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A SERIES OF ELEMENTS OF ORDER 4 IN THE SYMPLECTIC COBORDISM RING

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ABSTRACT. A series of elements of order 4 in the symplectic cobordism ring is constructed.

The classical cobordism graded rings consist of finitely generated abelian groups in each dimension. The complex cobordism ring have no elements of finite order and in the rings of the unoriented, oriented, special unitary and Spin cobordism all the elements of finite order have order 2 [9]. The symplectic cobordism ring $M\mathrm{Sp}_*$ is such that $M\mathrm{Sp}_* \otimes Z[\frac{1}{2}]$ is the polynomial algebra over $Z[\frac{1}{2}]$ with one $4k$-dimensional generator for any natural number k [7, 9]. The ideal of the elements of finite order $\mathrm{Tors}\, M\mathrm{Sp}_*$ contains the series of elements discovered by Nigel Ray: $\theta_1 \in M\mathrm{Sp}_1$, $\Phi_i \in M\mathrm{Sp}_{8i-3}$, $i = 1, 2, \ldots$ [8]. In small dimensions the ideal $\mathrm{Tors}\, M\mathrm{Sp}_*$ contains only elements of order 2 [10]. One of the principal tools used to study $M\mathrm{Sp}_*$ is the classical Adams spectral sequence. This thoroughly investigated by S. Kochman [4, 5].

The main result of this paper is the construction of a series of elements Γ_i, $i = 1, 2, \ldots, s$, of order 4 in the symplectic cobordism ring, where $\dim \Gamma_i = 8i + 95$. The key element of the series is Γ_1 in dimension 103. So, we are proving the following

MAIN THEOREM. *(i) There exists an indecomposable element $\Omega_1 \in M\mathrm{Sp}_{49}$ of order 2 in the symplectic cordism ring, such that the product $\theta_1\Phi_{6+i}\Omega_1 \neq 0$.*

(ii) Let $\Gamma_i \in \langle \Phi_{6+i}, 2, \Omega_1 \rangle$, for $i = 1, 2, \ldots$. Then the elements Γ_i have order 4 and $2\Gamma_i = \theta_1\Phi_{6+i}\Omega_1 \neq 0$.

The existence of the element Γ_2 was announced by Stanley Kochman in [4].

The main tool of the work is the Adams-Novikov spectral sequence (ANSS) and the algebraic spectral sequences connected with it [1, 7, 11]. The initial term of the ANSS is isomorphic to

$$\mathrm{Ext}_{A_*}\big(\mathrm{BP}_*, \mathrm{BP}_*(X)\big),$$

where $\mathrm{BP}_*(\ )$ is the Brown-Peterson homology theory, $A_* = \mathrm{BP}_*(\mathrm{BP})$ is dual to the Quillen algebra $A^* = \mathrm{BP}^*(\mathrm{BP})$ [1]. To study and compute this initial term algebraic spectral sequences can be used [7, 11]. Such spectral sequence arise from a multiplicative invariant (under the action of the Quillen algebra) filtration in $\mathrm{BP}_* = Z_p[v_1, \ldots, v_i, \ldots]$.

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This filtration generates a filtration in Adams resolution of $\mathrm{BP}_*(X)$ which in its turn gives rise to a spectral sequence converging to

$$\mathrm{Ext}_{A_*}\big(\mathrm{BP}_*, \mathrm{BP}_*(X)\big).$$

Its initial term is isomorphic to

$$\mathrm{Ext}_{\mathcal{A}/(Q_0)}\big(Z/p, \widetilde{\mathrm{BP}}_* \otimes H_*(X; Z/p)\big),$$

where $\mathcal{A}$ is the Steenrod algebra and $\widetilde{\mathrm{BP}}_*$ is an object associated to BP_* by the given filtration. In the classical case, considered by S. P. Novikov [7], filtration is given by the degrees of the maximal ideal $I = (v_0, v_1, \ldots, v_i, \ldots)$, $v_0 = p$.

Considering $M\mathrm{Sp}$, we are interested in the case $p = 2$. For our purposes the most convenient is the modified algebraic spectral sequence (MASS) [11]. The filtration of MASS on BP_* is defined by the following function $f(x)$:

$$f(v_i) = \begin{cases} 2, & \text{for } i = 0, \\ 1, & \text{for } i > 0. \end{cases}$$

The difference between the filtration defined by this function and the classical case is that v_0 has filtration degree equal to 2 in our case and equal to 1 in the classical one. We denote the object associated to BP_* by this filtration by $\overline{\overline{\mathrm{BP}}}_* = Z/2[h_0, h_1, \ldots, h_i, \ldots]$, $\deg h_0 = (2, 0)$, $\deg h_i = \big(1, 2(2^i - 1)\big)$, $i \geq 1$. The initial term of MASS is isomorphic to the polynomial algebra:

$$(Z/2)[c_2, \ldots, c_k, \ldots, u_1, \ldots, u_j, \ldots, h_0, h_1, \ldots, h_i, \ldots],$$

