) = [\Phi_\varphi(h)](k\overline{c}g) &= [f_\varphi(\overline{cg}h\overline{cgh}^{-1})](k\overline{c}g\overline{cg}^{-1}) \\ &= [\overline{c}g\overline{cg}^{-1} f_\varphi(\overline{cg}h\overline{cgh}^{-1})](k).\end{aligned}$$

Since f_φ is a 1-cocycle, we have

$$\overline{c}g\overline{cg}^{-1} f_\varphi(\overline{cg}h\overline{cgh}^{-1}) + f_\varphi(\overline{c}g\overline{cg}^{-1}) = f_\varphi(\overline{c}g\overline{cg}^{-1}\overline{cg}h\overline{cgh}^{-1}) = f_\varphi(\overline{c}gh\overline{cgh}^{-1}),$$

and hence, Φ_φ is a 1-cocycle. Since the representative of the trivial coset was chosen to be 1, we have

$$[\Phi_\varphi(k)](1) = [f_\varphi(k)](1) = \varphi(k), \quad \forall k \in \Gamma_2. \qquad \square$$

Remark 3.3. The map $\varphi \mapsto \Phi_\varphi$ depends on the choice of the coset representatives $\bar{c}$, and so it is not canonical. However, once these representatives have been fixed, it is immediate from the formulas that the map is functorial in V. If $\alpha : V \to W$ is a Γ_2-equivariant homomorphism of abelian groups equipped with Γ_2-action and $\psi \in Z^1(\Gamma_2, W)$ is the image of φ under the map $Z^1(\Gamma_2, V) \to Z^1(\Gamma_2, W)$ induced by α, then Φ_φ maps to $\Phi_\psi \in Z^1(\Gamma_1, \operatorname{Ind}_{\Gamma_2}^{\Gamma_1} W)$.

4. The space of 1-cocycles

We shall later identify our space of deformations with a space of 1-cocycles. In this section, we study the space of 1-cocycles in a more abstract situation. We fix the following notation:

- Γ_1 is an abstract group.
- M is an abelian group equipped with a linear left Γ_1-action with kernel Γ_2 of finite index in Γ_1.
- $\Delta := \Gamma_1/\Gamma_2$. We denote the projection $\pi : \Gamma_1 \to \Delta$.

We start by constructing an abelian group $\mathcal{E}$, which represents the functor

$$\mathrm{Ab} \to \mathrm{Ab}, \ V \mapsto Z^1(\Gamma_1, \operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V).$$

It turns out that $\mathcal{E}$ carries an action of Δ, such that $(\mathcal{E} \otimes M)_\Delta$ represents the functor

$$\mathrm{Ab} \to \mathrm{Ab}, \ V \mapsto Z^1(\Gamma_1, \operatorname{Hom}(M, V)).$$

Our strategy is to realize $\operatorname{Hom}(M, V)$ as Δ-invariants in $\operatorname{Ind}_{\Gamma_2}^{\Gamma_1} \operatorname{Hom}(M, V)$, and then use the explicit version of Shapiro's lemma of Section 3 to compute

$$Z^1(\Gamma_1, \operatorname{Ind}_{\Gamma_2}^{\Gamma_1} \operatorname{Hom}(M, V)).$$

At the end of this section, we study the functors $\mathrm{CRing} \to \mathrm{Ab}$ defined by

$$A \mapsto Z^1(\Gamma_1, \operatorname{Hom}(M, A^\times)), \quad A \mapsto H^1(\Gamma_1, \operatorname{Hom}(M, A^\times)).$$

Let V be an abelian group equipped with a trivial Γ_2-action. We may identify $\operatorname{Ind}_{\Gamma_2}^{\Gamma_1} V$ with the space of functions $f : \Delta \to V$ where the action is given by $[g \cdot f](c) = f(cg)$ for all $g \in \Gamma_1$ and

$c \in \Delta$. We also have an action of Δ on $\mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V$, which commutes with the action of Γ_1 and is given by $[d \cdot f](c) = f(d^{-1}c)$ for all $d \in \Delta$. This induces an action of Δ on $Z^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V)$, which is given explicitly by

$$[[d * \Phi](g)](c) = [\Phi(g)](d^{-1}c). \tag{4.1}$$

Since the action of Γ_2 on V is trivial, we have

$$Z^1(\Gamma_2, V) = H^1(\Gamma_2, V) = \mathrm{Hom}(\Gamma_2, V), \tag{4.2}$$

where Hom stands for group homomorphisms. For each $c \in \Delta$, we fix a coset representative $\overline{c} \in \Gamma_1$ as in the previous section.

Corollary 4.1. *Let $\varphi \in \mathrm{Hom}(\Gamma_2, V)$. Then the 1-cocycle Φ_φ constructed in Lemma 3.2 is given by*

$$\Phi_\varphi(g) = \sum_{c \in \Delta} \varphi(\overline{c} g \overline{cg}^{-1}) \mathbf{1}_c,$$

where $\mathbf{1}_c$ is the indicator function for the coset c.

Proof. Since the action of Γ_2 on V is trivial, by assumption, it follows from (3.1) that $[f_\varphi(h)](k) = \varphi(h)$ for all $k, h \in \Gamma_2$. The assertion then follows from Lemma 3.2. □

Proposition 3.1 and (4.2) give us an exact sequence of abelian groups:

$$0 \to B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V) \to Z^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V) \to \mathrm{Hom}(\Gamma_2, V) \to 0,$$

and Lemma 3.2 gives us a section so that

$$Z^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V) \cong B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V) \oplus \mathrm{Hom}(\Gamma_2, V). \tag{4.3}$$

We want to understand the action of Δ on $B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V) \oplus \mathrm{Hom}(\Gamma_2, V)$ via this isomorphism. It follows from (4.1) that $[[d * \Phi](g)](dc) = [\Phi(g)](c)$. Hence,

$$[d * \Phi_\varphi](g) = \sum_{c \in \Delta} \varphi(\overline{c} g \overline{cg}^{-1}) \mathbf{1}_{dc}$$

for $\varphi \in \mathrm{Hom}(\Gamma_2, V)$. Since Γ_2 is normal in Γ_1, $ck = c$ for all $k \in \Gamma_2$ and all $c \in \Delta$. Hence, $\overline{ck} = \overline{c}$, and we conclude that

$$[[d * \Phi_\varphi](k)](1) = \sum_c \varphi(\overline{c} k \overline{c}^{-1}) \mathbf{1}_{dc}(1) = \varphi(\overline{d^{-1}} k \overline{d^{-1}}^{-1}) = \varphi(\overline{d}^{-1} k \overline{d}) =: [d \cdot \varphi](k).$$

We note that $\varphi((\overline{d}h)^{-1} k (\overline{d}h)) = \varphi(\overline{d}^{-1} k \overline{d})$ for all $h, k \in \Gamma_2$, which justifies the third equality, and we take the last equality as the definition $d \cdot \varphi$. Since $[[\Phi_{d \cdot \varphi}](k)](1) = [d \cdot \varphi](k)$ by Lemma 3.2, Proposition 3.1 implies that the cocycle $d * \Phi_\varphi - \Phi_{d \cdot \varphi}$ is a 1-coboundary, and thus, there is a function $f : \Delta \to V$ depending on d such that for all $g \in \Gamma_1$, we have

$$[d * \Phi_\varphi](g) - \Phi_{d \cdot \varphi}(g) = (g - 1) f. \tag{4.4}$$

After subtracting a constant function, we may assume that $f(1) = 0$. By evaluating both sides of (4.4) at 1, we obtain

$$\begin{aligned} f(g) = f(g) - f(1) &= [[d * \Phi_\varphi](g)](1) - [\Phi_{d\cdot\varphi}(g)](1) \\ &= \varphi(\overline{d^{-1}}g\overline{d^{-1}g}^{-1}) - [d\cdot\varphi](g\overline{g}^{-1}) \\ &= \varphi(\overline{d^{-1}}g\overline{d^{-1}g}^{-1}) - \varphi(\overline{d^{-1}}g\overline{g}^{-1}\overline{d^{-1}}^{-1}) \\ &= \varphi((\overline{d^{-1}}g\overline{g}^{-1}\overline{d^{-1}}^{-1})^{-1}\overline{d^{-1}}g\overline{d^{-1}g}^{-1}) \\ &= \varphi(\overline{d^{-1}}\overline{g}\overline{d^{-1}g}^{-1}). \end{aligned} \tag{4.5}$$

We note that in the above calculation, $\overline{g}$ is the fixed coset representative of the coset $\Gamma_2 g$. In particular, $\overline{g} \in \Gamma_1$ and not in Δ. The function f satisfies $f(kg) = f(g)$ for all $k \in \Gamma_2$, so we may consider as a function on Δ. In this case, the formula (4.5) applied with $g = \overline{c}$ gives us

$$f(c) = \varphi(\overline{d^{-1}}\overline{c}\overline{d^{-1}c}^{-1}), \quad \forall c \in \Delta. \tag{4.6}$$

The function f corresponds to a group homomorphism $\alpha_f : \mathbb{Z}[\Delta] \to V$, given by $\alpha_f(c) = f(c)$. Since $f(1) = 0$, the restriction of α_f to the augmentation ideal I_Δ is given by

$$\alpha_f(c-1) = \varphi(\overline{d^{-1}}\overline{c}\overline{d^{-1}c}^{-1}), \quad \forall c \in \Delta. \tag{4.7}$$

Since V is abelian, we have $\mathrm{Hom}(\Gamma_2, V) = \mathrm{Hom}(\Gamma_2^{\mathrm{ab}}, V)$, where Γ_2^{ab} is the maximal abelian quotient of Γ_2. For $d, c \in \Delta$, we define $\kappa(d,c)$ to be the image of $\overline{d}\overline{c}\overline{dc}^{-1}$ in Γ_2^{ab}. The map $\kappa : \Delta \times \Delta \to \Gamma_2^{\mathrm{ab}}$ is the 2-cocycle corresponding to the extension

$$0 \to \Gamma_2^{\mathrm{ab}} \to \Gamma_1/[\Gamma_2, \Gamma_2] \to \Delta \to 0.$$

We define an action of Δ on $\Gamma_2^{\mathrm{ab}} \oplus I_\Delta$ by letting

$$d * (g, c-1) := (dgd^{-1} \cdot \kappa(d,c), dc - d). \tag{4.8}$$

This action defines an action of Δ on $\mathrm{Hom}(\Gamma_2^{\mathrm{ab}} \oplus I_\Delta, V)$ by

$$\begin{aligned} [d * (\varphi, \alpha)](g, c-1) &= (\varphi, \alpha)(d^{-1} * (g, c-1)) \\ &= \varphi(d^{-1}gd) + \varphi(\kappa(d^{-1}, c)) + \alpha(d^{-1}c - d^{-1}). \end{aligned} \tag{4.9}$$

We obtain an exact sequence of $\mathbb{Z}[\Delta]$-modules

$$0 \to \Gamma_2^{\mathrm{ab}} \to \mathcal{E} \to I_\Delta \to 0 \tag{4.10}$$

by letting $\mathcal{E} = \Gamma_2^{\mathrm{ab}} \oplus I_\Delta$ with the Δ-action defined as above.

Remark 4.2. Using the exact sequence $0 \to I_\Delta \to \mathbb{Z}[\Delta] \to \mathbb{Z} \to 0$ of $\mathbb{Z}[\Delta]$-modules, one obtains a canonical identification

$$\mathrm{Ext}^1_{\mathbb{Z}[\Delta]}(I_\Delta, \Gamma_2^{\mathrm{ab}}) \cong \mathrm{Ext}^2_{\mathbb{Z}[\Delta]}(\mathbb{Z}, \Gamma_2^{\mathrm{ab}}) \cong H^2(\Delta, \Gamma_2^{\mathrm{ab}});$$

see [4, Chapter XIV, §4, Remark]. The extension class of (4.10) gives an element in $\mathrm{Ext}^1_{\mathbb{Z}[\Delta]}(I_\Delta, \Gamma_2^{\mathrm{ab}})$. One can show that the image of this class in $H^2(\Delta, \Gamma_2^{\mathrm{ab}})$ is equal to the class of κ.

To $\alpha \in \mathrm{Hom}(I_\Delta, V)$, we may associate a function $f_\alpha : \Delta \to V$ by letting $f_\alpha(c) := \alpha(c-1)$. We note that $f_\alpha(1) = 0$. We then define a 1-coboundary $b_\alpha \in B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V)$ by $b_\alpha(g) := (g-1)f_\alpha$. We may

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recover f_α from b_α as the unique function $f : \Delta \to V$ satisfying $f(1) = 0$ and $b_\alpha(g) = (g-1)f$ for all $g \in \Gamma_1$. Thus, the map

$$\mathrm{Hom}(I_\Delta, V) \to B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V), \quad \alpha \mapsto b_\alpha \tag{4.11}$$

is an isomorphism. We now record the consequence of our calculations:

Proposition 4.3. *If V is an abelian group with the trivial Γ_2-action, then sending (φ, α) to $\Phi_\varphi + b_\alpha$ induces an isomorphism*

$$\mathrm{Hom}(\Gamma_2^{\mathrm{ab}} \oplus I_\Delta, V) \xrightarrow{\cong} Z^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V),$$

which is Δ-equivariant for the actions defined in (4.9) and (4.1).

Proof. It follows from (4.11) and (4.3) that the map is an isomorphism of abelian groups. We have to check that it is Δ-equivariant. We have

$$\begin{aligned}[][d * b_\alpha](g)](c) &= [b_\alpha(g)](d^{-1}c) = f_\alpha(d^{-1}cg) - f_\alpha(d^{-1}c)\\ &= \alpha(d^{-1}cg - 1) - \alpha(d^{-1}c - 1) = \alpha(d^{-1}cg - d^{-1}c),\end{aligned}$$

$$\begin{aligned}[b_{d*\alpha}(g)](c) &= f_{d*\alpha}(cg) - f_{d*\alpha}(c) = [d*\alpha](cg-1) - [d*\alpha](c-1)\\ &= \alpha(d^{-1}cg - d^{-1}) - \alpha(d^{-1}c - d^{-1}) = \alpha(d^{-1}cg - d^{-1}c).\end{aligned}$$

Hence, $d * b_\alpha = b_{d*\alpha}$. It follows from (4.4), (4.6) and (4.7) that $d * \Phi_\varphi = \Phi_{d\cdot\varphi} + b_\beta$, where $\beta(c-1) = \varphi(\kappa(d^{-1}, c))$ for all $c \in \Delta$. Moreover, it follows from (4.9) that $d * (\varphi, 0) = (d \cdot \varphi, \beta)$. Hence, $d * (\varphi, 0)$ is mapped to $\Phi_{d\cdot\varphi} + b_\beta = d * \Phi_\varphi$. □

Recall that M is an abelian group with an action of Δ, so Γ_2 acts trivially on $\mathrm{Hom}(M, V)$. We have natural isomorphisms of Γ_1-representations

$$\mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V) \cong \mathrm{Hom}(\mathbb{Z}[\Delta], \mathrm{Hom}(M, V)) \cong \mathrm{Hom}(\mathbb{Z}[\Delta] \otimes M, V), \tag{4.12}$$

where the action of Γ_1 on the last term is induced by its action on $\mathbb{Z}[\Delta] \otimes M$, which is given by $g \cdot (c \otimes m) = cg^{-1} \otimes m$, so that $[g \cdot f](x) = f(g^{-1}x)$ for all $x \in \mathbb{Z}[\Delta] \otimes M$. The diagonal action of Δ on $\mathbb{Z}[\Delta] \otimes M$ is given by $d \cdot (c \otimes m) = dc \otimes dm$, commutes with the action of Γ_1 and induces a (left) action of Δ on $\mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V)$, which commutes with the action of Γ_1. If $M = \mathbb{Z}$ with the trivial Δ-action, then the construction recovers the action of Δ on $\mathrm{Ind}_{\Gamma_2}^{\Gamma_1} V$ considered earlier.

Lemma 4.4. *As Γ_1-representations, $(\mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V))^\Delta$ is naturally isomorphic to $\mathrm{Hom}(M, V)$ with Γ_1 acting on it via Δ.*

Proof. The map $\mathbb{Z}[\Delta] \otimes M \to \mathbb{Z}[\Delta] \otimes M$, $c \otimes m \mapsto c \otimes c^{-1}m$ is an isomorphism of $\Delta \times \Gamma_1$-representations, where on the source, the action is the one considered above, and on the target, Δ acts by $d * (c \otimes m) = dc \otimes m$ and Γ_1 acts by $g * (c \otimes m) = cg^{-1} \otimes gm$. It is immediate by considering the $*$-action that $(\mathbb{Z}[\Delta] \otimes M)_\Delta \cong M$ and Γ_1 acts on M via Δ. Using (4.12), we can then translate this statement to a statement about invariants. □

Proposition 4.5. *Let $\mathcal{E}$ be the $\mathbb{Z}[\Delta]$-module constructed above. Then for all abelian groups V, we have an isomorphism of abelian groups functorial in V:*

$$\mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, V) \cong Z^1(\Gamma_1, \mathrm{Hom}(M, V)), \tag{4.13}$$

where the action of Δ on $\mathcal{E} \otimes M$ is diagonal and Γ_1 acts on $\mathrm{Hom}(M, V)$ via Δ.

