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ZT-SUBGROUPS OF SHARPLY 3-TRANSITIVE GROUPS

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A permutation group G operating on a set M is called a ZT-group (Zassenhaus transitive group) if G has the properties (i) and (ii):

(i) G operates 2-transitively on M;

(ii) $G_{a,b} \neq \{\mathrm{id}\}$ and $G_{a,b,c} = \{\mathrm{id}\}$ for distinct elements $a, b, c \in M$.

Here $G_{a,b} = \{\alpha \in G \mid \alpha(a) = a \text{ and } \alpha(b) = b\}$ denotes the stabilizer of $\{a, b\}$, and $G_{a,b,c}$ the stabilizer of $\{a, b, c\}$, respectively.

In this paper we are looking for all ZT-groups which are subgroups of sharply 3-transitive groups. It is shown that such ZT-groups can be uniquely described by means of certain subgroups B of the multiplicative group $(F^*, \cdot)$ of the KT-field $(F, +, \cdot, \sigma)$ which characterizes the underlying sharply 3-transitive group.

In §1 the basic notions and properties of sharply 3-transitive groups are given.

In §2 the above mentioned ZT-groups are described.

In §3 a method of constructing sharply 3-transitive groups and their ZT-subgroups is treated. It is shown that the smallest ZT-subgroups of these examples are all isomorphic to PSL$(2, K)$ even if the underlying sharply 3-transitive group is *not* isomorphic to PGL$(2, K)$. In the finite case this was already known by Zassenhaus (**6**), where he determined all finite sharply 3-transitive groups and their ZT-subgroups.

1. Basic notions and relations

Definition 1.1. A set F with two binary operations $(+)$ and $(\cdot)$ is called a *neardomain* ("Fastbereich") if the following axioms are valid:

Fb 1 $(F,+)$ is a loop (with neutral element 0)

Fb 2 $a + b = 0 \Rightarrow b + a = 0$

Fb 3 $(F^*, \cdot)$ is a group (with neutral element 1; $F^* := F\backslash\{0\}$).

Fb 4 $0 \cdot a = 0$, for every $a \in F$.

Fb 5 $a \cdot (b + c) = ab + ac$ for all $a, b, c \in F$.

Fb 6 For every pair of elements $a, b \in F$ there exists an element $d_{a,b} \in F^*$, such that

$$a + (b + x) = (a + b) + d_{a,b} \cdot x$$

for every $x \in F$.

Remark. Each sharply 2-transitive permutation group can be written as the group of linear transformations $x \to a + mx$, $m \neq 0$, of a neardomain $(F, +, \cdot)$. (Karzel (**2**)).

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Definition 1.2. $(F,+,\cdot,\sigma)$ is called a *KT-field*, if the axioms KT 1 and KT 2 are valid:

KT 1 $(F,+,\cdot)$ is a neardomain.

KT 2 σ is an involutory automorphism of the multiplicative group $(F^*,\cdot)$ which satisfies the functional equation:

$$\sigma(1+\sigma(x)) = 1-\sigma(1+x) \qquad \text{for all } x \in F\backslash\{0,1\}.$$

As one easily can verify the transformations $\alpha,\beta:\bar{F}:=F\cup\{\infty\}\to\bar{F}$, where ∞ denotes an element not in F and

$$\alpha:\begin{cases} \bar{F}\to\bar{F} \\ x\to a+mx, & a\in F, m\in F^* \\ \infty\to\infty \end{cases}$$

$$\beta:\begin{cases} \bar{F}\to\bar{F} \\ x\to a+\sigma(b+mx), & a,b\in F, m\in F^* \\ \infty\to a \\ -m^{-1}b\to\infty \end{cases}$$

form a group $T_3(\bar{F})$ which operates sharply 3-transitively on $\bar{F}$. Conversely each sharply 3-transitive group is isomorphic as a permutation group to the group $T_3(\bar{F})$ of a uniquely determined KT-field (see (4)).

Therefore in the following we will consider each sharply 3-transitive group as being represented in the form $T_3(\bar{F})$.

2. ZT-subgroups of $T_3(\bar{F})$

The main result will be the following theorem:

Theorem 2.1. *Let $(F,+,\cdot,\sigma)$ be a KT-field. If B is a subgroup of $(F^*,\cdot)$ such that $R\subseteq B$, $D\subseteq B$ and $\sigma(B)\subseteq B$, where $R:=\{a\sigma(a^{-1})\in F^* \mid a\in F^*\}$ and $D=\{d_{a,b}\in F^* \mid a,b\in F\}$ then the transformations of the form:*

$$\alpha:\begin{cases} x\to a+mx, & a\in F, m\in B \\ \infty\to\infty \end{cases}$$

$$\beta:\begin{cases} x\to a-\sigma(b+mx), & a,b\in F, m\in B \\ -m^{-1}b\to\infty \\ \infty\to a \end{cases}$$

constitute a subgroup U of $T_3(\bar{F})$ which is Zassenhaus transitive.