where $k = 2, 4, 5, \ldots$; $k \neq 2^n - 1$; $j = 1, 2, \ldots$; $i = 0, 1, \ldots$; $\deg c_k = (0, 0, 4k)$, $\deg u_j = \big(0, 1, 2(2^j - 1)\big)$, $\deg h_0 = (2, 0, 0)$, $\deg h_i = \big(1, 0, 2(2^i - 1)\big)$, $i \geq 1$. The generators u_j ($j > 1$) may be chosen as the projections of the Nigel Ray elements Φ_{2^j-2}, u_1 is the projection of $\theta_1 = \Phi_0$ and the elements h_i and c_k may be chosen so that the following formulae for the first differential are fulfilled:

$$d_1(h_i) = h_0 u_i,$$
$$d_1(c_k) = \sum_{i,j} (h_{k_i+1} u_{k_j+1} + h_{k_j+1} u_{k_i+1}) c_{2^{k_1}} \cdots \hat{c}_{2^{k_i}} \cdots \hat{c}_{2^{k_j}} \cdots c_{2^{k_s}},$$

where $k + 1 = 2^{k_1} + \cdots + 2^{k_s}$ is the binary representation of the number $k+1$. Moreover if k is odd then the projection of the Ray element $\Phi_{\frac{k+1}{2}}$ in the $E_1^{0,1,*}$-term of the MASS has the form

$$\Phi_{\frac{k+1}{2}} = u_1 c_k + \sum_{j=1}^{s} u_{k_j+1} c_{2^{k_1}} \cdots \hat{c}_{2^{k_j}} \cdots c_{2^{k_s}} + \sum_{0 < m < \frac{k+1}{2}} \Phi_m c_{J_m}, c_{J_m} \in E_2^{0,0,*}.$$

The coefficients c_{J_m} may be computed using S. Kochman's formula from [6] and simultaneously the action of the Landweber-Novikov operations S_ω on the elements c_k can be

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obtained. This is done up to Φ_{14} and c_{26}. Let us denote by $\tilde{\phi}_{k_1+1,\dots,k_s+1}$ the main part of the projection of the element $\Phi_{\frac{k+1}{2}}$:

$$\tilde{\phi}_{k_1+1,\dots,k_s+1} = u_1 c_k + \sum_{j=1}^{s} u_{k_j+1} c_{2^{k_1}} \cdots \hat{c}_{2^{k_j}} \cdots c_{2^{k_s}}.$$

Then the elements $\tilde{\phi}_{n_1,\dots,n_s}$ can be chosen as the generators of $E_2^{0,1,*}$ of the MASS as well as projections of Φ_n. The differentials d_r of the MASS don't change the third grading t, they increase the second grading s by 1 and the first grading q they increase by r [11].

Let ξ, η and ζ be elements of $E_1^{0,1,*}$ of the MASS. Suppose that they are cycles of d_1. We keep the same notations for their images in $E_2^{0,1,*}$. Direct computations show that all Massey products of the type $\langle \xi, h_0, \eta \rangle$ are defined and if the last grading t is less than 106 then almost all of them contain zero. In this case the matrix Massey products

$$\langle \xi, h_0, \eta, h_0 \rangle, \quad \left\langle \xi, h_0, (\eta, \zeta), \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \right\rangle$$

are defined. Let $c_{\xi,\eta}$ be the element in $E_1^{0,0,*}$ which is defined by the formula $d_1(c_{\xi,\eta}) \in \langle \xi, h_0, \eta \rangle$ uniquely up to cycles of the differential d_1. We denote by h_ξ the element in $E_1^{1,0,*}$ such that $d_1(h_\xi) = h_0 \xi$. Then we have

$$h_0 c_{\xi,\eta} + h_\xi h_\eta \in \langle \xi, h_0, \eta, h_0 \rangle, \quad \xi c_{\eta,\zeta} + \zeta c_{\xi,\eta} + \eta c_{\xi,\zeta} \in \left\langle \xi, h_0, (\eta, \zeta), \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \right\rangle.$$

Let us denote the first Massey product by $\mathcal{A}_{\xi,\eta}$ and the second by $\mathcal{F}_{\xi,\zeta,\eta}$. We choose $\tilde{\phi}_{i,j}$ as the canonical representative of $\mathcal{F}_{u_1,u_i,u_j}$ and $c_{2^{j-1}}$ as the canonical representative of c_{u_1,u_j}. If an element $\xi \in E_2^{0,1,*}$ has the decomposition $\xi = \sum_i u_i \tilde{c}_i$ in $E_1^{0,1,*}$ for some $\tilde{c}_i \in E_1^{0,0,*}$ then the element $\sum_i h_i \tilde{c}_i$ will be taken as the representative of h_ξ. We'll take h_ξ^2 as the canonical representative of $\mathcal{A}_{\xi,\xi}$. Under these conditions the elements $\mathcal{F}_{\xi,\zeta,\eta}$ and $\mathcal{A}_{\xi,\eta}$ are defined uniquely in $E_2^{0,1,t}$ of the MASS for $t < 108$. For simplicity we denote $\mathcal{F}_{u_i,u_j,u_k}$ by $\omega_{i,j,k}$, $\mathcal{F}_{u_1,u_j,\omega_{i,j,k}}$ by $\psi_{\hat{\imath},j,k}$ and $\mathcal{F}_{u_1,\tilde{\phi}_{i,j},\omega_{i,j,k}}$ by $\psi_{\hat{\imath},\hat{\jmath},k}$. The generators of $E_2^{0,1,t}$ for $t < 108$ are given in the Table 1. The generators of $E_2^{0,0,t}$ for $t < 108$ are given in the Table 2.

