Can. J. Math., Vol. XXXI, No. 5, 1979 pp. 961–985

BLOCK-FINITE ORTHOMODULAR LATTICES

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Introduction. Every orthomodular lattice (abbreviated: OML) is the union of its maximal Boolean subalgebras (blocks). The question thus arises how conversely Boolean algebras can be amalgamated in order to obtain an OML of which the given Boolean algebras are the blocks. This question we deal with in the present paper.

The problem was first investigated by Greechie [**6**, **7**, **8**, **9**]. His technique of pasting [**6**] will also play an important role in this paper. A case solved completely by Greechie [**9**] is the case that any two blocks intersect either in the bounds only or have the bounds, an atom and its complement in common. This is, of course, a very special situation. The more surprising it is that Greechie's methods, if skillfully applied, yield considerable insight into the structure of OMLs and provide a seemingly unexhaustible source for counter-examples.

A closely related problem was considered by G. Kalmbach [**11**]. Her notion of a bundle of Boolean algebras gives a necessary and sufficient condition for the union of Boolean algebras to be an OML and has the interesting consequence that every lattice is a sublattice of an OML. A drawback of her method for our present purposes is that the OML constructed from a bundle of Boolean algebras may have "hidden blocks", i.e. blocks which do not occur in the given bundle. For example, a totally non-atomic block may be hidden among the atomic blocks of the lattice of all closed subspaces of an infinite-dimensional Hilbert space. Thus a bundle of Boolean algebras may not directly describe the block-structure of the OML obtained from it.

In this paper we start investigating the interaction of the blocks of an arbitrary OML with finitely many blocks. Following a suggestion by B. Banaschewski we call such OMLs *block-finite*. The restriction to block-finite OMLs is essential since almost all our proofs proceed by induction on the number of blocks, making use of techniques developed in [**3**]. The key notion of this paper is that of a *path* (Section 4). This is a finite sequence of blocks the union of any two consecutive members of which form a subalgebra and hence intersect in a prescribed way. Depending on how "good" this intersection is we distinguish between proper and strictly proper paths. The main results of the general theory (Section 4, statements 4.4, 4.5 and 4.8) can then be described as follows: Any two blocks in a block-finite OML can be joined by a proper path. The relation "the blocks A and B can be joined by a strictly

Received March 16, 1978. The author gratefully acknowledges financial support of this work by the National Research Council of Canada, Grant A 2985.

https://doi.org/10.4153/CJM-1979-090-6 Published online by Cambridge University Press

proper path" splits the OML into subalgebras which, up to a common Boolean factor, intersect trivially. The third result allows us to single out certain subalgebras which can be represented as a non-trivial direct product.

The first three sections contain preliminary material. The results of these sections are essentially known, but I found it desirable to include them to provide the necessary frame for the later results. The next four sections solve the initially stated problem in special cases, Section 5 for a special type of OMLs which we call *line-like* and Sections 6, 7 and 8 in the case of OMLs with three, four and five blocks respectively. The final section contains two results about block-finite OMLs which are independent of the general techniques developed in this paper; they are both consequences of [**3**].

1. Generalities. In this section we recall some basic definitions and some techniques we have developed in [**3**].

An *ortholattice* (abbreviated: OL) is an algebra $(L; \vee, \wedge, ', 0, 1)$, where $(L; \vee, \wedge)$ is a lattice with bounds 0, 1 and where $a \to a'$ is an orthocomplementation, i.e. an anti-monotone complementation of period 2. An *orthomodular lattice* (abbreviated: OML) is an OL satisfying the orthomodular law:

$$\text{if } a \leqq b \text{ then } a \vee (a' \wedge b) = b.$$

For basic information about OMLs see [**1**, p. 55 ff; **5**; **10**].