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Proof. Proposition 4.3 gives us Δ-equivariant isomorphisms:

$$\mathrm{Hom}(\mathcal{E}\otimes M, V) \cong \mathrm{Hom}(\mathcal{E}, \mathrm{Hom}(M, V)) \cong Z^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V)).$$

The assertion follows after taking Δ-invariants and applying Lemma 4.4. □

The exact sequence $0 \to I_\Delta \to \mathbb{Z}[\Delta] \to \mathbb{Z} \to 0$ remains exact after tensoring with M. Taking Δ-coinvariants yields an exact sequence

$$0 \to H_1(\Delta, M) \to (I_\Delta \otimes M)_\Delta \to M \to M_\Delta \to 0.$$

This gives us a surjection $(I_\Delta \otimes M)_\Delta \twoheadrightarrow I_\Delta M$. By composing this map with the surjection $(\mathcal{E}\otimes M)_\Delta \twoheadrightarrow (I_\Delta \otimes M)_\Delta$ induced by (4.10), we obtain a surjection $(\mathcal{E}\otimes M)_\Delta \twoheadrightarrow I_\Delta M$. This in turn yields an injection

$$\mathrm{Hom}(I_\Delta M, V) \hookrightarrow \mathrm{Hom}((\mathcal{E}\otimes M)_\Delta, V).$$

Lemma 4.6. *The isomorphism (4.13) in Proposition 4.5 identifies the space of* 1*-coboundaries* $B^1(\Gamma_1, \mathrm{Hom}(M, V))$ *with a subgroup of* $\mathrm{Hom}(I_\Delta M, V)$. *Moreover, it induces an isomorphism between the two groups if V is divisible.*

Proof. The isomorphism $(\mathbb{Z}[\Delta]\otimes M)_\Delta \cong M$ for the diagonal action of Δ on $\mathbb{Z}[\Delta]\otimes M$ is realised by the map $c\otimes m \mapsto c^{-1}m$. The image of $I_\Delta \otimes M$ under this map is equal to $I_\Delta M$. Thus, the isomorphism

$$\vartheta : \mathrm{Hom}(M, V) \xrightarrow{\cong} (\mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V))^\Delta$$

in Lemma 4.4 is given explicitly by

$$[\vartheta(\alpha)(c)](m) = \alpha(c^{-1}m), \quad \forall c \in \Delta, \quad \forall m \in M.$$

Let $b \in B^1(\Gamma_1, \mathrm{Hom}(M, V))$ be a boundary with $b(g) = (g-1)\alpha$ for all $g \in \Gamma_1$. Its image in $B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V))$ is the boundary b' given by $b'(g) = (g-1)\vartheta(\alpha)$. The constant function $f'(c) = \alpha$ for all $c \in \Delta$ is Γ_1-invariant, and thus,

$$b'(g) = (g-1)(\vartheta(\alpha) - f'), \quad \forall g \in \Gamma_1.$$

Since $(\vartheta(\alpha) - f')(1) = \alpha - \alpha = 0$, by Proposition 4.3, $b' = b_\beta$, where $\beta : I_\Delta \to \mathrm{Hom}(M, V)$ is given by

$$[\beta(c-1)](m) = [(\vartheta(\alpha) - f')(c)](m) = \alpha(c^{-1}m) - \alpha(m) = \alpha(c^{-1}m - m).$$

We conclude that the image of $B^1(\Gamma_1, \mathrm{Hom}(M, V))$ under the isomorphism in Proposition 4.5 is contained in $\mathrm{Hom}(I_\Delta M, V)$.

Conversely, if we start with a homomorphism $\beta' : I_\Delta M \to V$, then by pulling it back under the surjection $I_\Delta \otimes M \twoheadrightarrow I_\Delta M$, we obtain a homomorphism $\beta : I_\Delta \otimes M \to V$ given by $\beta((c-1)\otimes m) = \beta'(c^{-1}m - m)$. The corresponding coboundary $b_\beta \in B^1(\Gamma_1, \mathrm{Ind}_{\Gamma_2}^{\Gamma_1} \mathrm{Hom}(M, V))$ is given by $b_\beta(g) = (g-1)f_\beta$, where

$$[f_\beta(c)](m) = \beta((c-1)\otimes m) = \beta'(c^{-1}m - m).$$

If V is divisible, then V is injective in Ab and the exact sequence

$$0 \to I_\Delta M \to M \to M_\Delta \to 0$$

gives rise to an exact sequence

$$0 \to \mathrm{Hom}(M_\Delta, V) \to \mathrm{Hom}(M, V) \to \mathrm{Hom}(I_\Delta M, V) \to 0,$$

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and we conclude that there is $\alpha \in \mathrm{Hom}(M, V)$ mapping to β', so that

$$\beta'(c^{-1}m - m) = \alpha(c^{-1}m) - \alpha(m).$$

Let f' be the constant function $f'(c) = \alpha$ for all $c \in \Delta$. Then the boundary $g \mapsto (g-1)(f_\beta + f')$ is equal to b_β and $f_\beta + f' = \vartheta(\alpha)$. □

By taking $V = (\mathcal{E} \otimes M)_\Delta$ in Proposition 4.5, we obtain a natural cocycle

$$\Phi_{\mathrm{nat}} \in Z^1(\Gamma_1, \mathrm{Hom}(M, (\mathcal{E} \otimes M)_\Delta)),$$

which corresponds to the identity in $\mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, (\mathcal{E} \otimes M)_\Delta)$.

Lemma 4.7. *If V is any abelian group, then for all Γ_1-modules N and all $i \geq 0$, we have a canonical map*

$$H^i(\Gamma_1, \mathrm{Hom}(N, V)) \to \mathrm{Hom}(H_i(\Gamma_1, N), V), \tag{4.14}$$

which is an isomorphism if V is divisible.

Proof. If $(C_\bullet, d_\bullet)$ is a complex of abelian groups, then we let $Z_n = \ker(d_n : C_n \to C_{n-1})$ and $Z^n = \ker(d^*_{n+1} : \mathrm{Hom}(C_n, V) \to \mathrm{Hom}(C_{n+1}, V))$. The evaluation pairing $Z_n \times Z^n \to V$ induces a bilinear map $H_n(C_\bullet) \times H^n(\mathrm{Hom}(C_\bullet, V)) \to V$, which induces a canonical map $H^n(\mathrm{Hom}(C_\bullet, V)) \to \mathrm{Hom}(H_n(C_\bullet), V)$. If V is divisible, then V is injective in Ab and the map is an isomorphism.

Let $P_\bullet \twoheadrightarrow \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}[\Gamma_1]$-modules. The complex $\mathrm{Hom}_{\Gamma_1}(P_\bullet, \mathrm{Hom}(N, V)) \cong \mathrm{Hom}((P_\bullet \otimes N)_{\Gamma_1}, V)$ computes the cohomology groups $H^i(\Gamma_1, \mathrm{Hom}(N, V))$. The complex $(P_\bullet \otimes N)_{\Gamma_1}$ computes the homology groups $H_i(\Gamma_1, N)$. We apply the previous discussion to $C_\bullet := (P_\bullet \otimes N)_{\Gamma_1}$ to obtain the required homomorphism. □

Applying Lemma 4.7 with $N = M$ and $V = (\mathcal{E} \otimes M)_\Delta$, we obtain a homomorphism $\varphi_{\mathrm{nat}} : H_1(\Gamma_1, M) \to (\mathcal{E} \otimes M)_\Delta$ corresponding to the cohomology class $[\Phi_{\mathrm{nat}}]$.

Lemma 4.8. *For all abelian groups V, the composition*

$$\begin{aligned} \mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, V) \xrightarrow{(4.13)} Z^1(\Gamma_1, \mathrm{Hom}(M, V)) \twoheadrightarrow \\ H^1(\Gamma_1, \mathrm{Hom}(M, V)) \xrightarrow{(4.14)} \mathrm{Hom}(H_1(\Gamma_1, M), V) \end{aligned} \tag{4.15}$$

is given by $\psi \mapsto \psi \circ \varphi_{\mathrm{nat}}$.

Proof. This follows from the fact that all our constructions are functorial in V. If we denote the four functors appearing in (4.15) with A, B, C, D, then for a homomorphism of abelian groups $\psi : W \to V$, we obtain a diagram:

$$\begin{array}{ccccccc} A(W) & \longrightarrow & B(W) & \longrightarrow & C(W) & \longrightarrow & D(W) \\ \downarrow{\scriptstyle \psi\circ} & & \downarrow{\scriptstyle \psi\circ} & & \downarrow{\scriptstyle \psi\circ} & & \downarrow{\scriptstyle \psi\circ} \\ A(V) & \longrightarrow & B(V) & \longrightarrow & C(V) & \longrightarrow & D(V) \end{array}$$

with commutative squares. If we take $W = (\mathcal{E} \otimes M)_\Delta$, then the identity map in $A((\mathcal{E} \otimes M)_\Delta)$ maps to φ_{nat} in $D((\mathcal{E} \otimes M)_\Delta)$ by construction and hence to $\psi \circ \varphi_{\mathrm{nat}}$ in $D(V)$. Since the identity maps to ψ in $A(V)$, we obtain the assertion. □

Proposition 4.9. *There is an exact sequence of abelian groups*

$$0 \to H_1(\Gamma_1, M) \xrightarrow{\varphi_{\mathrm{nat}}} (\mathcal{E} \otimes M)_\Delta \xrightarrow{q} I_\Delta M \to 0 \tag{4.16}$$

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which is functorial in M, and for all abelian groups V, the diagram

$$\begin{array}{ccccc} B^1(\Gamma_1, \mathrm{Hom}(M,V)) & \hookrightarrow & Z^1(\Gamma_1, \mathrm{Hom}(M,V)) & \xrightarrow{(4.14)} & \mathrm{Hom}(H_1(\Gamma_1, M), V) \\ \downarrow & & \downarrow \cong & & \downarrow = \\ \mathrm{Hom}(I_\Delta M, V) & \hookrightarrow & \mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, V) & \xrightarrow{\varphi_{\mathrm{nat}}^*} & \mathrm{Hom}(H_1(\Gamma_1, M), V) \end{array}$$

commutes.

Proof. Lemmas 4.6 and 4.8 give us the commutative diagram above. Moreover, if V is divisible, then by Lemma 4.7, the top row is exact, the last arrow in the top row is surjective and the first vertical arrow is an isomorphism. We deduce that for all divisible V, the maps $q : (\mathcal{E} \otimes M)_\Delta \twoheadrightarrow I_\Delta M$ and $\varphi_{\mathrm{nat}} : H_1(\Gamma, M) \to (\mathcal{E} \otimes M)_\Delta$ induce an exact sequence

$$0 \to \mathrm{Hom}(I_\Delta M, V) \to \mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, V) \to \mathrm{Hom}(H_1(\Gamma, M), V) \to 0. \tag{4.17}$$

Since divisible groups are precisely injective objects in Ab, which has enough injectives, an abelian group A is zero if and only if $\mathrm{Hom}(A, V) = 0$ for all divisible groups V. Using this and the exactness of (4.17), we obtain that $\mathrm{Im}(q \circ \varphi_{\mathrm{nat}}) = 0$; thus, (4.16) is a complex and a further application of the same argument shows that (4.16) is exact. □

Corollary 4.10. *The functor* CRing $\to$ Ab *given by*

$$A \mapsto Z^1(\Gamma_1, \mathrm{Hom}(M, A^\times))$$

is represented by the group algebra $\mathbb{Z}[(\mathcal{E} \otimes M)_\Delta]$.

Proof. If W is an abelian group, then we may identify W with a subgroup of units of the group ring $\mathbb{Z}[W]$. The map

$$\mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[W], A) \to \mathrm{Hom}(W, A^\times), \quad \psi \mapsto \psi|_W$$

is an isomorphism. Applying this observation to $W = (\mathcal{E} \otimes M)_\Delta$ and using Proposition 4.5 yields the assertion. □

Corollary 4.11. *If M is a free* $\mathbb{Z}$*-module of finite rank, then*

$$\mathbb{Z}[(\mathcal{E} \otimes M)_\Delta] \cong \mathbb{Z}[H_1(\Gamma_1, M)][t_1^{\pm 1}, \dots, t_s^{\pm 1}], \tag{4.18}$$

where $s = \mathrm{rank}_{\mathbb{Z}} M - \mathrm{rank}_{\mathbb{Z}} M_\Delta$.

Proof. Since $I_\Delta M$ is a submodule of M, which is free of finite rank over $\mathbb{Z}$, we have an isomorphism $I_\Delta M \cong \mathbb{Z}^s$, where s is as above. By choosing a section to the surjection $(\mathcal{E} \otimes M)_\Delta \twoheadrightarrow I_\Delta M$ in (4.16), we obtain an isomorphism $(\mathcal{E} \otimes M)_\Delta \cong H_1(\Gamma_1, M) \times \mathbb{Z}^s$, and this implies the assertion. □

Proposition 4.12. *The map (4.14) induces a map of presheaves* CRing $\to$ Set

$$H^1(\Gamma_1, \mathrm{Hom}(M, (-)^\times)) \to \mathrm{Spec}(\mathbb{Z}[H_1(\Gamma_1, M)]) \tag{4.19}$$

which exhibits the right-hand side as the fpqc sheafification of the left-hand side.

Proof. The right-hand side is an fpqc sheaf since it is representable [15, Tag 023Q]. So it suffices to find for every ring A and every homomorphism $f : \mathbb{Z}[H_1(\Gamma_1, M)] \to A$ an fpqc cover of A and a descent datum in the left-hand side, which maps to a descent datum in the right-hand side, which descends to f.

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By [17, Lemma 4.1.1], for every ring A, there is a faithfully flat map $A \to B$, such that $B^\times$ is divisible. By Lemma 4.7, the map

$$H^1(\Gamma_1, \mathrm{Hom}(M, B^\times)) \to \mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[H_1(\Gamma_1, M)], B)$$

is bijective. So the canonical descent datum associated with f and the map $A \to B$ come from a descent datum in the right-hand side. □

5. Admissible representations

In this section, let $\mathcal{O}$ be an arbitrary commutative ring, such that $\mathrm{Spec}\,\mathcal{O}$ is connected. Let G be a generalised torus over $\mathcal{O}$, such that G^0 is split and such that G/G^0 is constant. Let $\Pi : G \to G/G^0$ be the projection map. We define the character lattice of G^0 by

$$M := \mathrm{Hom}_{\mathcal{O}\text{-GrpSch}}(G^0, \mathbb{G}_m). \tag{5.1}$$

We may identify G^0 with the split torus $\mathfrak{D}(M)$ introduced in Section 2. We can write $G/G^0 = \underline{\Delta}$ for a finite group Δ, where $\underline{\Delta} := \mathrm{Spec}(\mathrm{Map}(\Delta, \mathcal{O}))$. For any $\mathcal{O}$-algebra A, we have a natural map $\Delta \to \underline{\Delta}(A) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathrm{Map}(\Delta, \mathcal{O}), A)$ defined by evaluation. The $\mathcal{O}$-group scheme G acts on G^0 by conjugation, and this action factors over $G/G^0 = \underline{\Delta}$. So we have a well-defined action map $G/G^0 \times G^0 \to G^0$, $(g, h) \mapsto ghg^{-1}$ with the property that for every $\mathcal{O}$-algebra A and every $g \in (G/G^0)(A)$, the map $G^0(A) \to G^0(A)$, $h \mapsto ghg^{-1}$ is a group automorphism of $G^0(A)$. This induces a (left) action of Δ on M via (5.1).

Let Γ_1 be an abstract group and let $\pi : \Gamma_1 \twoheadrightarrow \Delta$ be a surjective homomorphism with kernel Γ_2. So we are in the situation of Section 4.

Definition 5.1. We say that a representation $\rho : \Gamma_1 \to G(A)$ is *admissible* if $\Pi \circ \rho : \Gamma_1 \to \underline{\Delta}(A)$ sends $\gamma \in \Gamma_1$ to the image of $\pi(\gamma) \in \Delta$ in $\underline{\Delta}(A)$.

This terminology is motivated by admissible Galois representations into L-groups appearing in the Langlands correspondence; see [3, §9].

Let $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$ be the set of admissible representations $\rho : \Gamma_1 \to G(A)$. The group $\mathfrak{D}(M)(A)$ acts on $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$ by conjugation. This defines a scheme-theoretic action

$$\mathfrak{D}(M) \times \mathrm{Rep}^{\Gamma_1}_{G,\pi} \to \mathrm{Rep}^{\Gamma_1}_{G,\pi}. \tag{5.2}$$

Proposition 5.2. *Assume there is a representation $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. Then*

$$Z^1(\Gamma_1, \mathfrak{D}(M)(A)) \to \mathrm{Rep}^{\Gamma_1}_{G,\pi}(A), \quad \Phi \mapsto [\gamma \mapsto \Phi(\gamma)\rho_0(\gamma)] \tag{5.3}$$

is a bijection, which is natural in $A \in \mathcal{O}$-alg. In particular, $\mathrm{Rep}^{\Gamma_1}_{G,\pi}$ is representable by an $\mathcal{O}$-algebra, and (5.3) induces a natural isomorphism

$$\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi}) \xrightarrow{\cong} \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]. \tag{5.4}$$

Moreover, (5.3) is $\mathfrak{D}(M)(A)$-equivariant and induces a natural bijection between the set of $\mathfrak{D}(M)(A)$-orbits in $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$ and $H^1(\Gamma_1, \mathfrak{D}(M)(A))$. In particular, (5.4) is $\mathfrak{D}(M)$-equivariant.