Conversely, to each $U\leqslant T_3(\bar{F})$ which is a ZT-group, there exists a subgroup $B\leqslant F^*$ with $R\subseteq B$, $D\subseteq B$ and $\sigma(B)\subseteq B$ such that all elements of U have the form α or β.

Proof. For the first part of the theorem, let $\alpha_i: x\to a_i+m_ix$ and $\beta_i: x\to a_i-\sigma(b_i+m_ix)$ for $i=1,2$ with $a_i, b_i\in F$ and $m_i\in B$.

Then we have:

$$a_i^{-1}: x \to -m_i^{-1}a_i + m_i^{-1}x$$

$$\beta_i^{-1}: x \to -m_i^{-1}b_i - \sigma[\sigma(-m_i^{-1})a_i + \sigma(m_i^{-1})x]$$

$$\alpha_1\alpha_2: x \to (a_1 + m_1a_2) + d_{a_1,m_1a_2}m_1m_2x$$

$$\alpha_1\beta_2: x \to (a_1 + m_1a_2) \dot{-} \sigma[\sigma(d_{a_1,m_1a_2})\sigma(m_1)b_2 + \sigma(d_{a_1,m_1a_2})\sigma(m_1)m_2x]$$

$$\beta_1\alpha_2: x \to a_1 - \sigma((b_1 + m_1a_2) + d_{b_1,m_1a_2}m_1m_2x)$$

$$\beta_1\beta_2: x \to [a_1 - \sigma(t)] - \sigma[(-dt + dt\sigma(t^{-1})\sigma(d_{b_1,m_1a_2})\sigma(m_1)b_2) + d_{-dt,dt\sigma(t^{-1})\sigma(d_{b_1,m_1a_2})\sigma(m_1)b_2}dt\sigma(t^{-1})\sigma(d_{b_1,a_1m_2})\sigma(m_1)m_2x]$$

where $t := b_1 + m_1a_2 \neq 0$ and $d := \sigma(d_{a_1,\sigma(t)})$.

For $t = 0$ we get:

$$\beta_1\beta_2 :\to [a_1 + \sigma(m_1)b_2] + d_{a_1,\sigma(m_1)b_2}\sigma(m_1)m_2x.$$

Because of the properties of B the inverse and the products are all of the form α or β.

Conversely, let U be a subgroup of $T_3(\bar{F})$ which is Zassenhaus-transitive. Since $U_{\infty,0}$ is a subgroup consisting of permutations of the form $\alpha: x \to mx$ the set A

$$A := \{m \in F^* \mid m = \alpha(1) \text{ with } \alpha \in U_{\infty,0}\}$$

is a subgroup of F^*. We have to show that A possesses the required properties.

U_∞ consists of transformations of the form $x \to a + mx$ with $m \in A$. Because of the transitivity of U_∞ there exists to each $b \in F$ an $\alpha \in U_\infty$ such that $\alpha(0) = b$. Thus

$$U_\infty = \{x \to a + mx \mid a \in F \text{ and } m \in A\}.$$

We define now:

$$H := \{n \in F^* \mid \exists a, b \in F \text{ such that } \beta \in U, \beta: x \to a - \sigma(b + nx)\}.$$

Since U is a ZT-group there exists a transformation $\tau \in U$ with $\tau(0) = \infty$ and $\tau(\infty) = 0$. Therefore τ has the form $\tau: x \to -u\sigma(x)$ where $\tau(1) = u$ and $\sigma(u) \in H$.

Now take some $\beta \in U$, $\beta(x) = a - \sigma(b + nx)$ and define $\delta(x) = -a + x$. We have $\delta \in U_\infty$ and $\tau\delta\beta(x) = ub + unx$ with $\tau\delta\beta \in U_\infty$, i.e. $un \in A$. Hence $uH \subseteq A$. Also for each $m \in A$ the permutation $x \to \tau(mx) = -\sigma(\sigma(u)mx)$ lies in U from which $\sigma(u)A \subseteq H$. This implies that $u\sigma(u)A \subseteq uH \subseteq A$ whence $u\sigma(u) \in A$ and so $uH = A$.

Furthermore the above considerations show that for any $n \in H$ and each $a, b \in F$ the permutation $\beta(x) = a - \sigma(b + nx)$ belongs to U. The inclusion $D \subseteq A$ follows directly from $\alpha_1\alpha_2(x) = (a_1 + a_2) + d_{a_1,a_2}x$ for $\alpha_i(x) = \alpha_i + x$ and $i = 1,2$.