For an element a of and OML L define $a^0 = a$ and $a^1 = a'$. Define the commutator $\gamma(A)$ of a finite subset A of L by:

$$\gamma(A) = \bigwedge_{\alpha \in 2^A} \bigvee_{a \in A} a^{\alpha(a)}.$$

We write $\gamma(a_1, a_2, \ldots, a_n)$ instead of $\gamma(\{a_1, a_2, \ldots a_n\})$. The elements a, b of an OML L are said to commute, in symbols: aCb, if and only if $\gamma(a, b) = 0$. The relation C is reflexive, in fact satisfies the stronger condition that $a \leqq b$ implies aCb, it is symmetric and for every element $a \in L$ the set $C(a)$ of all elements commuting with a is a subalgebra of L. The center $C(L) = \bigcap \{C(a) | a \in L\}$ is the set of all elements of L which commute with every element of L. $C(L)$ is a Boolean subalgebra of L. L is irreducible if and only if $C(L) = \{0, 1\}$ and $0 \neq 1$. (Irreducible in this paper always means directly irreducible.)

A *block* of an OML L is a maximal Boolean subalgebra of L. The blocks can also be characterized as the maximal sets of pairwise commuting elements of L. $\mathfrak{A}(L)$ is the set of all blocks of L. Clearly $\bigcup \mathfrak{A}(L) = L$ and $\bigcap \mathfrak{A}(L) = C(L)$. L is said to be *block-finite* (Banaschewski) if and only if $\mathfrak{A}(L)$ is finite. $\Omega(L)$ is the set of all $\mathfrak{B} \subseteq \mathfrak{A}(L)$ satisfying $\bigcap \mathfrak{B} \nsubseteq \bigcup (\mathfrak{A}(L) - \mathfrak{B})$. Here we define the union of the empty subset of $\mathfrak{A}(L)$ to be $\{0, 1\}$ and the intersection of the empty subset of $\mathfrak{A}(L)$ to be L, so that $\mathfrak{A}(L) \in \Omega(L)$ if and only if $C(L) \neq \{0, 1\}$. In particular, if L is irreducible then $\mathfrak{A}(L) \notin \Omega(L)$.

The following two results ([**3**], (2.1) and Theorem 1) are the principal tools applied in this paper.

https://doi.org/10.4153/CJM-1979-090-6 Published online by Cambridge University Press

(1.1) If L is an OML, $\mathfrak{B} \in \Omega(L)$ and $a \in (\bigcap \mathfrak{B}) - \bigcup (\mathfrak{A}(L) - \mathfrak{B})$ then $C(a) = \bigcup \mathfrak{B}$, in particular $\bigcup \mathfrak{B}$ is a subalgebra of L. The blocks of this subalgebra are exactly the elements of $\mathfrak{B}$.

(1.2) Every block-finite OML L is isomorphic with a direct product $B \times L_1 \times L_2 \times \ldots \times L_n$ $(n \geqq 0)$, where B is a Boolean algebra and $L_1, L_2, \ldots L_n$ are irreducible OMLs with at least two blocks each.

These results in many cases provide the induction step in inductive proofs on the number of blocks of a block-finite OML. The first relevant fact for this is that the blocks of a product of two OMLs are the products of the blocks of the factors. Thus, if in the factorization (1.2) of L the number n is at least two, each of the factors L_i has fewer blocks than L and (1.2) allows the induction step provided the property to be proved is preserved under the formation of products. Boolean factors usually do not cause any difficulties. Thus if $n = 1$ in the direct factorization (1.2) we may usually assume that L is irreducible. The validity of the property to be proved then frequently depends on a set $\mathfrak{B}$ of blocks satisfying $\bigcap \mathfrak{B} \neq \{0, 1\}$ only. As is easily seen every such set $\mathfrak{B}$ is contained in a set $\mathfrak{B}' \in \Omega(L)$. By (1.1) and irreducibility of L, $\bigcup \mathfrak{B}'$ is a subalgebra with fewer blocks than L and this again allows an inductive argument.

A third useful observation is the following, which belongs to the folklore of the subject.

(1.3) If a Boolean algebra B is the (set-theoretical) union of the subalgebras B_1 and B_2 then $B = B_1$ or $B = B_2$.