LEMMA 1. *Let ξ, η and ζ be distinct elements of $E_2^{1,0,t}$ of the* MASS, *$t < 108$, and let i, j, k be distinct integers from the set $\{2,3,4,5\}$. Then the following list exhausts all the relations for the generators of $E_2^{0,1,t}$, $t < 108$:*

(1) $u_i\tilde{\phi}_{j,k} + u_j\tilde{\phi}_{i,k} + u_k\tilde{\phi}_{i,j} = u_1\omega_{i,j,k}$
(2) $u_i\tilde{\phi}_{i,j,k} + \tilde{\phi}_{i,j}\tilde{\phi}_{j,k} = u_1\psi_{\hat{\imath},j,k} + u_j u_k c_{2^{i-1}}^2$
(3) $\tilde{\phi}_{i,j}\tilde{\phi}_{i,j,k} = u_1\psi_{\hat{\imath},\hat{\jmath},k} + u_i\tilde{\phi}_{i,k}c_{2^{j-1}}^2 + u_j\tilde{\phi}_{j,k}c_{2^{i-1}}^2$
(4) $u_i\tilde{\psi}_{i,\hat{\jmath},k} + u_j\psi_{\hat{\imath},j,k} = \tilde{\phi}_{i,j}\omega_{i,j,k}$
(5) $\tilde{\phi}_{i,j}^2 = u_1^2 c_{2^{i-1}+2^{j-1}-1}^2 + u_i^2 c_{2^{j-1}}^2 + u_j^2 c_{2^{i-1}}^2$
(6) $u_i\psi_{\hat{\imath},\hat{\jmath},k} + \tilde{\phi}_{i,j}\psi_{\hat{\imath},j,k} = u_1\tilde{\phi}_{i,k}c_{2^{i-1}+2^{j-1}-1} + u_j\omega_{i,j,k,}c_{2^{i-1}}^2$
(7) $u_i\psi_{i,\hat{\jmath},\hat{k}} + \tilde{\phi}_{i,j}\psi_{i,j,\hat{k}} + \tilde{\phi}_{i,k}\psi_{i,\hat{\jmath},k} = \tilde{\phi}_{i,j,k}\omega_{i,j,k}$
(8) $\omega_{i,j,k,}^2 = u_i^2 c_{2^{j-1}+2^{k-1}-1}^2 + u_j^2 c_{2^{i-1}+2^{k-1}-1}^2 + u_k^2 c_{2^{i-1}+2^{j-1}-1}^2$
(9) $\tilde{\phi}_{i,j,k}^2 = u_1^2 c_{2^{i-1}+2^{j-1}+2^{k-1}-1}^2 + u_i^2 c_{2^{j-1}}^2 c_{2^{k-1}}^2 + u_j^2 c_{2^{i-1}}^2 c_{2^{k-1}}^2 + u_k^2 c_{2^{i-1}}^2 c_{2^{j-1}}^2$

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(10) $\xi\mathcal{F}_{\zeta,\eta,\theta}+\zeta\mathcal{F}_{\xi,\eta,\theta}+\eta\mathcal{F}_{\xi,\zeta,\eta}+\theta\mathcal{F}_{\xi,\zeta,\eta}=0.$

PROOF. It is done by using the decomposition of given elements through the generators of E_1 of the MASS.

LEMMA 2. *i) Let* $\xi,\eta,\zeta,\theta\in E_2^{0,1,t}$ *of the* MASS, $t<108$, *and such that* $\mathcal{A}_{\xi,\zeta}$ *is defined, then* $\mathcal{A}_{\theta\xi,\zeta}$ *and* $\mathcal{A}_{\xi,\theta\zeta}$ *are also defined and the following equalities hold:* $\theta\mathcal{A}_{\xi,\zeta}=\mathcal{A}_{\theta\xi,\zeta}=\mathcal{A}_{\xi,\theta\zeta}$.

ii) If $\mathcal{A}_{\xi,\eta}$ *and* $\mathcal{A}_{\xi,\zeta}$ *are defined, then* $\mathcal{A}_{\xi,\eta+\zeta}$ *is also defined and the following equality holds:* $\mathcal{A}_{\xi,\eta}+\mathcal{A}_{\xi,\zeta}=\mathcal{A}_{\xi,\eta+\zeta}$.

PROOF. (i) If $\mathcal{A}_{\xi,\zeta}$ is defined by the expression

$$\langle\xi,h_0,\zeta,h_0\rangle=h_0c_{\xi,\zeta}+h_\xi h_\zeta,$$

then $\mathcal{A}_{\theta\xi,\zeta}$ may be given by the formula:

$$\langle\theta\xi,h_0,\zeta,h_0\rangle=\theta(h_0c_{\xi,\zeta}+h_\xi h_\zeta).$$

The proof of (ii) may be given the same way.