Now for an arbitrary $\mu \in U_{\infty,0}$, say $\mu(x) = mx$, $m \in A$ we get

$$\tau^{-1}\mu\tau(x) = \sigma(u^{-1}mu\sigma(x)) = \sigma(u^{-1})\sigma(m)\sigma(u)x$$

whence $\sigma(u^{-1})\sigma(m)\sigma(u) \in A$ and so $\sigma(m) \in \sigma(u)A\sigma(u^{-1}) = u^{-1}Au$, on account of $u\sigma(u) \in A$. Thus $\sigma(A) \subseteq u^{-1}Au$.

Finally we show that $\mu \in A$:

For this we consider $\beta(x) = a + \tau(x)$. Then

$$\tau\beta(x) = -u\sigma(a - u\sigma(x))$$
$$= -u\sigma(a) - \sigma(-\sigma(u)a + \sigma(u)a\sigma(a^{-1})\sigma(u)x)$$

so that $\sigma(u)a\sigma(a^{-1})\sigma(u) \in H = u^{-1}A$ and hence $a\sigma(a^{-1})\sigma(u) \in \sigma(u^{-1})u^{-1}A$ for each $a \in F^*$.

If we put $a = u$ we get $u \in A$ and therefore $A = H$.
Together with $\sigma(A) \subseteq u^{-1}Au$ we get $\sigma(A) = A$. From this follows $a\sigma(a^{-1}) \in A\sigma(u)^{-1} = A$ for each $a \in F^*$, whence $R \subseteq A$.□

By computing one gets the

Corollary 2.2. *A ZT-subgroup U of $T_3(\bar{F})$ is normal if and only if the corresponding subgroup B of F^* is normal. In this case $T_3(\bar{F})/U \cong F^*/B$.*

3. Examples

The following theorem of Kerby (12.7 in (3)) shows the way to construct KT-fields. To my knowledge all examples so far known are made in this manner.

Theorem 3.1. *Let $(F, +, *)$ be a commutative field and let A be a subgroup of $(F^*, *)$ such that*

(i) $Q = \{a * a \mid a \in F^*\} \subseteq A$

(ii) *There exists a monomorphism $\pi: F^*/A \to \mathrm{Aut}(F, +, *)$.*

(iii) *$\tau(x) \in x * A$ for all $x \in F^*$ and all $\tau \in \pi(F^*/A)$.*

Let $\kappa: F^* \to F^*/A$ denote the canonical homomorphism. Then $(F, +, \circ)$

$$a \circ b = \begin{cases} 0 & \text{for } a = 0 \\ a * a_\varphi(b) & \text{with } a_\varphi = \pi\kappa(a) \end{cases}$$

is a (strongly coupled Dickson) nearfield and $(F, +, \circ, \sigma)$ is a KT-field with $\sigma(a) = a^{-1}$ (inverse with respect to $(*)$).

For instance (see (3), p. 67) take an arbitrary finite or infinite index set I. Further let K be a commutative field and $F = K(t_i)_{i \in I}$ the field of rational functions in $|I|$ transcendental indeterminates t_i and $\mathrm{grad}_i f = \mathrm{grad}_i\,(f_1/f_2) = \mathrm{grad}_i f_1(t_i) - \mathrm{grad}_i f_2(t_i)$ the degree of the polynomials f_1, f_2 with respect to t_i.

If we choose $\tau_i(k) = k$ for $k \in K$, $\tau_i(t_j) = t_j$ for $i \neq j$ and $\tau_i(t_i) = 1 - t_i$, then $F^\varphi := (F, +, \circ)$ with

$$f \circ g := f \cdot f_\varphi(g) \qquad \text{where } f_\varphi := \prod_{i \in I} \tau_i^{\mathrm{grad}_i f}$$

is a Dickson nearfield and $(F, +, \circ, \sigma)$ with $\sigma(f) = f^{-1}$ (inverse with respect to the multiplication $(\cdot)$ of the commutative field) is a KT-field.
The subgroup A of Th. 3.1 is here

$$A = \{f \in F \mid \mathrm{grad}_i f \equiv 0 \ (\mathrm{mod}\ 2) \text{ for all } i \in I\}.$$

For the rest of the paragraph let F^φ denote the KT-field which is constructed with the help of a commutative field F according to 3.1. Those ZT-subgroups $U \leq T_3(\overline{F^\varphi})$ which are at the same time subgroups of PGL(2, F) are characterised by the

Proposition 3.2. *Let F^φ be a KT-field derived according to* 3.1 *from a commutative field F. A ZT-subgroup $U \leq T_3(F^\varphi)$ is simultaneously a subgroup of PGL(2, F) if and only if the corresponding subgroup $B \leq (F^*, \circ)$ satisfies $B \subseteq$* $\operatorname{Ker}\varphi := \{z \in F^* \mid z_\varphi = \mathrm{id}\}$.