We will apply mainly the following consequence of (1.3).

(1.4) If B is a Boolean subalgebra of an OML L and L_1, L_2 are arbitrary subalgebras of L such that $B \subseteq L_1 \cup L_2$, then $B \subseteq L_1$ or $B \subseteq L_2$.

Finally, we will make use of the following main result of [**3**].

(1.5) Every finitely generated block-finite OML is finite.

2. Pasting. R. J. Greechie [**6**, **7**, **8**, **9**] has given several constructions to obtain OMLs by pasting simpler ones. Since his pasting construction [**6**], p. 212 ff restricted to the principal sections also plays a role in our present context we recall the main facts here. The construction presented here is, in fact, somewhat more general in that it includes the pasting of arbitarily many OMLs as opposed to Greechie's two. This requires some additional consideration.

(2.1) *Let L be an* OML, *$(L_i)_{i \in I}$ a family of subalgebras of L and $0 \neq c \in \bigcap_{i \in I} L_i$. Assume that the following two conditions are satisfied:*

(1) $\bigcup_{i \in I} L_i = L$,

(2) *for all $i, j \in I$ with $i \neq j$: $L_i \cap L_j = [0, c'] \cup [c, 1]$.*

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Then the following five conditions are equivalent:

(a) *if* $a, b \in L$ *and* $a \leqq b$ *then there exists* $i \in I$ *such that* $a, b \in L_i$,

(b) *if* $a, b \in L$ *and* aCb *then there exists* $i \in I$ *such that* $a, b \in L_i$,

(c) *if* $B \in \mathfrak{A}(L)$ *then there exists* $i \in I$ *such that* $B \subseteq L_i$,

(d) *for all* $i, j \in I$, $L_i \cup L_j$ *is a subalgebra of* L,

(e) *for every non-empty set* $J \subseteq I$, $\bigcup_{j \in J} L_j$ *is a subalgebra of* L.

Proof. (a) $\Rightarrow$(d). It is obviously enough to show that $a \in L_i - L_j, b \in L_j - L_i$ imply $a \vee b \in L_i \cup L_j$. By (a) there exist $k, l \in I$ such that $a, a \vee b \in L_k$ and $b, a \vee b \in L_l$. Assume first that $i \neq k$. Then we have by (2) that $a \leqq c'$ or $c \leqq a$. But $a \leqq c'$ would by (2) imply that $a \in L_j$, contrary to our assumption. It follows that $c \leqq a \leqq a \vee b$, hence by (2) that

$$a \vee b \in L_i \cap L_j \subseteq L_i \cup L_j.$$

The case $j \neq l$ follows by symmetry. We may thus assume that $i = k$ and $j = l$ and hence that $k \neq l$. But then $a \vee b \in L_k \cap L_l$ implies by (2) that $a \vee b \leqq c'$ or $c \leqq a \vee b$, in both cases, again by (2), that $a \vee b \in L_i \cap L_j \subseteq L_i \cup L_j$.

(d) $\Rightarrow$ (e). This is trivial since any two elements of $\bigcup_{j \in J} L_j$ belong to some union $L_i \cup L_j$ with $i, j \in J$.

(e) $\Rightarrow$ (c). If $B \subseteq \bigcap_{i \in I} L_i$ there is nothing to prove. If not there exists $a \in B - \bigcap_{i \in I} L_i$ and it follows from (1) and (2) that there exists exactly one index $i \in I$ with $a \in L_i$. Since

$$B \subseteq L_i \cup \bigcup \{L_j | j \neq i\}$$

and since by (e), $\bigcup \{L_j | j \neq i\}$ is a subalgebra, it follows from (1.4) that

$$B \subseteq L_i \quad \text{or} \quad B \subseteq \bigcup \{L_j | j \neq i\}.$$

But the second of these inclusions is impossible since $a \in B$ and $a \notin L_j$ for all $j \neq i$. We thus have $B \subseteq L_i$.