Proof. For $\Phi \in Z^1(\Gamma_1, \mathfrak{D}(M)(A))$, we have

$$\begin{aligned}\Phi(\gamma_1\gamma_2)\rho_0(\gamma_1\gamma_2) &= \Phi(\gamma_1)^{\gamma_1}\Phi(\gamma_2)\rho_0(\gamma_1)\rho_0(\gamma_2)\\ &= \Phi(\gamma_1)\rho_0(\gamma_1)\Phi(\gamma_2)\rho_0(\gamma_2),\end{aligned} \tag{5.5}$$

where in the last equality we use that ρ_0 is admissible. Thus, $\Phi\rho_0$ is a homomorphism. By applying the projection $\Pi : G(A) \to \underline{\Delta}(A)$, we verify that $\Phi\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$. It is clear that (5.3) is a bijection $\mathrm{Map}(\Gamma_1, G(A)) \to \mathrm{Map}(\Gamma_1, G(A))$ with inverse $\rho \mapsto \rho\rho_0^{-1}$. If $\rho \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$, we see that $\Phi := \rho\rho_0^{-1}$ is a 1-cocycle by reverting the computation in (5.5). The claim about representability follows from Corollary 4.10.

The $\mathfrak{D}(M)(A)$-conjugacy classes are the $B^1(\Gamma_1, \mathfrak{D}(M)(A))$-orbits: if we write an admissible homomorphism $\rho : \Gamma_1 \to G(A)$ as above as $\rho(\gamma) = \Phi(\gamma)\rho_0(\gamma)$ and $g \in \mathfrak{D}(M)(A)$, then

$$\begin{aligned} g\rho(\gamma)g^{-1} &= g\Phi(\gamma)({}^{\gamma}g^{-1})\rho_0(\gamma) \\ &= \beta_g(\gamma)\Phi(\gamma)\rho_0(\gamma), \end{aligned}$$

where $\beta_g \in B^1(\Gamma_1, \mathfrak{D}(M)(A))$ is the coboundary $\beta_g(\gamma) = g({}^{\gamma}g^{-1})$. It follows that the $\mathfrak{D}(M)(A)$-orbits are the classes in $H^1(\Gamma_1, \mathfrak{D}(M)(A))$. □

Remark 5.3. If $G = G^0 \rtimes \underline{\Delta}$, then the composition of π with the natural map $\Delta \to \underline{\Delta}(\mathcal{O}) \to G^0(\mathcal{O}) \rtimes \underline{\Delta}(\mathcal{O}) = G(\mathcal{O})$ is a canonical choice for a representation $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. In general, the isomorphisms (5.3) and (5.4) depend on ρ_0.

Proposition 5.4. *Assume there is a representation $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. Under the isomorphism (5.3), the action (5.2) corresponds to the ring homomorphism*

$$\mathcal{O}[(\mathcal{E} \otimes M)_\Delta] \to \mathcal{O}[(\mathcal{E} \otimes M)_\Delta] \otimes \mathcal{O}[M],$$

which sends $x \in (\mathcal{E} \otimes M)_\Delta$ to $x \otimes q(x)$, where $q : (\mathcal{E} \otimes M)_\Delta \twoheadrightarrow I_\Delta M$ is the map of (4.16).

Proof. We have seen in the proof of Proposition 5.2 that the action of $\mathfrak{D}(M)$ on $\mathrm{Rep}^{\Gamma_1}_{G,\pi}$ corresponds under the isomorphism (5.3) to the action via the boundary map

$$\mathfrak{D}(M)(A) \to Z^1(\Gamma_1, \mathfrak{D}(M)(A))$$

and group multiplication. By commutativity of the left square in Proposition 4.9, this map is induced by the composition $(\mathcal{E} \otimes M)_\Delta \xrightarrow{q} I_\Delta M \to M$. As the comultiplication on $\mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$ is given by $x \mapsto x \otimes x$ for all $x \in (\mathcal{E} \otimes M)_\Delta$, the claim follows by composing with $\mathcal{O}[(\mathcal{E} \otimes M)_\Delta] \to \mathcal{O}[M]$ in the second factor. □

Corollary 5.5. *Assume that there is a representation $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. The action of $\mathfrak{D}(M)$ on $\mathrm{Rep}^{\Gamma_1}_{G,\pi}$ factors through the action of $\mathfrak{D}(I_\Delta M)$, which acts freely.*

Proof. As in the proof of Proposition 5.4, we use the composition $(\mathcal{E} \otimes M)_\Delta \xrightarrow{q} I_\Delta M \to M$ to see that the boundary map $\mathfrak{D}(M)(A) \to Z^1(\Gamma_1, \mathfrak{D}(M)(A))$ factors through $\mathfrak{D}(I_\Delta M)$.

Let A be any $\mathcal{O}$-algebra and let $H(A)$ be the group $\mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, A^\times)$. Since $q : (\mathcal{E} \otimes M)_\Delta \to I_\Delta M$ is surjective, we may identify $\mathfrak{D}(I_\Delta M)(A)$ with a subgroup of $H(A)$. The isomorphism (5.4) identifies $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$ with $H(A)$, and the action of $\mathfrak{D}(I_\Delta M)(A)$ on $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$ is identified with the action of $\mathfrak{D}(I_\Delta M)(A)$ on $H(A)$ by multiplication, which is a free action. Hence, the action of $\mathfrak{D}(I_\Delta M)$ on $\mathrm{Rep}^{\Gamma_1}_{G,\pi}$ is free. □

Corollary 5.6. *Assume that there is a representation $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. The isomorphisms (4.19) and (5.3) induce an isomorphism*

$$\mathrm{Rep}^{\Gamma_1}_{G,\pi}/\mathfrak{D}(M) \xrightarrow{\cong} \mathrm{Spec}(\mathcal{O}[H_1(\Gamma_1, M)]), \tag{5.6}$$

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where $\mathrm{Rep}^{\Gamma_1}_{G,\pi}/\mathfrak{D}(M)$ *denotes the fpqc sheafification of the presheaf quotient. Moreover this quotient is a GIT quotient. In particular,* (5.6) *induces isomorphisms*

$$\mathcal{O}[H_1(\Gamma_1, M)] \xrightarrow{\cong} \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})^{G^0} \xrightarrow{\cong} \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]^{G^0}, \tag{5.7}$$

and the composition of the maps in (5.7) *is induced by* φ_{nat}.

Proof. The isomorphism (5.6) follows directly from Proposition 4.12 and Proposition 5.2. To see that $\mathrm{Rep}^{\Gamma_1}_{G,\pi}/\mathfrak{D}(M)$ is also a GIT quotient, we observe that every $\mathfrak{D}(M)$-equivariant map from $\mathrm{Rep}^{\Gamma_1}_{G,\pi}$ to an affine scheme equipped with the trivial $\mathfrak{D}(M)$-action factors through $\mathrm{Rep}^{\Gamma_1}_{G,\pi}/\mathfrak{D}(M)$ by the universal property of the fpqc sheaf quotient. Since $\mathrm{Rep}^{\Gamma_1}_{G,\pi}/\mathfrak{D}(M)$ is affine, it is indeed the GIT quotient, and this gives the first isomorphism in (5.7). The second isomorphism in (5.7) comes from (5.4).

Let $B \in \mathcal{O}$-alg. After removing the G^0-invariants in (5.7), we get two maps

$$\mathrm{Hom}((\mathcal{E} \otimes M)_\Delta, B^\times) \xrightarrow{\cong} \mathrm{Rep}^{\Gamma_1}_{G,\pi}(B) \to \mathrm{Hom}(H_1(\Gamma_1, M), B^\times).$$

The first map comes from (4.13) and (5.3). The second map comes from (4.14) and (5.3). Hence, the composition is (4.15) with $V = B^\times$, so the last assertion follows from Lemma 4.8. □

Lemma 5.7. *Let* $\tau : G \hookrightarrow \mathbb{A}^n$ *be a closed immersion of* $\mathcal{O}$*-group schemes and let* $\rho \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$. *Assume that* $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ *is nonempty. Then* $\tau(\rho(\Gamma_1))$ *is contained in a finitely generated* $\mathcal{O}[H_1(\Gamma_1, M)]$*-submodule of* $A^n = \mathbb{A}^n(A)$.

Proof. Proposition 5.2 implies that it is enough to prove the statement, when $A = \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$ and $\rho = \Phi_{\mathrm{nat}}\rho_0$, where ρ_0 is any representation in $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ and Φ_{nat} is the cocycle defined in Section 4.

Since Γ_2 is of finite index in Γ_1, it is enough to show that $\tau(\rho(\Gamma_2))$ is contained in a finitely generated $\mathcal{O}[H_1(\Gamma_1, M)]$-submodule of $(\mathcal{O}[(\mathcal{E} \otimes M)_\Delta])^n$. Since Γ_2 acts trivially on M, we have a canonical isomorphism

$$H_1(\Gamma_2, M) \cong \Gamma_2^{\mathrm{ab}} \otimes M,$$

and it follows from (4.16) that the image of $\mathcal{O}[\Gamma_2^{\mathrm{ab}} \otimes M]$ in $\mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$ is contained in $\mathcal{O}[H_1(\Gamma_1, M)]$. We thus may assume that $G = \mathfrak{D}(M)$ and $\Gamma_2 = \Gamma_1$. In this case, $\mathcal{O}[H_1(\Gamma, M)] = \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$, so there is nothing to prove. □

6. Lafforgue's *G*-pseudocharacters

We keep the notation of Section 5. We now recall Lafforgue's notion of *G-pseudocharacter* in the form of [13, Definition 3.1]. The difference to Lafforgue's original definition [9, Section 11] is that we work over the base ring $\mathcal{O}$ and allow G to be disconnected. The definition works for generalised reductive $\mathcal{O}$-group schemes as defined in [12], but this generality is not needed here.

Definition 6.1. Let A be a commutative $\mathcal{O}$-algebra. A *G-pseudocharacter* Θ of Γ over A is a sequence $(\Theta_n)_{n\geq 1}$ of $\mathcal{O}$-algebra maps

$$\Theta_n : \mathcal{O}[G^n]^{G^0} \to \mathrm{Map}(\Gamma^n, A)$$

for $n \geq 1$, satisfying the following conditions[1]:

[1]Here, G acts on G^n by $g \cdot (g_1, \dots, g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1})$. This induces a rational action of G on the affine coordinate ring $\mathcal{O}[G^n]$ of G^n. The submodule $\mathcal{O}[G^n]^{G^0} \subseteq \mathcal{O}[G^n]$ is defined as the rational invariant module of the G^0-representation $\mathcal{O}[G^n]$. It is an $\mathcal{O}$-subalgebra, since G acts by $\mathcal{O}$-linear automorphisms.

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(1) For each $n, m \geq 1$, each map $\zeta : \{1, \ldots, m\} \to \{1, \ldots, n\}$, $f \in \mathcal{O}[G^m]^{G^0}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \ldots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \ldots, \gamma_{\zeta(m)})$$

where $f^\zeta(g_1, \ldots, g_n) = f(g_{\zeta(1)}, \ldots, g_{\zeta(m)})$.

(2) For each $n \geq 1$, for each $\gamma_1, \ldots, \gamma_{n+1} \in \Gamma$ and each $f \in \mathcal{O}[G^n]^{G^0}$, we have

$$\Theta_{n+1}(\hat{f})(\gamma_1, \ldots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \ldots, \gamma_n\gamma_{n+1})$$

where $\hat{f}(g_1, \ldots, g_{n+1}) = f(g_1, \ldots, g_n g_{n+1})$.

We denote the set of G-pseudocharacters of Γ_1 over A by $\mathrm{PC}_G^{\Gamma_1}(A)$. The functor $A \mapsto \mathrm{PC}_G^{\Gamma_1}(A)$ is representable by an affine $\mathcal{O}$-scheme $\mathrm{PC}_G^{\Gamma_1}$ [13, Theorem 3.20]. When $\varphi : G \to H$ is a homomorphism of generalised tori over $\mathcal{O}$, the induced maps $\varphi_n^* : \mathcal{O}[H^n]^{H^0} \to \mathcal{O}[G^n]^{G^0}$ give rise to an H-pseudocharacter $(\Theta_n \circ \varphi_n^*)_{n \geq 1}$. By analogy, with the notation for representations, we denote this H-pseudocharacter by $\varphi \circ \Theta$. Thus, we also have an induced map $\mathrm{PC}_G^{\Gamma}(A) \to \mathrm{PC}_H^{\Gamma}(A)$. It is easy to verify that specialisation along $f : A \to B$ commutes with composition with φ, i.e. $(\varphi \circ \Theta) \otimes_A B = \varphi \circ (\Theta \otimes_A B)$.

Recall that for every homomorphism $\rho : \Gamma_1 \to G(A)$, there is an associated G-pseudocharacter $\Theta_\rho \in \mathrm{PC}_G^{\Gamma_1}(A)$, which depends on ρ only up to $G^0(A)$-conjugation. For $m \geq 1$, the homomorphism $(\Theta_\rho)_m : \mathcal{O}(G^m)^{G^0} \to \mathrm{Map}(\Gamma_1^m, A)$ is defined by

$$(\Theta_\rho)_m(f)(\gamma_1, \ldots, \gamma_m) \coloneqq f(\rho(\gamma_1), \ldots, \rho(\gamma_m)). \tag{6.1}$$

There is a natural $G^0(A)$-equivariant map

$$\mathrm{Rep}_G^{\Gamma_1}(A) \to \mathrm{PC}_G^{\Gamma_1}(A), \quad \rho \mapsto \Theta_\rho, \tag{6.2}$$

which induces maps of $\mathcal{O}$-schemes

$$\mathrm{Rep}_G^{\Gamma_1} \to \mathrm{PC}_G^{\Gamma_1}. \tag{6.3}$$

Since G^0 acts trivially on the target, (6.3) factors as

$$\mathrm{Rep}_G^{\Gamma_1} /\!\!/ G^0 \to \mathrm{PC}_G^{\Gamma_1}. \tag{6.4}$$

If $\rho \in \mathrm{Rep}_{G,\pi}^{\Gamma_1}(A)$, then Θ_ρ is *admissible* in the sense that the functorial image of Θ_ρ under the map $G \to \underline{\Delta}$ maps to the $\underline{\Delta}$-pseudocharacter attached to π in $\mathrm{PC}_{\underline{\Delta}}^{\Gamma_1}(A)$. We denote the set of admissible G-pseudocharacters by $\mathrm{PC}_{G,\pi}^{\Gamma_1}(A)$. The functor $A \mapsto \mathrm{PC}_{G,\pi}^{\Gamma_1}(A)$ is representable by an affine $\mathcal{O}$-scheme $\mathrm{PC}_{G,\pi}^{\Gamma_1}$. We have natural maps

$$\mathrm{Rep}_{G,\pi}^{\Gamma_1} \to \mathrm{PC}_{G,\pi}^{\Gamma_1} \tag{6.5}$$

$$\mathrm{Rep}_{G,\pi}^{\Gamma_1} /\!\!/ G^0 \to \mathrm{PC}_{G,\pi}^{\Gamma_1} \tag{6.6}$$

as above.

We will show, that (6.4) and (6.6) are isomorphisms. The result is specific to generalised tori[2] and might be of independent interest.

[2] We do not expect (6.4) and (6.6) to be isomorphisms when G is an arbitrary generalised reductive group in the sense of [13, Definition 2.3]. However, we do not know of an example where (6.4) is not an isomorphism.

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Proposition 6.2. *The map*

$$\mathcal{O}(\mathrm{PC}_G^{\Gamma_1}) \to \mathcal{O}(\mathrm{Rep}_G^{\Gamma_1})^{G^0} \tag{6.7}$$

corresponding to (6.4) is an isomorphism.