LEMMA 3. *Let* $\sum_i\xi_i\zeta_i=0$ *be one of the relations of Lemma 1 and let* $\xi_i,\zeta_i\in E_2^{0,1,t}$ *be such that the sum of their t-gradings is less then 108 and* $\mathcal{A}_{\xi_i,\zeta_i}$ *are defined, then* $\sum_i\mathcal{A}_{\xi_i,\zeta_i}=0$.

PROOF. Let us consider, for example, the first relation:

$$u_i\tilde{\phi}_{j,k}+u_j\tilde{\phi}_{i,k}+u_k\tilde{\phi}_{i,j}=u_1\omega_{i,j,k}.$$

We have the following decomposition:

$$\tilde{\phi}_{j,k}=u_1c_{j,k}+u_kc_{1,j}+u_jc_{1,k},$$

so:

$$h_{\tilde{\phi}_{j,k}}=h_1c_{u_j,u_k}+h_kc_{2^j-1}+h_jc_{2^k-1},$$

and

$$c_{u_i,\tilde{\phi}_{j,k}}=c_{2^{i-1}+2^{j-1}+2^{k-1}-1}+c_{2^{i-1}}c_{u_j,u_k}.$$

We have:

$$\omega_{i,j,k}=u_ic_{u_j,u_k}+u_jc_{u_i,u_k}+u_kc_{u_i,u_j},$$

so

$$h_{\omega_{i,j,k}}=h_ic_{u_j,u_k}+h_jc_{u_i,u_k}+h_kc_{u_i,u_j},$$

and

$$c_{u_1,\omega_{i,j,k}}=c_{2^{i-1}+2^{j-1}+2^{k-1}-1}+c_{2^{i-1}}c_{u_j,u_k}+c_{2^{j-1}}c_{u_i,u_k}+c_{2^{k-1}}c_{u_i,u_j}.$$

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Now we have the following decompositions:

$$
\begin{aligned}
\mathcal{A}_{u_i,\tilde{\phi}_{j,k}} &= h_0(c_{2^{i-1}+2^{j-1}+2^{k-1}-1} + c_{2^{i-1}}c_{u_j,u_k}) + h_i(h_1 c_{u_j,u_k} + h_k c_{2^{j-1}} + h_j c_{2^{k-1}}),\\
\mathcal{A}_{u_j,\tilde{\phi}_{i,k}} &= h_0(c_{2^{i-1}+2^{j-1}+2^{k-1}-1} + c_{2^{j-1}}c_{u_i,u_k}) + h_j(h_1 c_{u_i,u_k} + h_k c_{2^{i-1}} + h_i c_{2^{k-1}}),\\
\mathcal{A}_{u_k,\tilde{\phi}_{i,j}} &= h_0(c_{2^{i-1}+2^{j-1}+2^{k-1}-1} + c_{2^{k-1}}c_{u_i,u_j}) + h_k(h_1 c_{u_i,u_j} + h_i c_{2^{j-1}} + h_j c_{2^{i-1}}),\\
\mathcal{A}_{u_1,\omega_{i,j,k}} &= h_0(c_{2^{i-1}+2^{j-1}+2^{k-1}-1} + c_{2^{i-1}}c_{u_j,u_k} + c_{2^{j-1}}c_{u_i,u_k} + c_{2^{k-1}}c_{u_i,u_j})\\
&\quad + h_1(h_i c_{u_j,u_k} + h_j c_{u_i,u_k} + h_k c_{u_i,u_j}).
\end{aligned}
$$

Adding these equalities, we get the necessary relation. The rest of them may be proved by analogy.

LEMMA 4. *i) The Massey product $\langle \tilde{\phi}_{2,3,4}, h_0, \omega_{2,3,4} \rangle$ is defined in $E_2^{1,1,*}$, has indeterminacy equal to zero, and it defines an element $\varrho \in \langle \tilde{\phi}_{2,3,4}, h_0, \omega_{2,3,4} \rangle$ which is not equal to zero in $E_2^{1,1,104}$.*

ii) The following equalities hold: $\langle \tilde{\phi}_{2,3,4}, h_0, \omega_{2,3,4} \rangle = \langle \tilde{\phi}_{3,4}, h_0, \psi_{\hat{2},3,4} \rangle = \langle \tilde{\phi}_{2,4}, h_0, \psi_{2,\hat{3},4} \rangle = \langle u_4, h_0, \psi_{\hat{2},\hat{3},4} \rangle = \langle \tilde{\phi}_{2,3}, h_0, \psi_{2,3,\hat{4}} \rangle = \langle u_3, h_0, \psi_{\hat{2},3,\hat{4}} \rangle = \langle u_2, h_0, \psi_{2,\hat{3},\hat{4}} \rangle$ (the indeterminacy of each term is equal to zero).

iii) $h_0\varrho = 0$ in E_2-term of the MASS.