The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious, which finishes the proof of (2.1).

Note that if the family $(L_i)_{i \in I}$ in (2.1) consists of two subalgebras only then the condition (*c*) follows immediately from (1.4) and hence by what we have proved all five conditions are automatically fulfilled. On the other hand it is easy to give an example showing that even if $(L_i)_{i \in I}$ consists of three subalgebras only, the last five conditions above are not a consequence of the first two.

Definition. An OML L is said to be obtained by *pasting* a family $(L_i)_{i \in I}$ along the section $[0, c'] \cup [c, 1]$ if and only if all the conditions of (2.1) are satisfied.

Extending Greechie's construction slightly we want to show now that a family $(L_i)_{i \in I}$ of OMLs can under certain conditions be pasted in the above sense. Assume for this that the following conditions are satisfied.

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$(P1)$ $(L_i)_{i\in I}$ is a non-empty family of OMLs,
$(P2)$ for all $i, j \in I$, $L_i \cap L_j$ is a subalgebra of both L_i and L_j,
$(P3)$ $0 \neq c \in \bigcap_{i\in I} L_i$,
$(P4)$ for all $i, j \in I$ with $i \neq j$: $L_i \cap L_j = [0, c']_i \cup [c, 1]_i$.

Note that from $(P2)$ it follows in particular that all L_i have the same bounds so that we don't have to specify in $(P3)$ and $(P4)$ the algebras L_i in which the bounds are taken. For the same reason we don't have to specify in $(P4)$ in which L_i we take the orthocomplement; the result is always the same. The index i in $(P4)$ refers to the OML L_i in which the intervals are taken. Thus, if $\leqq_i$ is the partial ordering of L_i then $[0, c']_i = \{x \in L_i | x \leqq_i c'\}$ and $[c, 1]_i = \{x \in L_i | c \leqq_i x\}$.

(2.2) *Under the assumptions* $(P1)$, $(P2)$, $(P3)$, $(P4)$ *define* $L = \bigcup_{i\in I} L_i$ *and let* $\leqq$ *be the union of the partial orderings* $\leqq_i$ *of the* L_i. *Then* $\leqq$ *is a partial ordering of* L *and with this partial ordering and the obvious definition of orthocomplementation,* L *is an* OML. *It is obtained by pasting the family* $(L_i)_{i\in I}$ *along the section* $[0, c'] \cup [c, 1]$.

The proof of this requires only minor modifications of Greechie's proof and there is no need to give it here.

The special case $c = 1$ in the above definition of pasting was considered earlier by MacLaren [**12**]. The OML L is in this case called the *horizontal sum* of the family $(L_i)_{i\in I}$.

We are especially interested here in a construction which is more restricted than the general pasting as defined above but slightly more general than the horizontal sum. Assume for this that the OML L is obtained by pasting the family $(L_i)_{i\in I}$ along the section $[0, c'] \cup [c, 1]$ and let B be an arbitrary OML. It is then obvious that the product $B \times L$ is obtained by pasting the family $(B \times L_i)_{i\in I}$ along the section $[(0, 0), (1, c')] \cup [(0, c), (1, 1)]$. The special case of this is the case where L is actually the horizontal sum of the family $(L_i)_{i\in I}$ and B is a Boolean algebra. This gives rise to the following definition.

Definition. An OML L is said to be the *weak horizontal sum* of a family $(L_i)_{i\in I}$ of subalgebras if and only if there exists an isomorphism f of L onto a product of $B \times L'$ of a Boolean algebra B and an OML L' such that the subalgebras L_i of L correspond via f to subalgebras of the form $B \times L_i'$ and L' is the horizontal sum of the family $(L_i')_{i\in I}$.

The following statement describes internally those pastings which are weak horizontal sums.