Proof. As G^0 is linearly reductive the proof of [8, Proposition 2.11 (i)] applies, we recall the argument here. Let $\mathcal{O}(G^{\Gamma_1})$ be the $\mathcal{O}$-algebra which represents the functor

$$\mathcal{O}\text{-alg} \to \text{Set}, \ A \mapsto \mathrm{Map}(\Gamma_1, G(A)).$$

We have a natural surjection $\mathcal{O}(G^{\Gamma_1}) \twoheadrightarrow \mathcal{O}(\mathrm{Rep}_G^{\Gamma_1})$ with kernel J. We also have a natural surjection $\mathcal{O}(G^{\Gamma_1})^{G^0} \twoheadrightarrow \mathcal{O}(\mathrm{PC}_G^{\Gamma_1})$ with kernel I. The structure of J and I is described explicitly in [8, Proposition 2.5]; note that there G is assumed to be connected, but the description generalises easily to our situation. Namely, J is generated by the image of the G^0-equivariant maps

$$\varphi_{\gamma,\delta} : \mathcal{O}(G \times G^{\Gamma_1}) \to \mathcal{O}(G^{\Gamma_1})$$

for all $\gamma, \delta \in \Gamma_1$, where

$$\varphi_{\gamma,\delta}(f)((g_\alpha)_{\alpha\in\Gamma_1}) = f(g_{\gamma\delta}, (g_\alpha)_{\alpha\in\Gamma_1}) - f(g_\gamma g_\delta, (g_\alpha)_{\alpha\in\Gamma_1}).$$

The ideal I is generated by the image of $\varphi_{\gamma,\delta}(\mathcal{O}(G \times G^{\Gamma_1})^{G^0})$ for all $\gamma, \delta \in \Gamma_1$. Since taking G^0-invariants is exact, the natural map $\mathcal{O}(G^{\Gamma_1})^{G^0}/I = \mathcal{O}(\mathrm{PC}_G^{\Gamma_1}) \to \mathcal{O}(\mathrm{Rep}_G^{\Gamma_1})^{G^0} = \mathcal{O}(G^{\Gamma_1})^{G^0}/J^{G^0}$ is surjective. So it remains to show that $J^{G^0} \subseteq I$. Let $h \in J^G$ and write $h = \sum_{i=1}^n \varphi_{\gamma_i,\delta_i}(f_i)$, where $\gamma_i, \delta_i \in \Gamma_1$ and $f_i \in \mathcal{O}(G \times G^{\Gamma_1})$. Denote the Reynolds operator on G^0-modules by E. It commutes with the G^0-equivariant maps $\varphi_{\gamma_i,\delta_i}$, so that we have

$$h = E(h) = \sum_{i=1}^n \varphi_{\gamma_i,\delta_i}(E(f_i)) \in I.$$ □

Lemma 6.3. *If the conjugation action of G^0 on all $\mathcal{O}(G^m)$ is trivial, then the following hold:*

(1) (6.3) *is an isomorphism;*
(2) *If Γ_1 is a topological group and A is a topological $\mathcal{O}$-algebra, then* (6.2) *induces a bijection between continuous representations and continuous G-pseudocharacters;*

In particular, if $G = G^0$ or G^0 is trivial, then (6.3) *is an isomorphism, and hence,* $\mathrm{Rep}_{\underline{\Delta}}^{\Gamma_1} \cong \mathrm{PC}_{\underline{\Delta}}^{\Gamma_1}$.

Proof. Choose $\mathcal{O}$-algebra generators $f_1, \ldots, f_r \in \mathcal{O}(G)$ and let $\Theta \in \mathrm{PC}_G^{\Gamma_1}(A)$. The functions $\Theta_1(f_1), \ldots, \Theta_1(f_r)$ define a unique map $\rho : \Gamma_1 \to G(A)$ such that $\Theta_1(f_i) = f_i \circ \rho$ for all $i = 1, \ldots, r$. Let $\mu : \mathcal{O}(G) \to \mathcal{O}(G) \otimes_{\mathcal{O}} \mathcal{O}(G)$ be the comultiplication map. We can write

$$\mu(f_i) = \sum_{j,k} a_{ijk} f_j \otimes f_k$$

for some $a_{ijk} \in \mathcal{O}$. By (2) of Definition 6.1, we have

$$\begin{aligned} f_i(\rho(\gamma_1\gamma_2)) &= \Theta_1(f_i)(\gamma_1\gamma_2) = \Theta_2(\mu(f_i))(\gamma_1, \gamma_2) \\ &= \sum_{j,k} a_{ijk}\Theta_1(f_j)(\gamma_1)\Theta_1(f_k)(\gamma_2) \\ &= \sum_{j,k} a_{ijk} f_j(\rho(\gamma_1)) f_k(\rho(\gamma_2)) \\ &= \mu(f_i)(\rho(\gamma_1), \rho(\gamma_2)) = f_i(\rho(\gamma_1)\rho(\gamma_2)), \end{aligned}$$

so ρ is a homomorphism. For every $m \geq 1$, every function $f \in \mathcal{O}(G^m)$ can be written as a linear combination of functions of the form $(g_1, \ldots, g_m) \mapsto f_i(g_j)$. So by rule (1) of Definition 6.1, the function $\Theta_m(f)$ is determined by $\Theta_1(f_1), \ldots, \Theta_1(f_r)$. It follows that Θ is the only G-pseudocharacter satisfying $\Theta_1(f_i) = f_i \circ \rho$ for all i. By definition, $\Theta = \Theta_\rho$, so this shows surjectivity and injectivity of (6.3).

For the claim about continuity, we observe that continuity of $\Theta_1(f_1), \ldots, \Theta_1(f_r)$ is equivalent to continuity of ρ. □

Corollary 6.4. *The map*

$$\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi}) \to \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})^{G^0} \tag{6.8}$$

corresponding to (6.6) is an isomorphism.

Proof. Via the canonical projection $\Pi : G \to \underline{\Delta}$, we have a natural G^0-action on $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}})$, which is trivial, since $\underline{\Delta}^0 = 1$. So the natural G^0-equivariant map $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}}) \to \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G})$ induces a map $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}}) \to \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G})^{G^0}$. The representation $\pi : \Gamma_1 \to \Delta$ corresponds to a map $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}}) \to \mathcal{O}$. The tensor product $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G}) \otimes_{\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}})} \mathcal{O}$ represents $\mathrm{Rep}^{\Gamma_1}_{G,\pi}$. By virtue of Remark 2.3, we have a natural isomorphism

$$\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G})^{G^0} \otimes_{\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}})} \mathcal{O} \xrightarrow{\cong} \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})^{G^0}.$$

Moreover, $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})$ is representable by

$$\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G}) \otimes_{\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{\underline{\Delta}})} \mathcal{O} \cong \mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G}) \otimes_{\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}})} \mathcal{O}$$

by Lemma 6.3. We conclude that (6.8) is obtained from (6.7) by applying $- \otimes_{\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{\underline{\Delta}})} \mathcal{O}$. So by Proposition 6.2, (6.8) is an isomorphism as well. □

Corollary 6.5. *Assume there is a representation $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. Then we have isomorphisms $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi}) \cong \mathcal{O}[H_1(\Gamma_1, M)] \cong \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]^{G^0}$ via (5.7) and (6.8).*

Lemma 6.6. *Assume that $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ is nonempty. Then the map $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi}) \to \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})$ corresponding to (6.5) has a section as $\mathcal{O}$-algebras. In particular, for all $A \in \mathcal{O}$-alg and all $\Theta \in \mathrm{PC}^{\Gamma_1}_{G,\pi}(A)$, there is a representation $\rho \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$, such that $\Theta = \Theta_\rho$.*

Proof. Let $\rho_0 \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$. In the diagram

$$\begin{array}{ccccccc}
\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi}) & \xrightarrow{(6.8)} & \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})^{G^0} & \xrightarrow{\sim} & \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]^{G^0} & \xleftarrow{(5.7)} & \mathcal{O}[H_1(\Gamma_1, M)] \\
& \nwarrow_{\sigma} & \downarrow & & \downarrow & & \downarrow \uparrow \\
& & \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi}) & \xrightarrow{(5.4)} & \mathcal{O}[(\mathcal{E} \otimes M)_\Delta] & \xleftarrow{(4.18)} & \mathcal{O}[H_1(\Gamma_1, M)][t_1^{\pm 1}, \ldots, t_s^{\pm 1}]
\end{array}$$

the left solid square commutes since (5.4) is a G^0-equivariant isomorphism. The right solid square commutes since (5.7) is by Corollary 6.4 induced by φ_{nat}. The section of the right vertical arrow is defined by mapping $t_1, \ldots, t_s$ to 1 and, since all horizontal maps are isomorphisms, defines a section σ of the left vertical arrow. Given a G-pseudocharacter $\Theta \in \mathrm{PC}^{\Gamma_1}_{G,\pi}(A)$, it corresponds to a homomorphism $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi}) \to A$, and by restriction along σ, we get a representation $\rho \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(A)$, which by restriction along (6.8) recovers Θ, so $\Theta = \Theta_\rho$. □

7. Profinite completion

So far, we have worked with representations and pseudocharacters of an abstract group Γ_1, and topology did not play a role. In this section, we transfer some of the results proved in Sections 5 and 6 to the profinite completion $\widehat{\Gamma}_1$ of Γ_1 and impose continuity conditions on the representations and pseudocharacters of $\widehat{\Gamma}_1$.

Let $\mathcal{O}$ be the ring of integers in a finite extension of L with uniformiser ϖ and residue field k.

7.1. Deformations of representations

Let $\overline{\rho} : \widehat{\Gamma}_1 \to G(k)$ be a continuous representation, such that $\overline{\rho}|_{\Gamma_1} \in \operatorname{Rep}^{\Gamma_1}_{G,\pi}(k)$.

Let $D^{\square}_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to$ Set be the functor such that $D^{\square}_{\overline{\rho}}(A)$ is the set of continuous representations $\rho_A : \widehat{\Gamma}_1 \to G(A)$ such that $\rho_A \equiv \overline{\rho} \pmod{\mathfrak{m}_A}$. The functor $D^{\square}_{\overline{\rho}}$ is pro-representable by $R^{\square}_{\overline{\rho}} \in \widehat{\mathfrak{A}}_{\mathcal{O}}$.

Lemma 7.1. *There is a natural isomorphism in* $\widehat{\mathfrak{A}}_{\mathcal{O}}$*:*

$$R^{\square}_{\overline{\rho}} \cong \varprojlim_{I} \mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G)_{\mathfrak{m}}/I, \tag{7.1}$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G)$ corresponding to $\overline{\rho}$, and the limit is taken over all the ideals I, such that the quotient is finite.

Proof. If $A \in \mathfrak{A}_{\mathcal{O}}$ and $\rho \in D^{\square}_{\overline{\rho}}(A)$, then $\rho|_{\Gamma_1} \in \operatorname{Rep}^{\Gamma_1}_{G,\pi}(A)$, and hence, we obtain a natural homomorphism of $\mathcal{O}$-algebras $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G) \to A$. The condition $\rho \equiv \overline{\rho} \pmod{\mathfrak{m}_A}$ implies that this map extends to the localisation $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G)_{\mathfrak{m}} \to A$. Since A is finite, the kernel of this map is one of the ideals I appearing in the inductive system in (7.1).

Conversely, if I is an ideal of $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G)_{\mathfrak{m}}$ such that the quotient A is finite, then $A \in \mathfrak{A}_{\mathcal{O}}$. The map $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G) \to k$ corresponding to $\overline{\rho}$ factors through $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_{G,\pi}) \to k$ as $\overline{\rho}|_{\Gamma_1} \in \operatorname{Rep}^{\Gamma_1}_{G,\pi}(k)$ by assumption. Since G/G^0 is constant and A is a local ring, we have $(G/G^0)(A) = (G/G^0)(k)$, and hence, the map $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_G) \to A$ factors through $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_{G,\pi}) \to A$. By specialising the universal representation $\Gamma_1 \to G(\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_{G,\pi}))$ along this map, we obtain a representation $\rho : \Gamma_1 \to G(A)$. Since the target is finite, ρ extends uniquely to a continuous representation $\tilde{\rho} : \widehat{\Gamma}_1 \to G(A)$, which defines a point in $D^{\square}_{\overline{\rho}}(A)$.

Hence, the right-hand side of (7.1) pro-represents $D^{\square}_{\overline{\rho}}$ and the assertion follows. □

Lemma 7.2. *Let $\mathcal{A}$ be an abelian group and let $\mathfrak{m}$ be a maximal ideal of the group ring $\mathcal{O}[\mathcal{A}]$ with residue field k. Then there is an isomorphism in* $\widehat{\mathfrak{A}}_{\mathcal{O}}$

$$\mathcal{O}[\![N]\!] \cong \varprojlim_{I} \mathcal{O}[\mathcal{A}]_{\mathfrak{m}}/I, \tag{7.2}$$

where N is the pro-p completion of $\mathcal{A}$, and the limit is taken over all the ideals I, such that the quotient is finite.

Proof. Let $\overline{\psi} : \mathcal{A} \to k^{\times}$ be the character obtained by the composition $\mathcal{A} \to \mathcal{O}[\mathcal{A}]/\mathfrak{m} = k$ and let $\psi : \mathcal{A} \to \mathcal{O}^{\times}$ be any character lifting $\overline{\psi}$ (for example, the Teichmüller lift). Then $\mathfrak{m}$ is generated by ϖ and $(\psi(a)^{-1}a - 1)$ for all $a \in \mathcal{A}$. The map $\varphi : \mathcal{O}[\mathcal{A}] \to \mathcal{O}[\mathcal{A}]$, which sends $a \in \mathcal{A}$ to $\psi(a)a$, is an isomorphism of $\mathcal{O}$-algebras, which sends $\mathfrak{m}$ to the maximal ideal corresponding to the trivial character. We thus may assume that $\mathfrak{m}$ is generated by ϖ and the augmentation ideal of $\mathcal{O}[\mathcal{A}]$.

If P is a finite p-power order quotient of $\mathcal{A}$, then $\mathcal{O}/\varpi^n[P]$ is a finite local ring, and the surjection $\mathcal{O}[\mathcal{A}] \twoheadrightarrow \mathcal{O}/\varpi^n[P]$ maps $\mathfrak{m}$ to the maximal ideal of $\mathcal{O}/\varpi^n[P]$. Hence, the map factors through $\mathcal{O}[\mathcal{A}]_{\mathfrak{m}} \twoheadrightarrow \mathcal{O}/\varpi^n[P]$. Conversely, if A is a quotient of $\mathcal{O}[\mathcal{A}]_{\mathfrak{m}}$, which is finite (as a set), then the image of $\mathcal{A}$ is contained in $1 + \mathfrak{m}_A$ and hence is a finite group of p-power order. This implies the assertion. □

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Lemma 7.3. *Let $\mathcal{A}$ be an abelian group, let $\mathfrak{m}$ be a maximal ideal of the group ring $\mathcal{O}[\mathcal{A}]$ with residue field k and let N be the pro-p completion of $\mathcal{A}$. Then the following are equivalent:*

(1) *N is a finitely generated $\mathbb{Z}_p$-module;*
(2) *$\mathcal{A}/p\mathcal{A}$ is a finite p-group;*
(3) *$\mathcal{O}[\![N]\!]$ is noetherian.*

If the equivalent conditions hold, then $\mathcal{O}[\![N]\!]$ is naturally isomorphic to the $\mathfrak{m}$-adic completion of $\mathcal{O}[\mathcal{A}]$.

Proof. The equivalence of (1) and (2) follows from Nakayama's lemma and the fact that N/pN is the pro-p completion of $\mathcal{A}/p\mathcal{A} \cong \oplus_I \mathbb{F}_p$ for some set I.

If N is finitely generated as a $\mathbb{Z}_p$-module, then $N \cong \mu \oplus \mathbb{Z}_p^r$, where μ is a finite p-group and $r = \dim_{\mathbb{Q}_p} N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Hence, $\mathcal{O}[\![N]\!] \cong \mathcal{O}[\mu][\![x_1, \dots, x_r]\!]$ and hence, (1) implies (3).

If N is not finitely generated as a $\mathbb{Z}_p$-module, then we may find a strictly increasing nested family of finitely generated $\mathbb{Z}_p$-submodules $N_i \subset N$. Since they are finitely generated, they are closed in N. The kernels of $\mathcal{O}[\![N]\!] \twoheadrightarrow \mathcal{O}[\![N/N_i]\!]$ form a strictly increasing nested family of ideals in $\mathcal{O}[\![N]\!]$. Hence, (3) implies (1).

As in the proof of Lemma 7.2, we may assume that $\mathfrak{m}$ is generated by ϖ and the augmentation ideal of $\mathcal{O}[\mathcal{A}]$. The map $\mathcal{A} \to \mathfrak{m}$, $a \mapsto a - 1$ induces an isomorphism of k-vector spaces $k \otimes_{\mathbb{Z}} \mathcal{A} \cong \mathfrak{m}/(\varpi, \mathfrak{m}^2)$. If (2) holds, then $\dim_k \mathfrak{m}/(\varpi, \mathfrak{m}^2)$ is finite. Hence, the powers of $\mathfrak{m}$ form a cofinal system in the inverse system in (7.2), and the last assertion follows from Lemma 7.2. □

Lemma 7.4. *If $\Gamma_2^{\mathrm{ab}}/p\Gamma_2^{\mathrm{ab}}$ is a finite p-group and $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ is nonempty, then $R^{\square}_{\overline{\rho}}$ is naturally isomorphic to the $\mathfrak{m}$-adic completion of $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})$, where $\mathfrak{m}$ is the maximal ideal corresponding to $\overline{\rho}|_{\Gamma_1}$.*

Proof. Proposition 5.2 allows us to identify $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})$ with $\mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$. It follows from (4.10) that if $\Gamma_2^{\mathrm{ab}}/p\Gamma_2^{\mathrm{ab}}$ is finite, then $\mathcal{E}/p\mathcal{E}$ is also finite. Since M is a finite free $\mathbb{Z}$-module, the assertion follows from Lemma 7.1 and Lemma 7.3 applied to $\mathcal{A} = (\mathcal{E} \otimes M)_\Delta$. □

Lemma 7.5. *There is a natural isomorphism of local k-algebras between $R^{\square}_{\overline{\rho}}/\varpi$ and the completed group algebra $k[\![((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}]\!]$.*

Proof. The ring $R^{\square}_{\overline{\rho}}/\varpi$ represents the restriction of $D^{\square}_{\overline{\rho}}$ to $\mathfrak{A}_k$. Lemma 7.1 implies that

$$R^{\square}_{\overline{\rho}}/\varpi \cong \varprojlim_I \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_G)_{\mathfrak{m}}/(\varpi, I). \tag{7.3}$$

Since $\overline{\rho}|_{\Gamma_1} \in \mathrm{Rep}^{\Gamma_1}_{G,\pi}(k)$, we may apply Proposition 5.2 with the base ring $\mathcal{O}$ equal to k to obtain

$$\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi})/\varpi \cong k[(\mathcal{E} \otimes M)_\Delta]. \tag{7.4}$$

The assertion then follows from (7.3), (7.4) and Lemma 7.2. □

Let $\Gamma \twoheadrightarrow \widehat{\Gamma}_1$ be a surjection of profinite groups, such that for all $A \in \mathfrak{A}_{\mathcal{O}}$, every continuous representation $\rho_A : \Gamma \to G(A)$ factors through $\widehat{\Gamma}_1$.