PROOF. (i) The element ϱ belongs to the Massey product $\langle \tilde{\phi}_{2,3,4}, h_0, \omega_{2,3,4} \rangle$. So it has the following decomposition:

$$
\begin{aligned}
\varrho = \big((u_1h_2 + u_2h_1)c_{11} &+ (u_1h_3 + u_3h_1)c_9 + (u_1h_4 + u_4h_1)c_5\big)c_{13}\\
&+ \big((u_2h_3 + u_3h_2)c_2c_8 + (u_2h_4 + u_4h_2)c_2c_4\big)c_{11}\\
&+ \big((u_2h_3 + u_3h_2)c_4c_8 + (u_3h_4 + u_4h_3)c_2c_4\big)c_9\\
&+ (u_2h_4 + u_4h_2)c_4c_8 + (u_3h_4 + u_4h_3)c_2c_8)c_5.
\end{aligned}
$$

We have the following formulas for the first differential:

$$
\begin{aligned}
&d_1\big((c_2c_{11} + c_4c_9 + c_5c_8)c_{13}\big)\\
&\qquad = \varrho + \big((u_2h_3 + u_3h_2)c_8 + (u_2h_4 + u_4h_2)c_4 + (u_3h_4 + u_4h_3)c_2\big)c_{13}\\
&\qquad\qquad + (u_3h_4 + u_4h_3)c_{11}c_2^2 + (u_2h_4 + u_4h_2)c_9c_4^2 + (u_2h_3 + u_3h_2)c_5c_8^2,\\
&d_1(c_{11}) = (u_3h_4 + u_4h_3), \quad d_1(c_9) = (u_2h_4 + u_4h_2), \quad d_1(c_5) = (u_2h_3 + u_3h_2).
\end{aligned}
$$

Then the proof follows from these formulas.

(ii) We have:

$$
\begin{aligned}
d_1\big((c_5c_8 + c_4c_9)c_{13}\big) &= \varrho + \langle u_2, h_0, \psi_{2,\hat{3},\hat{4}} \rangle,\\
d_1\big((c_5c_8 + c_2c_{11})c_{13}\big) &= \varrho + \langle u_3, h_0, \psi_{\hat{2},3,\hat{4}} \rangle,\\
d_1\big((c_4c_9 + c_2c_{11})c_{13}\big) &= \varrho + \langle u_4, h_0, \psi_{\hat{2},\hat{3},4} \rangle.
\end{aligned}
$$

The other relations can be proved the same way.

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(iii) It follows from the formula:

$$d_1(h_1h_2c_{11}c_{13} + h_2h_3c_2c_8c_{11} + h_2h_4c_2c_4c_{11} + h_1h_3c_9c_{13} + h_2h_3c_4c_8c_9 + h_3h_4c_2c_4c_9 + h_1h_4c_5c_{13} + h_2h_4c_4c_5c_8 + h_3h_4c_2c_5c_8 + h_2^2c_4c_8c_{11} + h_3^2c_2c_8c_9 + h_4^2c_2c_4c_5) = h_0\varrho.$$

We call the pairs $(\tilde{\phi}_{2,3,4}, \omega_{2,3,4})$, $(\tilde{\phi}_{3,4}, \psi_{\hat{2},3,4})$, $(\tilde{\phi}_{2,4}, \psi_{2,\hat{3},4})$, $(u_4, \psi_{\hat{2},\hat{3},4})$, $(\tilde{\phi}_{2,3}, \psi_{2,3,\hat{4}})$, $(u_3, \psi_{\hat{2},3,\hat{4}})$, $(u_2, \psi_{2,\hat{3},\hat{4}})$ forbidden. For each forbidden pair (ξ, ζ) the element $\mathcal{A}_{\xi,\zeta}$ is not defined. Now we consider the set of all the elements $\mathcal{A}_{\xi,\zeta}$ for each not forbidden pair $(\xi, \zeta) \in E_2^{0,1,*}$ such that the sum of the t-gradings of ξ and ζ is less then 108. They are not linearly independent (Lemma 3). We choose a basis from them. Then we add one more element which we denote by $\mathcal{A}_{(u_2,\psi_{2,\hat{3},\hat{4}})+(\tilde{\phi}_{2,3,4},\omega_{2,3,4})}$, and which has the following decomposition:

$$\begin{aligned}\mathcal{A}_{(u_2,\psi_{2,\hat{3},\hat{4}})+(\tilde{\phi}_{2,3,4},\omega_{2,3,4})} &= h_0(c_5c_8c_{13} + c_4c_9c_{13}) + h_1h_3c_9c_{13} + h_1h_4c_5c_{13} \\ &\quad + h_2h_3c_8(c_{13} + c_8c_5 + c_4c_9) + h_3^2c_2c_8c_9 + h_4^2c_2c_4c_5 \\ &\quad + h_2h_4c_4(c_{13} + c_8c_5 + c_4c_9) + h_3h_4c_2(c_8c_5 + c_4c_9).\end{aligned}$$

We have obtained a complete system of generators of $E_2^{2,0,t}$, $t < 108$.