(2.3) *Let* L *be an* OML *obtained by pasting the family* $(L_i)_{i\in I}$ *along the section* $[0, c'] \cup [c, 1]$. *Then* L *is the weak horizontal sum of the family* $(L_i)_{i\in I}$ *if and only if the following three conditions are satisfied:*

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1. $c \in C(L)$,
2. $[0, c'] \cup [c, 1]$ *is a Boolean subalgebra of* L,
3. *for all* $i, j \in I$ *with* $i \neq j$, c *is an atom of* $L_i \cap L_j$.

The simple proof of this is left to the reader.

3. Orthomodular lattices with two blocks. As an immediate consequence of (1.2) we obtain a simple description of all OMLs with two blocks. Their structure is essentially known (see [**8**], p. 10), but since it is the starting point of all the following considerations we give here a detailed analysis of OMLs with two blocks.

(3.1) *Every* OML L *with two blocks is isomorphic with an* OML *of the form* $B \times (A_1 \dot{+} A_2)$, *where* B, A_1, A_2 *are Boolean algebras and* $A_1 \dot{+} A_2$ *is the horizontal sum of* A_1 *and* A_2. *In other words, every* OML L *with two blocks is the weak horizontal sum of its blocks.*

Proof. By (1.2) every OML with two blocks is isomorphic with a direct product of a Boolean algebra and an irreducible OML with two blocks. Since the irreducible OMLs with two blocks are obviously exactly the horizontal sums of two Boolean algebras the claim follows.

The next theorem describes in more detail how the section along which the blocks of an OML with two blocks are pasted can be explicitly calculated.

(3.2) *If* L *is an* OML *with two blocks* B_1 *and* B_2, $a \in B_1 - B_2$ *and* $b \in B_2 - B_1$ *then*

$$B_1 \cap B_2 = [0, \gamma'(a, b)] \cup [\gamma(a, b), 1].$$

If $B_1 \cap B_2 = [0, c'] \cup [c, 1]$ *then* $c = \gamma(a, b)$. L *is irreducible if and only if* $\gamma(a, b) = 1$.

Proof. By (3.1) we may assume that L is of the form $B \times (A_1 \dot{+} A_2)$ and that $B_1 = B \times A_1$ and $B_2 = B \times A_2$. By the remark of the last chapter, B_1 and B_2 are then pasted along the section $[(0, 0), (1, 0)] \cup [(0, 1), (1, 1)]$. But the elements a and b above are of the form $a = (a_1, a_2)$ and $b = (b_1, b_2)$ with $a_1, b_1 \in B$, $a_2 \in A_1 - A_2$ and $b_2 \in A_2 - A_1$. The first claim thus follows from the fact that $\gamma(a_1, b_1) = 0$ and $\gamma(a_2, b_2) = 1$, so that $\gamma(a, b) = (0, 1)$. To prove the second claim assume that $B_1 \cap B_2 = [0, c'] \cup [c, 1]$. Clearly $a \vee b \in B_1 \cap B_2$ and hence $a \vee b \leqq c'$ or $c \leqq a \vee b$. But $a \vee b \leqq c'$ would imply $a \in B_2$, a contradiction. We thus have $c \leqq a \vee b$ and hence also $a \vee b$; $a' \vee b$, $a' \vee b' \geqq c$, which gives $c \leqq \gamma(a, b)$. But $c < \gamma(a, b)$ would because of $c \in B_1 \cap B_2$ imply $c \leqq \gamma'(a, b)$, hence $c = 0$, hence $L = B_1 \cap B_2$, a contradiction. We thus have $c = \gamma(a, b)$. The third claim is an immediate consequence of the second.

The reader may find it instructive to apply (3.1) to prove the following result, which follows from (2.1) and the fact that an n-generated Boolean algebra has at most 2^{2^n} elements.

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(3.3) *An OML with two blocks which is generated by an n-element set has at most* $2^{2^n}(2^{2^{n-1}}+2)$ *elements and this bound is best possible for every* $n \geqq 2$.