Proof. We have to show that the mapping Ψ

$$\psi: \begin{cases} U & \rightarrow \quad \mathrm{PGL}(2, F) \\ \alpha: x \rightarrow a + m \circ x & \rightarrow \quad x \rightarrow a + m * x \\ \beta: x \rightarrow a - \sigma(b + m \circ x) & \rightarrow \quad x \rightarrow a - (b + m * x)^{-1} \end{cases}$$

is a homomorphism. If $B \subseteq \operatorname{Ker}\varphi$ then $m \circ x = m * x$ for all $m \in B$. Denoting the inverse of a with respect to ($\circ$) by $a^{\underline{-1}}$ and with respect to ($*$) by a^{-1} we get because of $a^{\underline{-1}} = a_\varphi^{-1}(a^{-1})$:

$$t \circ \sigma(t^{\underline{-1}}) = t * t_\varphi \sigma(t_\varphi^{-1}(t^{-1})) = t * t_\varphi[t_\varphi^{-1}(t^{-1})]^{-1}$$
$$= t * t_\varphi t_\varphi^{-1}(t) = t * t.$$

The formulae in the proof of Theorem 2.1 show that ψ is a homomorphism. If, on the other hand, ψ is a homomorphism then $B \subseteq \operatorname{Ker}_\varphi$.□

In all examples furnished by 3.1 the sets R and Q are equal:

$$R = \{a \circ \sigma(a^{\underline{-1}}) \mid a \in F^*\} = \{a * a \mid a \in F^*\} = Q$$

and $R = Q \subseteq A = \operatorname{Ker}_\varphi$. Moreover $Q \leq (F^*, *)$, $\sigma(Q) \subseteq Q$, $Q \leq (F^*, \circ)$. So $R = Q$ satisfies the conditions of 2.1 and $B = R$ supplies the smallest ZT-subgroup of $T_3(\overline{F^\varphi})$. It is well known that the smallest ZT-subgroup of PGL(2, F) is PSL(2, F). Thus 2.1, 3.1 and 3.2 give:

Proposition 3.3. *The smallest ZT-subgroup U of a sharply* 3-*transitive group $T_3(\overline{F^\varphi})$ where F^φ is constructed as in* 3.1, *is isomorphic to PSL*(2, *F*).

Finite KT-fields F^φ possess only two subgroups $B \leq (F^*, \circ)$ relevant to 2.1 namely F^* and R if $|F|$ is odd and only one such subgroup namely F^* if $|F|$ is even (**6**).

In order to get examples of ZT-groups which are not simultaneously ZT-subgroups of PGL(2, F) we have to look for subgroups $B \leq (F^*, \circ)$ which satisfy $\sigma(B) = B$, $R \subseteq B$ but $B \not\subseteq \operatorname{Ker}\varphi$.

We mention here only two possibilities:

I. Let $(F, +, \cdot)$ be a commutative field, $A \leq F^*$ and $\tau \in \operatorname{Aut}(F, +, \cdot)$ such that: $\tau^2 = \mathrm{id}$, $\tau \neq \mathrm{id}$, $\tau(A) = A$ and $[F^* : A] = 2$. Then $(F, +, \circ, \sigma)$ is a KT-field where $\sigma(a) = a^{-1}$ and

$$a \circ b = \begin{cases} a \cdot b & \text{if } a \in A \\ a \cdot \tau(b) & \text{if } a \notin A \end{cases} \qquad \text{(see (4), p. 232).}$$

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If we choose τ such that the fixed point field $F_\tau \not\subset A \cup \{0\}$ there is a $t \in F_\tau \backslash A$. We define $B = Q \cup Qt$. The set B is a group with $Q = R \subseteq B$ and $\sigma(B) \subseteq B$ but $B \not\subset A = \mathrm{Ker}_\varphi$.

II. Let F be a KT-field constructed according to 3.1 such that $A = \mathrm{Ker}\,\varphi$ is a commutative subgroup of index $[F^* : A] = 2^n$, $n \geq 2$. (More details of this construction can be found in Kerby (3; p. 67).) Then one can easily find a set B with the properties:

$$(B, \circ) \leq (F^*, \circ)$$
$$(B, \cdot) \leq (F^*, \cdot)$$
$$A \subseteq B$$
$$[F^* : B] = 2^m < 2^n = [F^* : A].$$

Thus $B \not\subset \mathrm{Ker}\,\varphi$, $\sigma(B) = B$, $R \subseteq B$ and B satisfies the conditions of 2.1.

For instance let $|I| \geq 2$ be as in the example following Theorem 3.1 and take for B:

$$B = \{f \in K(t_i)_{i \in I} \mid \mathrm{grad}_j f \equiv 0 \ (\mathrm{mod}\ 2) \text{ with } j \in J\},$$

where J is a proper subset of I.

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