Let $\mathrm{ad}\,\overline{\rho}$ be the representation of Γ on $\mathrm{Lie}\, G_k$ obtained by composing $\overline{\rho}$ with the adjoint representation. We write $H^i_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho})$ for the i-th continuous group cohomology and denote by $h^i_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho})$ its dimension as a k-vector space.

Proposition 7.6. *If $h^1_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho})$ is finite, then $((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}$ is a finitely generated $\mathbb{Z}_p$-module. Moreover, if $h^2_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho})$ is also finite and*

$$h^1_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho}) - h^0_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho}) - h^2_{\mathrm{cont}}(\Gamma, \mathrm{ad}\,\overline{\rho}) \geq \mathrm{rank}_{\mathbb{Z}_p}((\mathcal{E} \otimes M)_\Delta)^{\wedge,p} - \dim G_k, \tag{7.5}$$

then $R^{\square}_{\overline{\rho}}$ is complete intersection, $\mathcal{O}$-flat of relative dimension $\mathrm{rank}_{\mathbb{Z}_p}((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}$.

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Proof. The assumption on Γ implies that we could have equivalently defined the deformation problem $D^{\square}_{\overline{\rho}}$ with Γ instead of $\widehat{\Gamma}_1$.

By standard obstruction theory due to Mazur, we have a presentation

$$\frac{\mathcal{O}[\![x_1,\dots,x_r]\!]}{(f_1,\dots,f_s)} \cong R^{\square}_{\overline{\rho}}, \tag{7.6}$$

where $r=\dim_k Z^1_{\mathrm{cont}}(\Gamma,\operatorname{ad}\overline{\rho})$ and $s=h^2_{\mathrm{cont}}(\Gamma,\operatorname{ad}\overline{\rho})$. The exact sequence

$$0\to(\operatorname{ad}\overline{\rho})^{\Gamma}\to\operatorname{ad}\overline{\rho}\to Z^1_{\mathrm{cont}}(\Gamma,\operatorname{ad}\overline{\rho})\to H^1_{\mathrm{cont}}(\Gamma,\operatorname{ad}\overline{\rho})\to 0$$

implies that $r=h^1_{\mathrm{cont}}(\Gamma,\operatorname{ad}\overline{\rho})-h^0_{\mathrm{cont}}(\Gamma,\operatorname{ad}\overline{\rho})+\dim G_k$. Let $N:=((\mathcal{E}\otimes M)_\Delta)^{\wedge,p}$. By considering (7.6) modulo ϖ and using Lemma 7.5, we deduce that $\operatorname{rank}_{\mathbb{Z}_p}N\geq r-s$. Moreover, the assumption (7.5) implies that $r-s\geq\operatorname{rank}_{\mathbb{Z}_p}N$. Hence, $r-s=\operatorname{rank}_{\mathbb{Z}_p}N$, and $\varpi,f_1,\dots,f_s$ can be extended to a system of parameters of a regular ring $\mathcal{O}[\![x_1,\dots,x_r]\!]$. Thus, $\varpi,f_1,\dots,f_s$ are a part of a regular sequence, and hence, $R^{\square}_{\overline{\rho}}$ is complete intersection, flat over $\mathcal{O}$ of relative dimension of $\operatorname{rank}_{\mathbb{Z}_p}N$. □

Corollary 7.7. *If the assumptions of Proposition 7.6 hold, then there is an isomorphism of local $\mathcal{O}'$-algebras:*

$$\mathcal{O}'\otimes_{\mathcal{O}}R^{\square}_{\overline{\rho}}\cong\mathcal{O}'[\![((\mathcal{E}\otimes M)_\Delta)^{\wedge,p}]\!],$$

where $\mathcal{O}'$ is the ring of integers in a finite extension of L. In particular, $\operatorname{Rep}^{\widehat{\Gamma}_1}_{G,\pi}(\mathcal{O}')$ *and* $\operatorname{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O}')$ *are nonempty.*

Proof. It follows from Proposition 7.6 that there is a homomorphism of $\mathcal{O}$-algebras $x:R^{\square}_{\overline{\rho}}\to\overline{\mathbb{Q}}_p$. Then $\kappa(x)$ is a finite extension of L, and the image of x is contained in the ring of integers of $\kappa(x)$, which we denote by $\mathcal{O}'$. Let $\rho_x:\widehat{\Gamma}_1\to G(\mathcal{O}')$ be the specialisation of the universal deformation along $x:R^{\square}_{\overline{\rho}}\to\mathcal{O}'$. Then $\rho_x\in\operatorname{Rep}^{\widehat{\Gamma}_1}_{G,\pi}(\mathcal{O}')$ and its restriction to Γ_1 defines a point in $\operatorname{Rep}^{\Gamma_1}_{G}(\mathcal{O}')$.

Since $\mathcal{O}'\otimes_{\mathcal{O}}R^{\square}_{\overline{\rho}}$ represents the functor $D^{\square}_{\overline{\rho}_{k'}}:\mathfrak{A}_{\mathcal{O}'}\to\mathrm{Set}$, where $\overline{\rho}_{k'}$ is the composition of $\overline{\rho}$ with $G(k)\hookrightarrow G(k')$, where k' is the residue field of $\mathcal{O}'$, we may assume that $\mathcal{O}'=\mathcal{O}$.

Proposition 5.2 allows us to identify $\mathcal{O}(\operatorname{Rep}^{\Gamma_1}_{G,\pi})$ with $\mathcal{O}[(\mathcal{E}\otimes M)_\Delta]$. This identification depends on the lift ρ_x. The assertion follows from Lemmas 7.1 and 7.2. □

7.2. Deformations of pseudocharacters

If A is a topological ring, then a G-pseudocharacter $\Theta\in\mathrm{PC}^{\widehat{\Gamma}_1}_{G,\pi}(A)$ is *continuous*, if every Θ_m takes values in the subset $\mathcal{C}(\widehat{\Gamma}_1^m,A)\subseteq\operatorname{Map}(\widehat{\Gamma}_1^m,A)$ of continuous maps. We write $\mathrm{cPC}^{\widehat{\Gamma}_1}_{G,\pi}(A)\subseteq\mathrm{PC}^{\widehat{\Gamma}_1}_{G,\pi}(A)$ for the subset of continuous G-pseudocharacters.

Lemma 7.8. *Assume that* $\operatorname{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ *is nonempty. Let A be a finite discrete $\mathcal{O}$-algebra. Then the natural map* $\mathrm{cPC}^{\widehat{\Gamma}_1}_{G,\pi}(A)\to\mathrm{PC}^{\Gamma_1}_{G,\pi}(A),\Theta\mapsto\Theta|_{\Gamma_1}$ *is bijective.*

Proof. As the image of Γ_1 in $\widehat{\Gamma}_1$ is dense, injectivity follows from [13, Lemma 3.2]. Let $\Theta\in\mathrm{PC}^{\Gamma_1}_{G,\pi}(A)$. By Lemma 6.6, there is a representation $\rho\in\operatorname{Rep}^{\Gamma_1}_{G,\pi}(A)$, such that $\Theta=\Theta_\rho$. Since A is finite, ρ extends to a continuous representation $\tilde{\rho}:\widehat{\Gamma}_1\to G(A)$, and we have $\Theta_{\tilde{\rho}}|_{\Gamma_1}=\Theta$. □

Let $\overline{\Theta}$ be the G-pseudocharacter of $\widehat{\Gamma}_1$ associated to $\overline{\rho}$. Let $D^{\mathrm{ps}}_{\overline{\Theta}}:\mathfrak{A}_{\mathcal{O}}\to\mathrm{Set}$ be the deformation functor which sends A to the set of continuous G-pseudocharacters Θ of $\widehat{\Gamma}_1$ valued in A such that $\Theta\otimes_A k=\overline{\Theta}$. The functor $D^{\mathrm{ps}}_{\overline{\Theta}}$ is pro-represented by $R^{\mathrm{ps}}_{\overline{\Theta}}\in\widehat{\mathfrak{A}}_{\mathcal{O}}$ by [13, Theorem 5.4].

Proposition 7.9. *Assume that* $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ *is nonempty. Then there is an isomorphism in* $\widehat{\mathfrak{A}}_{\mathcal{O}}$*:*

$$R^{\mathrm{ps}}_{\overline{\Theta}} \cong \varprojlim_{I} \mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})_{\mathfrak{m}}/I, \tag{7.7}$$

where $\mathfrak{m}$ *is the maximal ideal of* $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})$ *corresponding to* $\overline{\Theta}|_{\Gamma_1}$*, and the limit is taken over all ideals* I *such that the quotient is finite.*

Proof. If $A \in \mathfrak{A}_{\mathcal{O}}$, then we have natural bijections

$$\mathrm{Hom}_{\widehat{\mathfrak{A}}_{\mathcal{O}}}(R^{\mathrm{ps}}_{\overline{\Theta}}, A) \cong \{\Theta \in \mathrm{cPC}^{\widehat{\Gamma}_1}_{G,\pi}(A) \mid \Theta \otimes_A k = \overline{\Theta}\},$$
$$\mathrm{Hom}_{\mathrm{local}\ \mathcal{O}\text{-alg}}(\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})_{\mathfrak{m}}, A) \cong \{\Theta \in \mathrm{PC}^{\Gamma_1}_{G,\pi}(A) \mid \Theta \otimes_A k = \overline{\Theta}\}.$$

As $\mathrm{cPC}^{\widehat{\Gamma}_1}_{G,\pi}(A) \cong \mathrm{PC}^{\Gamma_1}_{G,\pi}(A)$ by Lemma 7.8, the claim follows. □

Lemma 7.10. *Assume that* $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ *is nonempty. Then* $R^{\mathrm{ps}}_{\overline{\Theta}}$ *is isomorphic to the completed group algebra* $\mathcal{O}[\![H_1(\Gamma_1, M)^{\wedge,p}]\!]$.

Proof. Since $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ is nonempty, Corollary 6.5 allows us to identify $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})$ with $\mathcal{O}[H_1(\Gamma, M)]$. The assertion follows from Proposition 7.9 and Lemma 7.2. □

Lemma 7.11. *If* $\Gamma_2^{\mathrm{ab}}/p\Gamma_2^{\mathrm{ab}}$ *is a finite p-group and* $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ *is nonempty, then* $R^{\mathrm{ps}}_{\overline{\Theta}}$ *is noetherian and is naturally isomorphic to the* $\mathfrak{m}$*-adic completion of* $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})$*, where* $\mathfrak{m}$ *is the maximal ideal of* $\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})$ *corresponding to* $\overline{\Theta}|_{\Gamma_1}$.

Proof. The exact sequence of Γ_1-modules $0 \to I_\Delta \otimes M \to \mathbb{Z}[\Delta] \otimes M \to M \to 0$ induces an exact sequence in homology:

$$H_1(\Gamma_2, M) \to H_1(\Gamma_1, M) \to (I_\Delta \otimes M)_\Delta.$$

Since the action of Γ_2 on M is trivial, we have a canonical isomorphism $H_1(\Gamma_2, M) \cong \Gamma_2^{\mathrm{ab}} \otimes M$. Since I_Δ and M are free $\mathbb{Z}$-modules of finite rank, we conclude that the assumption $\Gamma_2^{\mathrm{ab}}/p\Gamma_2^{\mathrm{ab}}$ is finite implies that $\mathcal{A}/p\mathcal{A}$ is finite, where $\mathcal{A} = H_1(\Gamma_1, M)$. The assertion follows from Lemmas 7.10 and 7.3. □

7.3. Moduli space of representations

We study the following schemes.

Definition 7.12. For $\mathcal{G} = \Gamma_1$ or $\widehat{\Gamma}_1$, let $X^{\mathrm{gen},\mathcal{G}}_{G,\overline{\Theta}} : R^{\mathrm{ps}}_{\overline{\Theta}}\text{-alg} \to \mathrm{Set}$ be the functor

$$X^{\mathrm{gen},\mathcal{G}}_{G,\overline{\Theta}}(A) := \{\rho \in \mathrm{Rep}^{\mathcal{G}}_{G,\pi}(A) : \Theta_\rho = \Theta^u \otimes_{R^{\mathrm{ps}}_{\overline{\Theta}}} A\},$$

where $\Theta^u \in D^{\mathrm{ps}}_{\overline{\Theta}}(R^{\mathrm{ps}}_{\overline{\Theta}})$ is the universal deformation of $\overline{\Theta}$, and we consider its restriction to Γ_1 if $\mathcal{G} = \Gamma_1$.

Proposition 7.13. *Assume that* $\mathrm{Rep}^{\widehat{\Gamma}_1}_{G,\pi}(\mathcal{O})$ *is nonempty. The restriction to* Γ_1 *induces an isomorphism*

$$X^{\mathrm{gen},\widehat{\Gamma}_1}_{G,\overline{\Theta}} \xrightarrow{\cong} X^{\mathrm{gen},\Gamma_1}_{G,\overline{\Theta}}.$$

In particular, $X^{\mathrm{gen},\widehat{\Gamma}_1}_{G,\overline{\Theta}}$ *is representable by the* $R^{\mathrm{ps}}_{\overline{\Theta}}$*-algebra isomorphic to*

$$R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}(\mathrm{PC}^{\Gamma_1}_{G,\pi})} \mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi}) \cong R^{\mathrm{ps}}_{\overline{\Theta}}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}],$$

where $s = \mathrm{rank}_{\mathbb{Z}} M - \mathrm{rank}_{\mathbb{Z}} M_\Delta$.

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Proof. We let $\mathcal{G}$ be either Γ_1 or its profinite completion $\widehat{\Gamma}_1$ in the proof. The functor $X^{\mathrm{gen},\mathcal{G}}_{G,\overline{\Theta}}$ is representable by $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}(\mathrm{PC}^{\mathcal{G}}_{G,\pi})} \mathcal{O}(\mathrm{Rep}^{\mathcal{G}}_{G,\pi})$. Let $\rho_0 \in \mathrm{Rep}^{\widehat{\Gamma}_1}_{G,\pi}(\mathcal{O})$. Then $\rho_0|_{\Gamma_1}$ is in $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$, and using these representations, we may identify $\mathrm{Rep}^{\mathcal{G}}_{G,\pi}$ with the space of 1-cocycles $Z^1(\mathcal{G}, \mathfrak{D}(M)(-))$ by Proposition 5.2. Corollaries 4.11 and 6.5 imply that after choosing a basis of $I_\Delta M$ as a $\mathbb{Z}$-module, we may identify

$$\mathcal{O}(\mathrm{Rep}^{\mathcal{G}}_{G,\pi}) \cong \mathcal{O}(\mathrm{PC}^{\mathcal{G}}_{G,\pi})[t_1^{\pm 1}, \dots, t_s^{\pm 1}].$$

Thus, for both $\mathcal{G} = \Gamma_1$ and $\mathcal{G} = \widehat{\Gamma}_1$, we have

$$R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}(\mathrm{PC}^{\mathcal{G}}_{G,\pi})} \mathcal{O}(\mathrm{Rep}^{\mathcal{G}}_{G,\pi}) \cong R^{\mathrm{ps}}_{\overline{\Theta}}[t_1^{\pm 1}, \dots, t_s^{\pm 1}], \tag{7.8}$$

and under these isomorphisms, the restriction to Γ_1 is just the identity map. □

The following Lemma will allow us to relate the scheme $X^{\mathrm{gen},\widehat{\Gamma}_1}_{G,\overline{\Theta}}$ to the scheme $X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}}$ introduced in [12].