4. Paths. We start out with some definitions which will be the main topic of interest in the rest of the paper.

Definition. For blocks A, B of an OML L define

$A \sim B$ if and only if $A \neq B$ and $A \cup B$ is a subalgebra of L,

$A \approx B$ if and only if $A \sim B$ and $A \cap B \neq C(L)$.

A ***link*** (***strong link***) in L is an unordered pair $\{A, B\}$ of blocks of L satisfying $A \sim B$ ($A \approx B$). A path in L is a finite sequence $B_0, B_1, \ldots B_n$ ($n \geqq 0$) in $\mathfrak{A}(L)$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leqq i < n$. The path is said to *join* the blocks B_0 and B_n. The number n is said to be the *length* of the path. The path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leqq i < n$. The path is said to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leqq i < n$. The *distance* $d(A, B)$ of blocks A, $B \in \mathfrak{A}(L)$ is defined to be the minimum of the lengths of all strictly proper paths joining A and B if such path exists and to be ∞ if there is no strictly proper path joining A and B.

If $A \sim B$ holds then by (3.2) there exists exactly one element $e \in A \cap B$ satisfying

$$A \cap B = ([0, e'] \cup [e, 1]) \cap (A \cup B).$$

We say that A and B are *linked* (*strongly linked* if $A \approx B$) *at* e and use the notation $A \sim {}_e B$ or $A \approx {}_e B$.

If $A \sim B$ and $C \in \mathfrak{A}(L)$ then $(A \cup B) \cap C$ is clearly a Boolean subalgebra of L and $(A \cup B) \cap C \subseteq A \cup B$. It follows from (1.4) that $(A \cup B) \cap C$ is contained in either A or B. This gives the following simple but very useful remark.

(4.1) *If* $A \sim B$ *then for every* $C \in \mathfrak{A}(L)$ *either* $A \cap C \subseteq B$ *or* $B \cap C \subseteq A$ *holds.*

(4.2) *If* L_1, L_2 *are* OMLs, $L = L_1 \times L_2$, $A, B \in \mathfrak{A}(L_1)$ *and* $C, D \in \mathfrak{A}(L_2)$ *then* $A \times C \sim B \times D$ *holds in* L *if and only if either* $A = B$ *and* $C \sim D$ *or* $A \sim B$ *and* $C = D$. *If* A *and* B *are linked at* a *then* $A \times C$ *and* $B \times C$ *are linked at* $(a, 0)$ *If* C *and* D *are linked at* c *then* $A \times C$ *and* $A \times D$ *are linked at* $(0, c)$.

We leave the simple proof of this to the reader.

Let again, L_1 and L_2 be OMLs, $A, B \in \mathfrak{A}(L_1)$ and $D, E \in \mathfrak{A}(L_2)$. Let $A = A_0 \sim A_1 \sim \ldots \sim A_n = B$ be a path in L_1. It then follows from (4.2) that $A \times D = A_0 \times D \sim A_1 \times D \sim \ldots \sim A_n \times D = B \times D$ is a path in $L = L_1 \times L_2$. If the first path is proper (strictly proper) then the second path is proper (strictly proper). If, however, $D \neq C(L_2)$, i.e. if L_2 has at least two blocks, then the second path is strictly proper regardless of whether the first

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path is proper (strictly proper) or not. If L_2 has only one block then clearly the second path is proper (strictly proper) if and only if the first path has this property. Similarly, if $D = D_0 \sim D_1 \sim \ldots \sim D_m = E$ is a path in L_2 then $B \times D = B \times D_0 \sim B \times D_1 \sim \ldots \sim B \times D_m = B \times E$ is a path in L and the same remarks hold. Thus the composite of the two paths in L is again a path in L. We thus obtain

(4.3) *Let L_1, L_2 be* OMLs *and $L = L_1 \times L_2$. If each of L_1 and L_2 has at least two blocks, if any two blocks in L_1 can be joined by a path of length at most n and if any two blocks in L_2 can be joined by a path of length at most m then any two blocks in L can be joined by a strictly proper path of length at most $n + m$. If one of L_1, L_2 is Boolean then any two blocks in the other can be joined by a path (proper path, strictly proper path) if and only if any two blocks in L have this property.*

We are now in position to prove the first main theorem.