LEMMA 5. *The following list :*

1) $\xi\mathcal{A}_{\xi,\eta} = \eta\mathcal{A}_{\xi,\xi}$*;*
2) $\xi\mathcal{A}_{\zeta,\eta} = \zeta\mathcal{A}_{\xi,\eta} = \eta\mathcal{A}_{\xi,\zeta}$*;*
3) $\mathcal{A}_{\xi,\eta}^2 = h_0c^2_{2^{\xi-1}+2^{\eta-1}-1} + h_\xi^2h_\eta^2$*;*
4) $\mathcal{A}_{\xi,\eta}\mathcal{A}_{\xi,\zeta} = h_\xi^2\mathcal{A}_{\eta,\zeta} + h_0\mathcal{A}_{\xi,\mathcal{F}_{\xi,\zeta,\eta}}$*;* $\xi, \eta, \zeta \in E_2^{0,1,t}$, $t < 108$ *(under the condition that all the elements of the formulae are defined) exhausts all the relations between the generators of* $E_2^{0,1,t}$ *and* $E_2^{2,0,t}$*, if sum of their t-gradings is less than 108.*

PROOF. 1) We have:

$$d_1(h_\xi c_{\xi,\eta}) = \xi\mathcal{A}_{\xi,\eta} + \eta\mathcal{A}_{\xi,\xi}.$$

To prove 2) we have analogously:

$$d_1(h_\xi c_{\zeta,\eta} + h_\zeta c_{\xi,\eta}) = \xi\mathcal{A}_{\zeta,\eta} + \zeta\mathcal{A}_{\xi,\eta}.$$

3) and 4) are proved by direct computations.

Now we consider the ANSS for $M\mathrm{Sp}$ in small dimensions continuing the computations of [10].

PROPOSITION 1. *There exists an indecomposable element* $\Omega_1 \in E_2^{1,50}$ *in the* ANSS *whose projection to* E_∞ *of the* MASS *is equal to* $\omega_{2,3,4}$. *It is permanent cycle of the* ANSS *and defines an element* $\Omega_1 \in M\mathrm{Sp}_{49}$ *of order* 2 *in the symplectic cordism ring.*

PROOF. It may be done by direct computations in the ANSS. The fact that Ω_1 is indecomposable follows from its form in the term E_1 of MASS and the fact that it has MASS-filtration degree equal to 1.

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t	Generators
2	u_1
6	u_2
14	u_3
22	ϕ_3 $(= u_1c_5 + u_2c_4 + u_3c_2)$
30	u_4
38	$\tilde{\phi}_{2,4}$ $(= u_1c_9 + u_2c_8 + u_4c_2)$
46	$\tilde{\phi}_{2,4}$ $(= u_1c_{11} + u_3c_8 + u_4c_4)$
50	$\omega_{2,3,4}$ $(= u_2c_{11} + u_3c_9 + u_4c_5)$
54	$\tilde{\phi}_{2,3,4}$ $(= u_1c_{13} + u_2c_4c_8 + u_3c_2c_8 + u_4c_2c_4)$
58	$\psi_{\hat{2},3,4}$ $(= u_1c_5c_9 + u_2(c_{13} + c_4c_9 + c_5c_8) + u_3c_2c_9 + u_4c_2c_5)$
62	u_5
66	$\psi_{2,\hat{3},4}$ $(= u_1c_5c_{11} + u_2c_4c_{11} + u_3(c_{13} + c_2c_{11} + c_5c_8) + u_4c_4c_5)$
70	$\tilde{\phi}_{2,5}$ $(= u_1c_{17} + u_2c_{16} + u_5c_2)$
74	$\psi_{\hat{2},\hat{3},4}$ $(= u_1c_5c_{13} + u_2c_4(c_{13} + c_4c_9 + c_5c_8) + u_3c_2(c_{13} + c_2c_{11} + c_5c_8)$ $+u_4c_2c_4c_5)$
78	$\tilde{\phi}_{3,5}$ $(= u_1c_{19} + u_3c_{16} + u_5c_4)$
82	$\omega_{2,3,5}$ $(= u_2c_{19} + u_3c_{17} + u_5c_5)$, $\psi_{2,3,\hat{4}}$ $(= u_1c_9c_{11} + u_2c_8c_{11} + u_3c_8c_9 + u_4(c_{13} + c_2c_{11} + c_4c_9))$
86	$\tilde{\phi}_{2,3,5}$ $(= u_1c_{21} + u_2c_4c_{16} + u_3c_2c_{16} + u_5c_2c_4)$
90	$\psi_{\hat{2},3,\hat{4}}$ $(= u_1c_9c_{13} + u_2c_8(c_{13} + c_4c_9 + c_5c_8)$ $+u_3c_2c_8c_9 + u_4c_2(c_{13} + c_2c_{11} + c_4c_9))$, $\psi_{\hat{2},3,5}$ $(= u_1c_5c_{17} + u_2(c_{21} + c_4c_{17} + c_5c_{16}) + u_3c_2c_{17} + u_5c_2c_5)$
94	$\tilde{\phi}_{4,5}$ $(= u_1c_{23} + u_4c_{16} + u_5c_8)$
98	$\psi_{2,\hat{3},\hat{4}}$ $(= u_1c_{11}c_{13} + u_2c_4c_8c_{11} + u_3c_8(c_{13} + c_2c_{11} + c_5c_8)$ $+u_4c_4(c_{13} + c_2c_{11} + c_4c_9))$, $\psi_{2,\hat{3},5}$ $(= u_1c_5c_{19} + u_2c_4c_{19} + u_3(c_{21} + c_2c_{19} + c_5c_{16}) + u_5c_4c_5)$, $\omega_{2,4,5}$ $(= u_2c_{23} + u_4c_{17} + u_5c_9)$
102	$\tilde{\phi}_{2,4,5}$ $(= u_1c_{25} + u_2c_8c_{16} + u_4c_2c_{16} + u_5c_2c_8)$
106	$\psi_{\hat{2},\hat{3},5}$ $(= u_1c_5c_{21} + u_2c_4(c_{21} + c_4c_{17} + c_5c_{16}) + u_3c_2(c_{21} + c_2c_{19} + c_5c_{16})$ $+u_5c_2c_4c_5)$, $\psi_{2,3,\hat{5}}$ $(= u_1c_9c_{17} + u_2(c_{25} + c_8c_{17} + c_9c_{16}) + u_4c_2c_{17} + u_5c_2c_9)$, $\omega_{3,4,5}$ $(= u_3c_{23} + u_4c_{19} + u_5c_{11})$