Lemma 7.14. *Let $\tau : G \hookrightarrow \mathbb{A}^n$ be a closed immersion of $\mathcal{O}$-schemes, let A be an $R^{\mathrm{ps}}_{\overline{\Theta}}$-algebra and let $\rho \in X^{\mathrm{gen},\widehat{\Gamma}_1}_{G,\overline{\Theta}}(A)$. Assume that $\mathrm{Rep}^{\widehat{\Gamma}_1}_{G,\pi}(\mathcal{O})$ is nonempty. Then $\tau(\rho(\widehat{\Gamma}_1))$ is contained in a finitely generated $R^{\mathrm{ps}}_{\overline{\Theta}}$-submodule of $A^n = \mathbb{A}^n(A)$.*

Proof. Since $X^{\mathrm{gen},\widehat{\Gamma}_1}_{G,\overline{\Theta}}$ is represented by $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}(\mathrm{PC}^{\widehat{\Gamma}_1}_{G,\pi})} \mathcal{O}(\mathrm{Rep}^{\widehat{\Gamma}_1}_{G,\pi})$, the assertion follows from Lemma 5.7 applied to $\widehat{\Gamma}_1$, and Corollary 6.5. □

7.4. *Irreducible components*

Let N be a finitely generated $\mathbb{Z}_p$-module and let $\mathcal{O}[\![N]\!]$ be the completed group algebra of N, where $\mathcal{O}$ is the ring of integers in a finite extension L of $\mathbb{Q}_p$. Let $\widehat{\mathfrak{D}}(N) : \widehat{\mathfrak{A}}_{\mathcal{O}} \to \mathrm{Ab}$ be a formal group scheme, given by

$$\widehat{\mathfrak{D}}(N)(A) := \mathrm{Hom}_{\widehat{\mathfrak{A}}_{\mathcal{O}}}(\mathcal{O}[\![N]\!], A) = \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Group}}(N, A^\times). \tag{7.9}$$

Multiplication in $\widehat{\mathfrak{D}}(N)$ is induced by the map

$$c : \mathcal{O}[\![N]\!] \to \mathcal{O}[\![N]\!] \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![N]\!], \quad n \mapsto n \widehat{\otimes} n, \quad \forall n \in N. \tag{7.10}$$

Let μ be the torsion subgroup of N. Then we have a non-canonical isomorphism $N \cong \mu \oplus \mathbb{Z}_p^r$, where $r = \dim_{\mathbb{Q}_p} N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This induces an isomorphism

$$\mathcal{O}[\![N]\!] \cong \mathcal{O}[\mu][\![x_1, \dots, x_r]\!]. \tag{7.11}$$

We assume that L contains all the p^m-th roots of unity, where p^m is the order of μ. Then the group $\mathrm{X}(\mu)$ of characters $\chi : \mu \to \mathcal{O}^\times$ also has order p^m. The following Lemma is an immediate consequence of (7.11):

Lemma 7.15. *The following hold:*

(1) *the irreducible components of* $\mathrm{Spec}\, \mathcal{O}[\![N]\!]$ *are in canonical bijection with* $\mathrm{X}(\mu)$*, so that the irreducible component corresponding to* $\chi \in \mathrm{X}(\mu)$ *is given by* $\mathrm{Spec}(\mathcal{O}[\![N]\!] \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O})$*;*

(2) *every irreducible component of* $\mathrm{Spec}\, \mathcal{O}[\![N]\!]$ *contains an* $\mathcal{O}$*-rational point* $\psi \in \widehat{\mathfrak{D}}(N)(\mathcal{O})$*;*

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(3) *a point* $\psi \in \widehat{\mathfrak{D}}(N)(\mathcal{O}) = \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Group}}(N, \mathcal{O}^\times)$ *lies on the irreducible component corresponding to* $\chi \in \mathrm{X}(\mu)$ *if and only if* $\psi(x) = \chi(x)$ *for all* $x \in \mu$.

Let R be a complete local noetherian $\mathcal{O}$-algebra with residue field k. Let $\mathcal{X} : \widehat{\mathfrak{A}}_{\mathcal{O}} \to \mathrm{Set}$ be the functor $\mathcal{X}(A) = \mathrm{Hom}_{\widehat{\mathfrak{A}}_{\mathcal{O}}}(R, A)$. This functor is represented by a formal scheme Spf R. Let us suppose that we have a faithful and transitive action of $\widehat{\mathfrak{D}}(N)$ on $\mathcal{X}$. Concretely, this means that for all $A \in \widehat{\mathfrak{A}}_{\mathcal{O}}$, we have a faithful and transitive action of the group $\widehat{\mathfrak{D}}(N)(A)$ on the set $\mathcal{X}(A)$, which is functorial in A. It is enough to restrict to $A \in \mathfrak{A}_{\mathcal{O}}$ as the general case follows by continuity. The action map $\widehat{\mathfrak{D}}(N) \times \mathcal{X} \to \mathcal{X}$ gives us a morphism in $\widehat{\mathfrak{A}}_{\mathcal{O}}$:

$$\alpha : R \to \mathcal{O}[\![N]\!] \widehat{\otimes}_{\mathcal{O}} R. \tag{7.12}$$

Let us assume that $\mathcal{X}(\mathcal{O})$ is nonempty and choose $x \in \mathcal{X}(\mathcal{O})$. Since the action map is faithful and transitive, for every $A \in \widehat{\mathfrak{A}}_{\mathcal{O}}$, the map

$$\widehat{\mathfrak{D}}(N)(A) \to \mathcal{X}(A), \quad \psi \mapsto \psi \cdot x_A \tag{7.13}$$

is bijective, where x_A is the image of x in $\mathcal{X}(A)$. This implies that the composition

$$\alpha_x : R \xrightarrow{\alpha} \mathcal{O}[\![N]\!] \widehat{\otimes}_{\mathcal{O}} R \to (\mathcal{O}[\![N]\!] \widehat{\otimes}_{\mathcal{O}} R) \widehat{\otimes}_{R,x} \mathcal{O} \cong \mathcal{O}[\![N]\!]$$

is an isomorphism.

Example 7.16. If $\mathcal{X} = \widehat{\mathfrak{D}}(N)$ and the action is given by left translations, then it follows from (7.9), (7.10) that α_ψ corresponds to the character $N \to \mathcal{O}[\![N]\!]^\times$, $n \mapsto \psi(n)n$.

Lemma 7.17. *Assume that* $\mathcal{X}(\mathcal{O})$ *is nonempty. Then the following hold:*

(1) *every irreducible component of* Spec R *has an* $\mathcal{O}$*-rational point* $x \in \mathcal{X}(\mathcal{O})$*;*
(2) *every* $x \in \mathcal{X}(\mathcal{O})$ *lies on a unique irreducible component of* Spec R*;*
(3) *if* $x, y \in \mathcal{X}(\mathcal{O})$*, then there exists a unique* $\psi \in \widehat{\mathfrak{D}}(N)(\mathcal{O})$ *such that* $\psi \cdot x = y$*;*
(4) *let* x*,* y *and* ψ *be as in part* (3)*. Then* x*,* y *lie on the same irreducible component of* Spec R *if and only if* $\psi|_\mu = 1$.

Proof. If $\mathcal{X} = \widehat{\mathfrak{D}}(N)$ with the action given by left translations, then the assertions follow readily from Lemma 7.15.

In the general case, we pick $x \in \mathcal{X}(\mathcal{O})$. The isomorphism $\alpha_x : R \to \mathcal{O}[\![N]\!]$ of $\mathcal{O}$-algebras allows us to reduce the question to the case above. For parts (3) and (4), we note that the bijection (7.13) is $\widehat{\mathfrak{D}}(N)(A)$-equivariant for the action by left translations on the source. □

Lemma 7.18. *Assume that* $\mathcal{X}(\mathcal{O})$ *is nonempty. Then the action of* $\widehat{\mathfrak{D}}(N)(\mathcal{O})$ *on* $\mathcal{X}(\mathcal{O})$ *induces a canonical action of* $\mathrm{X}(\mu)$ *on the set of irreducible components of* Spec R*. This action is faithful and transitive.*

Proof. Since μ is a direct summand of N, the map $\psi \mapsto \psi|_\mu$ induces a surjective group homomorphism $\widehat{\mathfrak{D}}(N)(\mathcal{O}) \twoheadrightarrow \mathrm{X}(\mu)$. Let K be the kernel of this map. Lemma 7.17 implies that there is a natural bijection between the set of irreducible components of Spec R and the set of K-orbits in $\mathcal{X}(\mathcal{O})$. The action $\widehat{\mathfrak{D}}(N)(\mathcal{O})$ on the set of K-orbits factors through the action of $\mathrm{X}(\mu)$, which induces the sought-after action on the set of irreducible components. Since the action of $\widehat{\mathfrak{D}}(N)(\mathcal{O})$ on $\mathcal{X}(\mathcal{O})$ is faithful and transitive, the same applies to the action of $\mathrm{X}(\mu)$. □

We will now get back to our particular example. Let $\widehat{Z}^1 : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Ab}$ be the functor such that $\widehat{Z}^1(A)$ is the set of continuous 1-cocycles $\Phi : \widehat{\Gamma}_1 \to \mathrm{Hom}(M, 1 + \mathfrak{m}_A)$ for the discrete topology on the target.

Lemma 7.19. *The functor* $\widehat{Z}^1$ *is pro-represented by* $\mathcal{O}[\![((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}]\!]$.

Proof. Let $G = \mathfrak{D}(M) \rtimes \underline{\Delta}$, $\overline{\rho} : \widehat{\Gamma}_1 \to G(k), \gamma \mapsto (1, \pi(\gamma))$ and let $\rho_0 : \widehat{\Gamma}_1 \to G(\mathcal{O}), \gamma \mapsto (1, \pi(\gamma))$. The map $\Phi \mapsto \Phi\rho_0$ induces a natural bijection between $\widehat{Z}^1(A)$ and $D^{\square}_{\overline{\rho}}(A)$ for all $A \in \mathfrak{A}_{\mathcal{O}}$; this can be shown by the same argument as in the proof of Proposition 5.2. Thus, $\widehat{Z}^1$ is pro-represented by $R^{\square}_{\overline{\rho}}$, and the assertion follows from Proposition 5.2, Lemma 7.1 and Lemma 7.2. □

It follows from Proposition 5.2 that if $\Phi \in \widehat{Z}^1(A)$ and $\rho \in D^{\square}_{\overline{\rho}}(A)$, then $\gamma \mapsto \Phi(\gamma)\rho(\gamma)$ defines a representation $\Phi\rho \in D^{\square}_{\overline{\rho}}(A)$, and the map $\widehat{Z}^1 \times D^{\square}_{\overline{\rho}} \to D^{\square}_{\overline{\rho}}$, $(\Phi, \rho) \mapsto \Phi\rho$ defines a faithful and transitive action of $\widehat{Z}^1$ on $D^{\square}_{\overline{\rho}}$.

Proposition 7.20. *Assume that $\Gamma_2^{ab}/p\Gamma_2^{ab}$ is finite and let μ be the torsion subgroup of $((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}$. Assume further that $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ is nonempty and $\mathcal{O}$ contains all the p^m-th roots of unity, where p^m is the order of μ. Then there is a canonical action of the character group $\mathrm{X}(\mu)$ on the set of irreducible components of* $\operatorname{Spec} R^{\square}_{\overline{\rho}}$, $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$ *and* $X^{\mathrm{gen}}_{G,\overline{\Theta}}$*, respectively. Moreover, the following hold:*

(1) *this action is faithful and transitive;*
(2) *irreducible components of* $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ *and* $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$ *are formally smooth over* $\mathcal{O}$*;*
(3) *irreducible components of* $X^{\mathrm{gen}}_{G,\overline{\Theta}}$ *are of the form*

$$\operatorname{Spec} \mathcal{O}[\![x_1, \dots, x_r]\!][t_1^{\pm 1}, \dots, t_s^{\pm 1}],$$

where $r = \operatorname{rank}_{\mathbb{Z}_p} H_1(\Gamma_1, M)^{\wedge,p}$ *and* $r + s = \operatorname{rank}_{\mathbb{Z}_p} ((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}$.

Proof. The assumption that $\Gamma_2^{ab}/p\Gamma_2^{ab}$ is finite implies that $((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}$ is a finitely generated $\mathbb{Z}_p$-module by Lemma 7.3, and hence, μ is a finite p-group.

The assumption that $\mathrm{Rep}^{\Gamma_1}_{G,\pi}(\mathcal{O})$ is nonempty implies via Proposition 5.2 that $\mathcal{O}(\mathrm{Rep}^{\Gamma_1}_{G,\pi}) \cong \mathcal{O}[(\mathcal{E} \otimes M)_\Delta]$. It follows from Lemmas 7.1 and 7.2 that $R^{\square}_{\overline{\rho}} \cong \mathcal{O}[\![((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}]\!]$. In particular, $D^{\square}_{\overline{\rho}}(\mathcal{O})$ is nonempty. It follows from Lemma 7.18 that the action of $\widehat{Z}^1(\mathcal{O})$ on $D^{\square}_{\overline{\rho}}(\mathcal{O})$ induces a transitive and faithful action of $\mathrm{X}(\mu)$ on the set of irreducible components of $R^{\square}_{\overline{\rho}}$.

Let us spell out Lemma 7.17 in our context. Given an irreducible component X of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$, $X(\mathcal{O})$ is nonempty, and we pick any $\rho \in X(\mathcal{O})$; given $\chi \in \mathrm{X}(\mu)$, we pick any $\Phi \in \widehat{Z}^1(\mathcal{O})$ such that $\Phi(\gamma) = \chi(\gamma)$ for all $\gamma \in \mu$. Then $\chi \cdot X$ is the unique irreducible component of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ such that $\Phi\rho \in (\chi \cdot X)(\mathcal{O})$.

It follows from Proposition 4.9 that

$$((\mathcal{E} \otimes M)_\Delta)^{\wedge,p} \cong H_1(\Gamma_1, M)^{\wedge,p} \oplus \mathbb{Z}_p^s$$

for some $s \geq 0$. Hence, $R^{\square}_{\overline{\rho}} \cong R^{\mathrm{ps}}_{\overline{\Theta}}[\![t_1, \dots, t_s]\!]$ and μ is the torsion subgroup of $H_1(\Gamma_1, M)^{\wedge,p}$. Thus, the map $\operatorname{Spec} R^{\square}_{\overline{\rho}} \to \operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$ is $\mathrm{X}(\mu)$-equivariant and induces an $\mathrm{X}(\mu)$-equivariant bijection between the irreducible components.

Since $A^{\mathrm{gen}}_{G,\overline{\Theta}} \cong R^{\mathrm{ps}}_{\overline{\Theta}}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$, by Proposition 7.13, the map $X^{\mathrm{gen}}_{G,\overline{\Theta}} \to \operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$ is $\mathrm{X}(\mu)$-equivariant and induces an $\mathrm{X}(\mu)$-equivariant bijection between the sets of irreducible components.

The isomorphism $R^{\mathrm{ps}}_{\overline{\Theta}} \cong \mathcal{O}[\![H_1(\Gamma, M)^{\wedge,p}]\!]$ allows us to consider $R^{\mathrm{ps}}_{\overline{\Theta}}$ as an $\mathcal{O}[\mu]$-algebra. This is non-canonical: it amounts to distinguishing one irreducible component of $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$. Once we do this, the other irreducible components are given by $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ for $\chi \in \mathrm{X}(\mu)$, and the special component corresponds to the trivial character. These are isomorphic to $\mathcal{O}[\![H_1(\Gamma, M)^{\wedge,p}/\mu]\!]$ and hence are formally smooth.

Similarly, irreducible components of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ and $X^{\mathrm{gen}}_{G,\overline{\Theta}}$ are given by

$$R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O} \cong \mathcal{O}[\![((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}/\mu]\!]$$

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and

$$A^{\mathrm{gen}}_{G,\overline{\Theta}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O} \cong \mathcal{O}[\![H_1(\Gamma, M)^{\wedge,p}/\mu]\!][t_1^{\pm 1}, \dots, t_s^{\pm 1}],$$

respectively. □

8. Rank calculations

Let E be a finite Galois extension of F, let $\Delta := \mathrm{Gal}(E/F)$ and let M be a free $\mathbb{Z}$-module of finite rank with an action of Δ. In this section, we compute the $\mathbb{Z}_p$-rank and the torsion subgroup of the pro-p completion of $(E^\times \otimes M)^\Delta$ and related modules. These calculations are used in the next section.

If $\mathcal{A}$ is an abelian group, we denote its pro-p completion by $\mathcal{A}^{\wedge,p}$. If N is a $\mathbb{Z}_p$-module, we define $\mathrm{rank}_{\mathbb{Z}_p} N := \dim_{\mathbb{Q}_p}(N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.

Let Γ_E^{ab} be the maximal abelian pro-finite quotient of Γ_E and let $\Gamma_E^{\mathrm{ab},p}$ be the maximal abelian pro-p quotient of Γ_E. The Artin map $\mathrm{Art}_E : E^\times \to \Gamma_E^{\mathrm{ab}}$ induces an isomorphism between the profinite completion of $E^\times$ and Γ_E^{ab}. Thus, $(E^\times \otimes M)^{\wedge,p} \cong \Gamma_E^{\mathrm{ab},p} \otimes M$.

Lemma 8.1. *Let N be a finitely generated $\mathbb{Z}[\Delta]$ (resp. $\mathbb{Z}_p[\Delta]$) module. Then the Tate cohomology groups $\widehat{H}^i(\Delta, N)$ are finite for all $i \in \mathbb{Z}$.*

Proof. If N is finitely generated over $\mathbb{Z}$, then the statement is proved in [5, Corollary 2, p. 105]. The same argument carries over if N is finitely generated over $\mathbb{Z}_p$: the cohomology groups are finitely generated $\mathbb{Z}_p$-modules since the complex computing cohomology consists of finitely generated $\mathbb{Z}_p$-modules. Moreover, they are annihilated by the order of Δ. □

Lemma 8.2. $\mathrm{rank}_{\mathbb{Z}_p}(M^{\wedge,p} \otimes I_\Delta)_\Delta = \mathrm{rank}_{\mathbb{Z}} M - \mathrm{rank}_{\mathbb{Z}} M_\Delta$.