(4.4) *Any two blocks in a block-finite* OML *L can be joined by a proper path.*

Proof. (By induction on the number n of blocks of L.) If $n = 1$ the claim is trivial. Assume $n \geqq 2$. By (1.2) L is either the direct product of two OMLs with at least two blocks each or it is the direct product of a Boolean algebra and an irreducible OML. In the first case the claim follows from (4.3) by induction hypothesis. In the second case we may, again by (4.3), restrict our attention to the case that L is irreducible, i.e. $C(L) = \{0, 1\}$. Let $A, B \in \mathfrak{A}(L)$ and $A \cap B \neq C(L) = \{0, 1\}$. Then by the remarks following (1.2) there exists $\mathfrak{B} \in \Omega(L)$ with $A, B \in \mathfrak{B}$. By (1.1) $\bigcup\mathfrak{B}$ is a subalgebra with $\mathfrak{A}(\bigcup\mathfrak{B}) = \mathfrak{B}$ and since L is irreducible, $\bigcup\mathfrak{B}$ has fewer blocks than L. By inductive hypothesis A and B can be joined by a proper path in $\bigcup\mathfrak{B}$. Since $C(\bigcup\mathfrak{B}) = \bigcap \mathfrak{B} \neq \{0, 1\} = C(L)$, every such path in $\bigcup\mathfrak{B}$ is a strictly proper path in L and hence we have even shown that A and B can be joined by a strictly proper path in L. Assume finally that $A, B \in \mathfrak{A}(L)$ and $A \cap B = \{0, 1\}$. If $A \cup B$ is a subalgebra then $A \sim B$ is a proper path and the claim is again proved. If $A \cup B$ is not a subalgebra then there exist $a \in A - B$ and $b \in B - A$ such that $a \vee b \notin A \cup B$. Since $a, b \leqq a \vee b$ there exist blocks C, D such that $a, a \vee b \in C$ and $b,, a \vee b \in D$. Since $a, b, a \vee b \neq 0, 1$, each of the intersections $A \cap C$, $C \cap D$ and $D \cap B$ is different from $\{0, 1\}$. By what we have already shown any two consecutive blocks of the sequence A, C, D, B can be joined by a strictly proper path. It follows that A and B can be joined by a strictly proper path, completing the proof.

We investigate next the question under which conditions any two blocks can be joined by a strictly proper path.

Definition. For blocks A, B of an OML L define $A \equiv B$ if and only if A and B can be joined by a strictly proper path.

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Clearly $\equiv$ is an equivalence relation in $\mathfrak{A}(L)$.

(4.5) *Let* L *be a block-finite* OML *containing at least two blocks which cannot be joined by a strictly proper path. Let* $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ *be the equivalence classes of* $\mathfrak{A}(L)$ *modulo* $\equiv$. *Then each* $\bigcup\mathfrak{B}_i$ $(1 \leqq i \leqq n)$ *is a subalgebra of* L *with* $\mathfrak{A}(\bigcup\mathfrak{B}_i) = \mathfrak{B}_i$ *and* L *is the weak horizontal sum of the family* $(\bigcup\mathfrak{B}_i)_{1\leqq i\leqq n}$.