TABLE 1. $E_2^{0,1,t}$ OF THE MASS FOR $t < 108$ (GENERATORS)

PROPOSITION 2. *If $t < 108$ then we have the isomorphism: $E_2^{*,*,t} \cong E_\infty^{*,*,t}$ of the terms of the* MASS.

PROOF. All the elements of $E_2^{*,*,t}$, $t < 108$, except $\varrho \in E^{1,1,104}$, are cycles of higher

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t	16	24	32	40	48	56	64
Generators	$e_4\ (=c_2^2)$	c_6	$e_8\ (=c_4^2)$	$c_{10}, e_4\ (=c_5^2)$	c_{12}	c_{14}	$e_{16}\ (=c_8^2)$

t	72	80	88	96	104
Generators	$c_{18}, e_{18}\ (=c_9^2)$	c_{20}	$c_{22}, e_{22}\ (=c_{11}^2)$	c_{24}	$c_{26}, e_{26}\ (=c_{13}^2)$

TABLE 2. $E_2^{0,0,t}$ OF THE MASS FOR $t < 108$ (GENERATORS)

differentials by dimension reasons. For ϱ it follows because it belongs to the Massey product $\langle \tilde{\phi}_{2,3,4}, h_0, \omega_{2,3,4}\rangle$.

We denote by $\pi_s^2(x)$ the projection of an element $x \in E_2^{s,t}$ of the ANSS to E_2 of the MASS. We choose an element $z_{13} \in E_2^{0,52}$ of the ANSS such that $\pi_0^2(z_{13}) = \mathcal{A}_{u_1,\Omega_1}$. Using the action of the Landweber-Novikov operations we prove that $d_3(z_{13}) = u_1^2\Omega_1$.

We choose the generators of E_2-term of the ANSS:

$$y_{10} \in E_2^{0,40},\ y'_{10} \in E_2^{0,40},\ y_{12} \in E_2^{0,48},\ y_{14} \in E_2^{0,56},\ y_{16} \in E_2^{0,64},\ y_{18} \in E_2^{0,72},\ y'_{18} \in E_2^{0,72},$$
$$y_{20} \in E_2^{0,80},\ y_{22} \in E_2^{0,88},\ y'_{22} \in E_2^{0,88},\ y_{26} \in E_2^{0,104},\ y'_{26} \in E_2^{0,104},$$

such that

$$\pi_0^2(y_{10}) = c_{10},\ \pi_0^2(y'_{10}) = c_5^2 + c_{10} + c_2^2c_6,\ \pi_0^2(y_{12}) = c_{12},\ \pi_0^2(y_{14}) = c_{14},\ \pi_0^2(y_{16}) = c_8^2,$$
$$\pi_0^2(y_{18}) = c_{18},\ \pi_0^2(y'_{18}) = c_9^2 + c_2^2c_{14} + c_4^2(c_{10} + c_5^2) + c_6^3,\ \pi_0^2(y_{20}) = c_{20},\ \pi_0^2(y_{22}) = c_{22},$$
$$\pi_0^2(y'_{22}) = c_{11}^2 + c_{14}(c_2^4 + c_4^2) + c_{10}(c_{12} + c_2^2c_4^2 + c_2^6) + c_6(c_2^4c_4^2 + c_8^2 + c_2^8),$$
$$\pi_0^2(y_{26}) = c_{26} + c_{10}c_2^8,\ \pi_0^2(y'_{26}) = c_{13}^2 + c_{11}^2c_2^2 + c_5^2c_8^2 + c_4^2c_9^2.$$