Proof. The long exact sequence in homology attached to

$$0 \to M^{\wedge,p} \otimes I_\Delta \to M^{\wedge,p} \otimes \mathbb{Z}[\Delta] \to M^{\wedge,p} \to 0$$

yields an exact sequence

$$H_1(\Delta, M^{\wedge,p}) \to (M^{\wedge,p} \otimes I_\Delta)_\Delta \to M^{\wedge,p} \to (M^{\wedge,p})_\Delta \to 0,$$

as $M \otimes \mathbb{Z}[\Delta]$ is induced by the projection formula. Since we may swap coinvariants with completions and M_Δ is finitely generated, we have

$$\mathrm{rank}_{\mathbb{Z}_p}(M^{\wedge,p})_\Delta = \mathrm{rank}_{\mathbb{Z}_p}(M_\Delta)^{\wedge,p} = \mathrm{rank}_{\mathbb{Z}} M_\Delta.$$

Since $M^{\wedge,p}$ is finitely generated over $\mathbb{Z}_p$ and Δ is finite, the group is $H_1(\Delta, M^{\wedge,p})$ is finite by Lemma 8.1. This implies the assertion. □

Lemma 8.3. $\mathrm{rank}_{\mathbb{Z}_p}(\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta = \mathrm{rank}_{\mathbb{Z}_p}((E^\times \otimes M)^\Delta)^{\wedge,p}$.

Proof. Since M is a free $\mathbb{Z}$-module, we have an exact sequence of Δ-modules

$$0 \to (1 + \mathfrak{p}_E) \otimes M \to E^\times \otimes M \to (E^\times/(1 + \mathfrak{p}_E)) \otimes M \to 0.$$

Since $1 + \mathfrak{p}_E$ is a finitely generated $\mathbb{Z}_p$-module and $E^\times/(1 + \mathfrak{p}_E)$ is a finitely generated $\mathbb{Z}$-module, we deduce that $\widehat{H}^i(\Delta, E^\times \otimes M)$ are finite for all $i \in \mathbb{Z}$. From the exact sequence

$$0 \to \widehat{H}^{-1}(\Delta, E^\times \otimes M) \to (E^\times \otimes M)_\Delta \to (E^\times \otimes M)^\Delta \to \widehat{H}^0(\Delta, E^\times \otimes M) \to 0,$$

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we deduce that the pro-p completions of $(E^\times \otimes M)_\Delta$ and of $(E^\times \otimes M)^\Delta$ have the same $\mathbb{Z}_p$-rank. It follows from the universal property of pro-p completion that it commutes with taking Δ-coinvariants. Hence, the pro-p completion of $(E^\times \otimes M)_\Delta$ is isomorphic to $((E^\times)^{\wedge,p} \otimes M)_\Delta \cong (\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta$. □

Lemma 8.4. $\mathrm{rank}_{\mathbb{Z}_p} (\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta = \mathrm{rank}_{\mathbb{Z}} M \cdot [F : \mathbb{Q}_p] + \mathrm{rank}_{\mathbb{Z}} M_\Delta$.

Proof. We consider the long exact sequence in homology

$$H_1(\Delta, M^{\wedge,p}) \to ((\mathcal{O}_E^\times)^{\wedge,p} \otimes M)_\Delta \to ((E^\times)^{\wedge,p} \otimes M)_\Delta \to (M^{\wedge,p})_\Delta \to 0.$$

Since completion commutes with taking Δ-coinvariants we have

$$\mathrm{rank}_{\mathbb{Z}_p} (M^{\wedge,p})_\Delta = \mathrm{rank}_{\mathbb{Z}} M_\Delta.$$

Since $H_1(\Delta, M^{\wedge,p})$ is finite by Lemma 8.1, we are left to compute the $\mathbb{Z}_p$-rank of $((\mathcal{O}_E^\times)^{\wedge,p} \otimes M)_\Delta$. We note that $(\mathcal{O}_E^\times)^{\wedge,p}$ is equal to $1 + \mathfrak{p}_E$, and another application of Lemma 8.1 shows that the rank does not change if we replace $(\mathcal{O}_E^\times)^{\wedge,p}$ with any open Δ-stable subgroup V of $1 + \mathfrak{p}_E$.

We choose V to be the image of a p-adic exponential function defined on $\mathfrak{p}_E^n$ for some large enough $n \geq 1$. We then have an isomorphism $\mathfrak{p}_E^n \otimes M \cong V \otimes M$ of Δ-modules. Since $\mathfrak{p}_E^n \cong \mathcal{O}_E$ is isomorphic to $\mathcal{O}_F[\Delta]$ as $\mathbb{Z}_p[\Delta]$-module (see the proof of [14, Section 1.4]), $\mathcal{O}_E \otimes M$ is free and thus $(\mathcal{O}_E \otimes M)_\Delta \cong \mathcal{O}_F \otimes M$. Thus, $\mathrm{rank}_{\mathbb{Z}_p} ((\mathcal{O}_E^\times)^{\wedge,p} \otimes M)_\Delta = [F : \mathbb{Q}_p] \cdot \mathrm{rank}_{\mathbb{Z}} M$, and the assertion follows. □

Lemma 8.5. *The torsion subgroup of* $((E^\times \otimes M)^\Delta)^{\wedge,p}$ *is equal to* $(\mu_{p^\infty}(E) \otimes M)^\Delta$.

Proof. The image of $(E^\times \otimes M)^\Delta \to (E^\times/\mathcal{O}_E^\times) \otimes M$ is a free $\mathbb{Z}$-module of finite rank, which we denote by s, as the target is a free $\mathbb{Z}$-module of finite rank. The kernel of this map is equal to $(\mathcal{O}_E^\times \otimes M)^\Delta$. The Teichmüller lift gives an isomorphism of Δ-modules $\mathcal{O}_E^\times \cong (1 + \mathfrak{p}_E) \oplus k_E^\times$, and hence, we have an isomorphism of abelian groups

$$(E^\times \otimes M)^\Delta \cong ((1 + \mathfrak{p}_E) \otimes M)^\Delta \oplus (k_E^\times \otimes M)^\Delta \oplus \mathbb{Z}^s. \tag{8.1}$$

Since the order of $k_E^\times$ is prime to p and $((1 + \mathfrak{p}_E) \otimes M)^\Delta$ is closed in $(1 + \mathfrak{p}_E) \otimes M$ and hence p-adically complete, we conclude that the torsion subgroup in the pro-p completion of $(E^\times \otimes M)^\Delta$ coincides with the torsion subgroup in $((1 + \mathfrak{p}_E) \otimes M)^\Delta$, which is equal to $(\mu_{p^\infty}(E) \otimes M)^\Delta$. □

Lemma 8.6. *If* $\mathcal{A}$ *is a finitely generated* $\mathbb{Z}[\Delta]$*-module, then* $(\mathcal{A}^{\wedge,p})^\Delta \cong (\mathcal{A}^\Delta)^{\wedge,p}$.

Proof. We have $\mathcal{A}^{\wedge,p} \cong \mathcal{A} \otimes \mathbb{Z}_p$ and $(\mathcal{A}^\Delta)^{\wedge,p} \cong \mathcal{A}^\Delta \otimes \mathbb{Z}_p$ by [15, Tag 00MA]. We may express $\mathcal{A}^\Delta$ as the kernel of

$$\mathcal{A} \to \bigoplus_{\delta \in \Delta} \mathcal{A}, \quad a \mapsto (\delta a - a)_{\delta \in \Delta}.$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module, we conclude that $\mathcal{A}^\Delta \otimes \mathbb{Z}_p \cong (\mathcal{A} \otimes \mathbb{Z}_p)^\Delta$. □

Lemma 8.7. $((E^\times \otimes M)^\Delta)^{\wedge,p} \cong (\Gamma_E^{\mathrm{ab},p} \otimes M)^\Delta$.

Proof. Let n_0 be an integer such that $\exp : \mathfrak{p}_E^n \to 1 + \mathfrak{p}_E^n$ converges for all $n \geq n_0$ and let $V_n := \exp(\mathfrak{p}_E^n)$. Then V_n for $n \geq n_0$ form a basis of open neighbourhoods of 1 in $1 + \mathfrak{p}_E$. Since $\mathfrak{p}_E^n \cong \mathcal{O}_E \cong \mathcal{O}_F[\Delta]$ as Δ-modules and exp is Δ-equivariant, we have an isomorphism $V_n \otimes M \cong \mathrm{Ind}_1^\Delta(\mathcal{O}_F \otimes M)$, and hence, $H^1(\Delta, V_n \otimes M) = 0$. Thus, for all $n \geq n_0$, we obtain an exact sequence

$$0 \to (V_n \otimes M)^\Delta \to (E^\times \otimes M)^\Delta \to ((E^\times/V_n) \otimes M)^\Delta \to 0. \tag{8.2}$$

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The completion and $\varprojlim_n$ are both limits and hence commute with each other. Lemma 8.6 and (8.2) imply that

$$((E^\times \otimes M)^\Delta)^{\wedge,p} \cong \varprojlim_n ((E^\times/V_n) \otimes M)^\Delta)^{\wedge,p} \cong \varprojlim_n ((E^\times/V_n) \otimes M)^{\wedge,p})^\Delta. \tag{8.3}$$

The isomorphism $(E^\times \otimes M)^{\wedge,p} \cong \Gamma_E^{\mathrm{ab},p} \otimes M$ induces an isomorphism

$$((E^\times/V_n) \otimes M)^{\wedge,p} \cong (\Gamma_E^{\mathrm{ab},p}/\mathrm{Art}_E(V_n)) \otimes M. \tag{8.4}$$

Since $H^1(\Delta, V_n \otimes M) = 0$, we have an exact sequence

$$0 \to (V_n \otimes M)^\Delta \to (\Gamma_E^{\mathrm{ab},p} \otimes M)^\Delta \to ((\Gamma_E^{\mathrm{ab},p}/\mathrm{Art}_E(V_n)) \otimes M)^\Delta \to 0. \tag{8.5}$$

We thus have

$$(\Gamma_E^{\mathrm{ab},p} \otimes M)^\Delta \cong \varprojlim_n ((\Gamma_E^{\mathrm{ab},p}/\mathrm{Art}_E(V_n)) \otimes M)^\Delta,$$

and the assertion follows from (8.4) and (8.3). □

Corollary 8.8. *There is an isomorphism of* $\mathbb{Z}_p$*-modules:*

$$(\Gamma_E^{\mathrm{ab},p} \otimes M)^\Delta \cong (\mu_{p^\infty}(E) \otimes M)^\Delta \times \mathbb{Z}_p^r,$$

where $r = \mathrm{rank}_{\mathbb{Z}} M \cdot [F : \mathbb{Q}_p] + \mathrm{rank}_{\mathbb{Z}} M_\Delta$.

Proof. This follows from Lemmas 8.7, 8.5, 8.4, 8.3. □

Proposition 8.9. *Let* $0 \to E^\times \to \mathcal{E} \to I_\Delta \to 0$ *be any extension of* $\mathbb{Z}[\Delta]$*-modules.*

$$\mathrm{rank}_{\mathbb{Z}_p}((\mathcal{E} \otimes M)_\Delta)^{\wedge,p} = ([F : \mathbb{Q}_p] + 1)\,\mathrm{rank}_{\mathbb{Z}} M. \tag{8.6}$$

Proof. Since $I_\Delta \otimes M$ is a free $\mathbb{Z}$-module, the surjection $\mathcal{E} \otimes M \twoheadrightarrow I_\Delta \otimes M$ has a section, and hence, we have an exact sequence of $\mathbb{Z}_p[\Delta]$-modules

$$0 \to (E^\times \otimes M)^{\wedge,p} \to (\mathcal{E} \otimes M)^{\wedge,p} \to (I_\Delta \otimes M)^{\wedge,p} \to 0.$$

Since M is a free $\mathbb{Z}$-module of finite rank and $(E^\times)^{\wedge,p} \cong \Gamma_E^{\mathrm{ab},p}$, we have

$$(E^\times \otimes M)^{\wedge,p} \cong (E^\times)^{\wedge,p} \otimes M \cong \Gamma_E^{\mathrm{ab},p} \otimes M.$$

Since I_Δ is a free $\mathbb{Z}$-module $(I_\Delta \otimes M)^{\wedge,p} \cong I_\Delta \otimes M^{\wedge,p}$, so that we obtain an exact sequence of $\mathbb{Z}_p[\Delta]$-modules:

$$0 \to \Gamma_E^{\mathrm{ab},p} \otimes M \to (\mathcal{E} \otimes M)^{\wedge,p} \to I_\Delta \otimes M^{\wedge,p} \to 0.$$

Taking Δ-coinvariants and observing that $H_1(\Delta, I_\Delta \otimes M^{\wedge,p}) \cong H_2(\Delta, M^{\wedge,p})$, we obtain an exact sequence:

$$H_2(\Delta, M^{\wedge,p}) \to (\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta \to ((\mathcal{E} \otimes M)^{\wedge,p})_\Delta \to (I_\Delta \otimes M^{\wedge,p})_\Delta \to 0. \tag{8.7}$$

Since $H_2(\Delta, M^{\wedge,p})$ is a torsion module by Lemma 8.1, we deduce that

$$\mathrm{rank}_{\mathbb{Z}_p}((\mathcal{E} \otimes M)^{\wedge,p})_\Delta = \mathrm{rank}_{\mathbb{Z}_p}(\Gamma_E^{\mathrm{ab},p} \otimes M)_\Delta + \mathrm{rank}_{\mathbb{Z}_p}(I_\Delta \otimes M^{\wedge,p})_\Delta.$$

The assertion (8.6) follows from Lemmas 8.2 and 8.4. □

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9. Galois deformations

We will apply the machinery developed in the previous sections in an arithmetic situation. Let F be a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F}$ and let $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$. Let $\overline{\rho} : \Gamma_F \to G(k)$ be a continuous representation, where G is a generalised torus over $\mathcal{O}$, and let $\overline{\Theta}$ be the G-pseudocharacter associated to $\overline{\rho}$. Let Γ_E be the kernel of the composition $\Gamma_F \xrightarrow{\overline{\rho}} G(k) \to (G/G^0)(k)$ and let $\Delta := \mathrm{Gal}(E/F)$. Let $\pi : \Gamma_F \to \Delta$ and $\Pi : G \to G/G^0$ be the projection maps.

We let $D^{\mathrm{ps}}_{\overline{\Theta}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ be the functor

$$D^{\mathrm{ps}}_{\overline{\Theta}}(A) = \{\Theta \in \mathrm{cPC}^{\Gamma_F}_{G,\pi}(A) : \Theta \otimes_A k = \overline{\Theta}\}$$

and let $R^{\mathrm{ps}}_{\overline{\Theta}} \in \widehat{\mathfrak{A}}_{\mathcal{O}}$ be the ring pro-representing $D^{\mathrm{ps}}_{\overline{\Theta}}$ and let Θ^u be the universal deformation of $\overline{\Theta}$.

We let $X^{\mathrm{gen}}_{G,\overline{\Theta}} : R^{\mathrm{ps}}_{\overline{\Theta}}\text{-alg} \to \mathrm{Set}$ be the functor

$$X^{\mathrm{gen}}_{G,\overline{\Theta}}(A) = \{\rho \in \mathrm{Rep}^{\Gamma_F}_{G,\pi}(A) : \Theta_\rho = \Theta^u \otimes_{R^{\mathrm{ps}}_{\overline{\Theta}}} A\}.$$

Let $A^{\mathrm{gen}}_{G,\overline{\Theta}}$ be the $R^{\mathrm{ps}}_{\overline{\Theta}}$-algebra representing $X^{\mathrm{gen}}_{G,\overline{\Theta}}$.

We let $D^{\square}_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ be the functor such that $D^{\square}_{\overline{\rho}}(A)$ is the set of continuous representations $\rho : \Gamma_F \to G(A)$ such that $\rho \equiv \overline{\rho} \bmod \mathfrak{m}_A$. Let $R^{\square}_{\overline{\rho}} \in \widehat{\mathfrak{A}}_{\mathcal{O}}$ be the the ring pro-representing $D^{\square}_{\overline{\rho}}$.

The Weil group $W_{E/F}$ fits into a short exact sequence

$$0 \to E^\times \to W_{E/F} \to \Delta \to 0,$$

corresponding to the fundamental class $[u_{E/F}] \in H^2(\Delta, E^\times)$ by [16, (1.2)]. Let $\widehat{W}_{E/F}$ be the profinite completion of $W_{E/F}$. The Artin map $\mathrm{Art}_E : E^\times \to \Gamma_E^{\mathrm{ab}}$ induces an isomorphism between the profinite completion of $E^\times$ and Γ_E^{ab}. This yields a natural isomorphism

$$\widehat{W}_{E/F} \cong \Gamma_F/[\Gamma_E, \Gamma_E], \tag{9.1}$$

where $[\Gamma_E, \Gamma_E]$ is the closure of the subgroup generated by the commutators in Γ_E.

Lemma 9.1. *Every continuous representation $\rho : \Gamma_F \to G(A)$ satisfying $\Pi \circ \rho = \pi$, where A is a finite discrete $\mathcal{O}$-algebra, factors through the quotient $\Gamma_F \twoheadrightarrow \widehat{W}_{E/F}$.*

Proof. Since $\rho(\Gamma_E) \subseteq G^0(A)$ and $G^0(A)$ is commutative and Hausdorff, the assertion follows from (9.1). □

Lemma 9.2. *Let $A \in R^{\mathrm{ps}}_{\overline{\Theta}}$-alg. Then every $\rho \in X^{\mathrm{gen}}_{G,\overline{\Theta}}(A)$ factors through the quotient $\Gamma_F \twoheadrightarrow \widehat{W}_{E/F}$.*

Proof. The restriction of $\overline{\rho}$ to Γ_E takes values in $G^0(k)$. Let $\overline{\Psi}$ be the G^0-pseudocharacter of $\overline{\rho}|_{\Gamma_E}$ and let $\Psi^u \in D^{\mathrm{ps}}_{\overline{\Psi}}(R^{\mathrm{ps}}_{\overline{\Psi}})$ be the universal deformation of $\overline{\Psi}$. Since G^0 is commutative, Lemma 6.3 implies that there is a continuous group homomorphism $\psi^u : \Gamma_E \to G^0(R^{\mathrm{ps}}_{\overline{\Psi}})$ such that $\Psi^u = \Theta_{\psi^u}$. In particular,

$$\psi^u(\gamma) = 1, \quad \forall \gamma \in [\Gamma_E, \Gamma_E]. \tag{9.2}$$

Since $\Theta^u|_{\Gamma_E}$ is a deformation of $\overline{\Psi}$ to $R^{\mathrm{ps}}_{\overline{\Theta}}$, there is a homomorphism $R^{\mathrm{ps}}_{\overline{\Psi}} \to R^{\mathrm{ps}}_{\overline{\Theta}}$ such that

$$\Theta^u|_{\Gamma_E} = \Psi^u \otimes_{R^{\mathrm{ps}}_{\overline{\Psi}}} R^{\mathrm{ps}}_{\overline{\Theta}}. \tag{9.3}$$

If $\rho \in X^{\mathrm{gen}}_{G,\overline{\Theta}}(A)$, then $\Theta_\rho = \Theta^u \otimes_{R^{\mathrm{ps}}_{\overline{\Theta}}} A$, and it follows from (9.3) that $\rho|_{\Gamma_E} = \psi^u \otimes_{R^{\mathrm{ps}}_{\overline{\Psi}}} A$. The assertion follows from (9.2). □

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Theorem 9.3. *There is a finite extension L' of L with the ring of integers $\mathcal{O}'$, such that the following hold:*

(1) *$G^0_{\mathcal{O}'}$ is split and $(G/G^0)_{\mathcal{O}'}$ is a constant group scheme;*
(2) *$\mathrm{Rep}^{\Gamma_F}_{G,\pi}(\mathcal{O}')$ is nonempty.*

Moreover, if (1) *and* (2) *hold, then there are isomorphisms of $\mathcal{O}'$-algebras:*

(3) $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}} \mathcal{O}' \cong \mathcal{O}'[\![(\Gamma_E^{\mathrm{ab},p} \otimes M)^{\Delta}]\!] \cong \mathcal{O}'[(\mu_{p^\infty}(E) \otimes M)^{\Delta}][\![x_1, \dots, x_r]\!]$;
(4) $A^{\mathrm{gen}}_{G,\overline{\Theta}} \otimes_{\mathcal{O}} \mathcal{O}' \cong R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}} \mathcal{O}'[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$;
(5) $R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}' \cong (R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}} \mathcal{O}')[\![t_1, \dots, t_s]\!] \cong \mathcal{O}'[(\mu_{p^\infty}(E) \otimes M)^{\Delta}][\![z_1, \dots, z_{r+s}]\!]$,

where M is the character lattice of $G^0_{\mathcal{O}'}$, $r = \mathrm{rank}_{\mathbb{Z}} M \cdot [F : \mathbb{Q}_p] + \mathrm{rank}_{\mathbb{Z}} M_{\Delta}$, $s = \mathrm{rank}_{\mathbb{Z}} M - \mathrm{rank}_{\mathbb{Z}} M_{\Delta}$.

Proof. If L' is a finite extension of L with the ring of integers $\mathcal{O}'$ and residue field k', then the functor $D^{\mathrm{ps}}_{\overline{\Theta}_{k'}} : \mathfrak{A}_{\mathcal{O}'} \to \mathrm{Set}$ is pro-representable by $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_{\mathcal{O}} \mathcal{O}'$, and analogous statements hold for $X^{\mathrm{gen}}_{G,\overline{\Theta}_{k'}}$ and $D^{\square}_{\overline{\rho}_{k'}}$. Thus, it is enough to prove the statement for $\mathcal{O}' = \mathcal{O}$, after replacing L by a finite extension.

Since G is a generalised torus, after replacing L by a finite unramified extension, we may assume that G^0 is split and G/G^0 is a constant group scheme. We may assume that $G/G^0 = \underline{\Delta}$, as replacing G by the preimage of $\underline{\Delta}$ does not change the functors under consideration. The character lattice M of G^0 does not change if we further replace L by a finite extension. As explained at the beginning of Section 5, we have an action of Δ on M.

We will apply the results of previous sections with $\Gamma_1 = W_{E/F}$ and $\Gamma_2 = E^\times$. It is a fundamental result of Langlands proved in [11] (see also a nice exposition by Birkbeck [2, Proposition 2.0.3]) that there are natural isomorphisms:

$$H_1(W_{E/F}, M) \cong H_1(E^\times, M)^{\Delta} \cong (E^\times \otimes M)^{\Delta}. \tag{9.4}$$

For each $c \in \Delta$, we choose a coset representative $\bar{c} \in W_{E/F}$. We have constructed a Δ-action on $\mathcal{E} := E^\times \oplus I_\Delta$, which depends on this choice, such that we have an extension of $\mathbb{Z}[\Delta]$-modules

$$0 \to E^\times \to \mathcal{E} \to I_\Delta \to 0,$$

and the image of the extension class in $H^2(\Delta, E^\times)$ under the isomorphisms

$$\mathrm{Ext}^1_{\mathbb{Z}[\Delta]}(I_\Delta, E^\times) \cong \mathrm{Ext}^2_{\mathbb{Z}[\Delta]}(\mathbb{Z}, E^\times) \cong H^2(\Delta, E^\times)$$

is equal to $[u_{E/F}]$.

Let N be the pro-p completion of $(\mathcal{E} \otimes M)_\Delta$. It follows from Proposition 8.9 and Euler–Poincaré characteristic formula that

$$h^1_{\mathrm{cont}}(\Gamma_F, \mathrm{ad}\,\overline{\rho}) - h^0_{\mathrm{cont}}(\Gamma_F, \mathrm{ad}\,\overline{\rho}) - h^2_{\mathrm{cont}}(\Gamma_F, \mathrm{ad}\,\overline{\rho}) = \mathrm{rank}_{\mathbb{Z}_p} N - \dim G_k. \tag{9.5}$$

Lemma 9.1 and (9.5) ensure that the assumptions of Proposition 7.6 are satisfied, and hence, after replacing L by a finite extension, we may ensure that $\mathrm{Rep}^{\widehat{W}_{E/F}}_{G,\pi}(\mathcal{O})$ is nonempty by Corollary 7.7. In particular, $\mathrm{Rep}^{\Gamma_F}_{G,\pi}(\mathcal{O})$ and $\mathrm{Rep}^{W_{E/F}}_{G,\pi}(\mathcal{O})$ are also nonempty.

Lemma 6.6 implies that there is $\rho \in X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}}(R^{\mathrm{ps}}_{\overline{\Theta}})$ such that $\Theta^u = \Theta_\rho$. Since ρ factors through the quotient $\Gamma_F \twoheadrightarrow \widehat{W}_{E/F}$ by Lemma 9.2, we conclude that Θ^u is obtained from a G-pseudocharacter of $\widehat{W}_{E/F}$ via inflation to Γ_F. This together with Lemmas 9.1 and 9.2 implies that in the definitions of the functors $D^{\mathrm{ps}}_{\overline{\Theta}}$, $D^{\square}_{\overline{\rho}}$ and $X^{\mathrm{gen}}_{G,\overline{\Theta}}$, we can replace Γ_F with $\widehat{W}_{E/F}$ without changing the functors themselves.

Lemma 7.10 and (9.4) imply that $R^{\mathrm{ps}}_{\overline{\Theta}}$ is isomorphic to $\mathcal{O}[\![N]\!]$, where N is the pro-p completion of $(E^\times \otimes M)^{\Delta}$. It follows from Lemma 8.7 that $N \cong (\Gamma_E^{\mathrm{ab},p} \otimes M)^{\Delta}$, which is a finitely generated

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$\mathbb{Z}_p$-module of rank r and torsion subgroup isomorphic to $(\mu_{p^\infty}(E) \otimes M)^\Delta$ by Corollary 8.8. This yields the isomorphisms in part (3). Part (4) follows from Proposition 7.13.

The map $\rho \mapsto \Theta_\rho$ induces a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\square}_{\overline{\rho}}$ and hence a map of $R^{\mathrm{ps}}_{\overline{\Theta}}$-algebras $A^{\mathrm{gen}}_{G,\overline{\Theta}} \to R^{\square}_{\overline{\rho}}$. Since $E^\times/(E^\times)^p$ is finite, Lemma 7.4 implies that $R^{\square}_{\overline{\rho}}$ is the completion of $\mathcal{O}(\mathrm{Rep}^{W_{E/F}}_{G,\pi})$ with respect to the maximal ideal corresponding to $\overline{\rho}$. By considering the composition $\mathcal{O}(\mathrm{Rep}^{W_{E/F}}_{G,\pi}) \to A^{\mathrm{gen}}_{G,\overline{\Theta}} \to R^{\square}_{\overline{\rho}}$, we deduce that it induces a natural isomorphism between $R^{\square}_{\overline{\rho}}$, and the completion of $A^{\mathrm{gen}}_{G,\overline{\Theta}}$ with respect to the maximal ideal corresponding to $\overline{\rho}$. Part (5) then follows from parts (3) and (4). □

We will now deduce some corollaries, which hold without extending the scalars.

Corollary 9.4. *The map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\square}_{\overline{\rho}}$ is formally smooth. In particular, it is flat and induces a bijection between the sets of irreducible components.*

Proof. Since the map $R^{\mathrm{ps}}_{\overline{\Theta}} \to \mathcal{O}' \otimes_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}}$ is faithfully flat, the assertion follows from [15, Tag 06CM] and part (5) of Theorem 9.3. □

Corollary 9.5. *The map $R^{\mathrm{ps}}_{\overline{\Theta}} \to A^{\mathrm{gen}}_{G,\overline{\Theta}}$ is smooth and induces a bijection between the sets of irreducible components. Moreover, $A^{\mathrm{gen}}_{G,\overline{\Theta}}$ is flat over $\mathcal{O}$ of relative dimension $\dim G_k \cdot ([F : \mathbb{Q}_p] + 1)$.*

Proof. The map $R^{\mathrm{ps}}_{\overline{\Theta}} \to A^{\mathrm{gen}}_{G,\overline{\Theta}}$ is formally smooth by the same argument as in Corollary 9.4 using part (4) of Theorem 9.3. Since it is of finite type, it is smooth by [15, Tag 00TN]. The map $X^{\mathrm{gen}}_{G,\overline{\Theta}} \to X^{\mathrm{ps}}_{\overline{\Theta}} := \operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$ is flat and of finite presentation and hence open by [15, Tag 00I1]. By [15, Tag 004Z], it is enough to show the fibres of $X^{\mathrm{gen}}_{G,\overline{\Theta}} \to X^{\mathrm{ps}}_{\overline{\Theta}}$ are irreducible.

To ease the notation, we let $R = R^{\mathrm{ps}}_{\overline{\Theta}}$, $A = A^{\mathrm{gen}}_{G,\overline{\Theta}}$, $R' = R \otimes_{\mathcal{O}} \mathcal{O}'$ and $A' = A \otimes_{\mathcal{O}} \mathcal{O}'$. Let $x : \operatorname{Spec} \kappa \to \operatorname{Spec} R$ be a geometric point and let $x' : \operatorname{Spec} \kappa \to \operatorname{Spec} R'$ be a point above x. The map

$$A \otimes_{R,x} \kappa \to A' \otimes_{R',x'} \kappa$$

is an isomorphism. Part (2) of Theorem 9.3 implies that

$$A' \otimes_{R',x'} \kappa \cong \kappa[t_1^{\pm 1}, \dots, t_s^{\pm 1}].$$

Thus, the fibres of $X^{\mathrm{gen}}_{G,\overline{\Theta}} \to X^{\mathrm{ps}}_{\overline{\Theta}}$ are irreducible, and the result follows.

The last assertion follows from the fact that $\mathcal{O}'$ is finite and free over $\mathcal{O}$, parts (3) and (4) of Theorem 9.3 and $\dim G_k = \operatorname{rank}_{\mathbb{Z}} M$. □

Lemma 9.6. *Let $A \in R^{\mathrm{ps}}_{\overline{\Theta}}$-alg, let $\rho \in X^{\mathrm{gen}}_{G,\overline{\Theta}}(A)$ and let $\tau : G \hookrightarrow \mathbb{A}^n$ be a closed immersion of $\mathcal{O}$-schemes. Then $\tau(\rho(\Gamma_F))$ is contained in a finitely generated $R^{\mathrm{ps}}_{\overline{\Theta}}$-submodule of $A^n = \mathbb{A}^n(A)$.*

Proof. It is enough to verify the assertion after extending the scalars to $\mathcal{O}'$ given by Theorem 9.3. This follows from Lemma 9.2 and Lemma 9.6. □

Remark 9.7. Lemma 9.6 implies that the scheme $X^{\mathrm{gen}}_{G,\overline{\Theta}}$ coincides with the scheme $X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}}$ defined in [12] for a generalised reductive group G; see [12, Proposition 7.3].

Corollary 9.8. *Let p^m be the order of $(\mu_{p^\infty}(E) \otimes M)^\Delta$. Assume that $\mathcal{O}$ contains all the p^m-th roots of unity and (1) and (2) in Theorem 9.3 hold with $\mathcal{O}' = \mathcal{O}$. Then there is a canonical action of the character group $\mathrm{X}((\mu_{p^\infty}(E) \otimes M)^\Delta)$ on the set of irreducible components of $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$, $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ and $X^{\mathrm{gen}}_{G,\overline{\Theta}}$, respectively. Moreover, the following hold:*

(1) *this action is faithful and transitive;*
(2) *irreducible components of* $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ *and* $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\Theta}}$ *are formally smooth over* $\mathcal{O}$*;*
(3) *irreducible components of* $X^{\mathrm{gen}}_{G,\overline{\Theta}}$ *are of the form*

$$\operatorname{Spec} \mathcal{O}[\![x_1, \dots, x_r]\!][t_1^{\pm 1}, \dots, t_s^{\pm 1}],$$

where r and s are as in Theorem 9.3.

Proof. The assertion follows from Proposition 7.20. □

Corollary 9.9. *Let* $\varphi : G \to H$ *be a surjection of generalised tori over* $\mathcal{O}$ *and let* $\overline{\rho} : \Gamma_F \to G(k)$ *be a continuous representation. Then the map* $(R^{\square}_{\varphi\circ\overline{\rho}}/\varpi)^{\mathrm{red}} \to (R^{\square}_{\overline{\rho}}/\varpi)^{\mathrm{red}}$ *is flat, where* red *indicates reduced rings, and the fibre at the closed point has dimension* $([F : \mathbb{Q}_p] + 1)(\dim G_k - \dim H_k)$*.*

Proof. If $\mathcal{P}$ is a finitely generated $\mathbb{Z}_p$-module and $R = k[\![\mathcal{P}]\!]$ is the completed group algebra of $\mathcal{P}$, then $R^{\mathrm{red}} \cong k[\![\mathcal{P}^{\mathrm{tf}}]\!]$, where $\mathcal{P}^{\mathrm{tf}}$ is the maximal torsion-free quotient of $\mathcal{P}$. In particular, $R^{\mathrm{red}} \cong k[\![x_1, \dots, x_d]\!]$, where $d = \dim_{\mathbb{Q}_p}(\mathcal{P} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.

Let $f : \mathcal{P}_1 \to \mathcal{P}_2$ be a homomorphism of finitely generated $\mathbb{Z}_p$-modules and let $R_i = k[\![\mathcal{P}_i]\!]$. If $\ker f$ is torsion, then $\mathcal{P}_1^{\mathrm{tf}}$ is a submodule of $\mathcal{P}_2^{\mathrm{tf}}$ and the fibre $k \otimes_{R_1^{\mathrm{red}}} R_2^{\mathrm{red}}$ is isomorphic to $k[\![\mathcal{P}_2^{\mathrm{tf}}/\mathcal{P}_1^{\mathrm{tf}}]\!]$. Since the dimension of the fibre is equal to $d_2 - d_1$, where $d_i = \dim_{\mathbb{Q}_p}(\mathcal{P}_i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, we deduce from [15, Tag 00R4] that $R_1^{\mathrm{red}} \to R_2^{\mathrm{red}}$ is flat.

Let M be the character lattice of G^0 and let N be the character lattice of H^0. Then $N \subseteq M$, and it follows from (8.7) and Lemma 8.1 that the kernel of the map $((\mathcal{E} \otimes N)^{\wedge,p})_\Delta \to ((\mathcal{E} \otimes M)^{\wedge,p})_\Delta$ is torsion. Since $R^{\square}_{\overline{\rho}}/\varpi \cong k[\![((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}]\!]$ by Lemma 7.5 and we may swap pro-p completion with Δ-coinvariants, we obtain the assertion by letting $\mathcal{P}_1 = ((\mathcal{E} \otimes N)_\Delta)^{\wedge,p}$ and $\mathcal{P}_2 = ((\mathcal{E} \otimes M)_\Delta)^{\wedge,p}$. The assertion about the dimension of the fibre follows from Proposition 8.9. □

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