Using again the Landweber-Novikov operations we prove the following formulae for the differential d_3 modulo elements having nonzero MASS-filtration degree and monomials containing u_1:

$$d_3(y_{10}) = u_3^3,\quad d_3(y'_{10}) = u_2^2u_4 + u_3^3 + u_2u_3\Phi_3,\quad d_3(y_{12}) = u_2u_3u_4 + u_3^2\Phi_3,$$
$$d_3(y_{14}) = u_2^2\Phi_6 + u_2u_3\Phi_5 + u_2\Phi_3u_4,\quad d_3(y_{16}) = u_2u_4^2,\quad d_3(y_{18}) = u_3u_4^2,$$
$$d_3(y'_{18}) = u_3u_4^2 + u_2^2u_5 + u_2u_3\Phi_7 + u_2\Phi_3\Phi_6 + u_2u_4\Phi_5,$$
$$d_3(y_{20}) = u_2u_3u_5 + u_2u_4\Phi_6 + \Phi_3u_4^2 + u_2u_3u_4y_4^2,$$
$$d_3(y_{22}) = u_3^2u_5 + u_3u_4\Phi_6 + u_4^3 + u_3^2u_4y_4^2,$$
$$d_3(y'_{22}) = u_3^2u_5 + u_3u_4\Phi_6 + u_4^3 + u_3^2u_4y_4^2 + u_3\Phi_5^2,\quad d_3(y_{26}) = u_3\Phi_6^2,\quad d_3(y'_{26}) = u_2\Omega_1^2.$$

PROPOSITION 3. *In $M\mathrm{Sp}_*$ the element $\theta_1\Phi_7\Omega_1$ of dimension* 103 *is not equal to zero.*

PROOF. Let $x \in E_2^{0,104}$ be an arbitrary element with the MASS-filtration degree at least 2 and such that $d_3(x) = u_1\Phi_7\Omega_1$. Denote by χ the projection of this element into the term E_2 of MASS. From the fact that $S_{13}(u_1\Phi_7\Omega_1) = u_1\Omega_1$ and $d_3(z_{13}) = u_1^2\Omega_1$ we

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obtain that $S_{13}(\chi) = A_{u_1,\Omega_1}$. It follows from the description of E_2-term of the MASS given earlier that there is no such element. If the element killing $u_1\Phi_7\Omega_1$ has the MASS-filtration degree equal to zero then it may be only y_{26} or y'_{26} as the only multiplicatively indecomposable. From the formulae $d_3(y_{26}) = u_3\Phi_6^2$, $d_3(y'_{26}) = u_2\Omega_1^2$ valid modulo elements of the MASS-filtration degree greater than zero and monomials containing u_1, it follows that this is impossible.

Let α and β be two elements of order 2 in $M\mathrm{Sp}_*$ so that the Massey product $\langle\alpha, 2, \beta\rangle$ is defined.

PROPOSITION 4. *If α and β both have the Adams filtration equal to 1 and $h_0\alpha = 0, h_0\beta = 0$ in E_2 of the Adams spectral sequence for $M\mathrm{Sp}$, then we have: $2\langle\alpha, 2, \beta\rangle = \theta_1\alpha\beta$.*

PROOF. If follows easy from the description of the term E_2 of the classical Adams spectral sequence for $M\mathrm{Sp}$ given in [3] and convergence of the Massey products [2].

PROOF OF THE MAIN THEOREM. Let Γ_1 belongs to the Massey product $\langle\Phi_7, 2, \Omega_1\rangle$. It follows from the Propositions 3 and 4 that it has order 4 and $2\Gamma_1 = \theta_1\Phi_7\Omega_1 \neq 0$. Let $\Gamma_i, i = 2, 3, \ldots$, belongs to the Massey product $\langle\Phi_{6+i}, 2, \Omega_1\rangle$, then it has order 4 and $2\Gamma_i = \theta_1\Phi_{6+i}\Omega_1 \neq 0$. It follows from the action of the operation $S_{2(i-1)}$ on $\theta_1\Phi_{6+i}\Omega_1$ and for small values of i from the computations in low dimensions.

REFERENCES

1. J. F. Adams, *Stable homotopy and generalised homology*, The University of Chicago Press, Chicago and London, 1974.
2. J. C. Alexander, *Cobordism Massey products,* Trans. Amer. Math. Soc. **166**(1972), 197–214.
3. L. N. Ivanovskii, *Cohomology of algebras with simple systems of generators,* Siberian Math. J. **14**(1973), 864–874.
4. S. O. Kochman, *The symplectic cobordism ring I,* Mem. Amer. Math. Soc. **228**(1980).
5. ———, *The symplectic cobordism ring II,* Mem. Amer. Math. Soc. **271**(1982).
6. ———, *The Hurewicz image of Ray's elements in $M\mathrm{Sp}_*$,* Proc. Amer. Math. Soc. **94**(1985), 715–717.
7. S. P. Novikov, *The methods of algebraic topology from the viewpoint of cobordism theories,* Math. USSR-Izv. **1**(1967), 827–913.
8. Nigel Ray, *Indecomposable in* Tors $M\mathrm{Sp}_*$, Topology **10**(1971), 261–270.
9. R. E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, 1968.
10. V. V. Vershinin, *Computation of the symplectic cobordism ring in dimensions up to 32 and nontriviality of majority of triple products of Ray elements,* Siberian Math. J. **24**(1983), 41–51.
11. ———, *Cobordism and spectral sequences*, Transl. Math. Monographs **130**, Amer. Math. Soc., Providence, 1993.

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