Proof. Since by (4.3) and (4.4) any two blocks in the product of two OMLs with at least two blocks each can be joined by a strictly proper path the assumptions of (4.5) imply by (1.2) that L is the direct product of a Boolean algebra and an irreducible OML. By (4.3) it is thus enough to prove the claim under the assumption that L is irreducible. To show that the sets $\bigcup\mathfrak{B}_i$ are subalgebras it is obviously enough to show that $a, b \in \bigcup\mathfrak{B}_i$ implies $a \vee b \in \bigcup\mathfrak{B}_i$. For this assume $a \in A \in \mathfrak{B}_i$ and $b \in B \in \mathfrak{B}_i$. If $a \vee b \in A \cup B$ this is clear. If $a \vee b \notin A \cup B$ there exists $C \in \mathfrak{A}(L)$ such that $a, a \vee b \in C$ and since $0, 1 \neq a \in A \cap C$ we have by (4.4) that $A \equiv C$, i.e. $a \vee b \in C \in \mathfrak{B}_i$ i.e. $a \vee b \in \bigcup\mathfrak{B}_i$, proving that $\bigcup\mathfrak{B}_i$ is a subalgebra. By (4.4) any two blocks A, B with $A \cap B \neq \{0, 1\}$ can be joined by a strictly proper path. It follows from this that $(\bigcup\mathfrak{B}_i) \cap (\bigcup\mathfrak{B}_j) = \{0, 1\}$ holds whenever $i \neq j$. Clearly every block of L belongs to one of the $\mathfrak{B}_i$, which implies that L is the horizontal sum of the family $(\bigcup\mathfrak{B}_i)_{1\leqq i\leqq n}$ and that the blocks of $\bigcup\mathfrak{B}_i$ are exactly the elements of $\mathfrak{B}_i$, completing the proof.

The next simple observation will be used later on.

(4.6) *If blocks* A, B, C *of an* OML L *satisfy* $A \sim_e B \sim_f C$ *and if* $A \cap C \nsubseteq B$ *then* $e = f$ *and* $A \cap B = B \cap C$.

Proof. The assumptions imply by (4.1) that $B \cap C \subseteq A$ and $A \cap B \subseteq C$, which gives the second claim. To prove the first claim pick $a \in (A \cap C) - B$ and $b \in B - (A \cup C)$. Then by (3.2), $e = \gamma(a, b) = f$.

(4.7) *Let* L_1, L_2 *be* OML*s*, $A, B \in \mathfrak{A}(L_1)$, $C, D \in \mathfrak{A}(L_2)$ *and*

$$A \times C \sim_e B \times C \sim_f B \times D.$$

Then this path is strictly proper and there exists exactly one block F *in* $L_1 \times L_2$, *namely* $F = A \times D$, *such that*

$$A \times C \sim_f F \sim_e B \times D.$$

Proof. It is obvious that the given path is strictly proper. By (4.1) the assumptions imply $A \sim_a B$, $C \sim_c D$, $e = (a, 0)$ and $f = (0, c)$. It follows from this immediately that $F = A \times D$ has the desired property. To show uniquess assume

$$A \times C \sim_f E \times G \sim_e B \times D.$$

From (4.1) it follows that either $A = E$ or $C = G$. But $C = G$ would imply

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$A \sim_b E$ for some $b \neq 0$, which would give $(b, 0) = f = (0, c)$, a contradiction. We thus obtain $A = E$ and, by symmetry, $G = D$.

The following theorem seems rather technical. It describes, however, an important feature of the interaction of blocks in a block-finite OML in that it allows to single out certain subalgebras which admit a representation as a non-trivial direct product.

(4.8) *Let L be a block-finite* OML, *A, B, $C \in \mathfrak{A}(L)$, $A \sim_e B \sim_f C$ and $e \leqq f'$. Then the path $A \sim B \sim C$ is strictly proper and there exists exactly one block D such that $A \sim_f D \sim_e C$. This D is different from B. Furthermore, $A \cup B \cup C \cup D$ is a subalgebra of L isomorphic with a direct product of two* OMLs *with two blocks each.*

Proof. We show first that if the OMLs L_1 and L_2 satisfy (4.8) then the product $L = L_1 \times L_2$ does. By symmetry and (4.1) we may assume that the given path is either of the form

(1) $A_1 \times A_2 \sim_e A_1 \times B_2 \sim_f A_1 \times C_2$

or of the form

(2) $A_1 \times A_2 \sim_e B_1 \times A_2 \sim_f B_1 \times C_2$.

In the case (2) the claim follows from (4.7). In the first case there exist $a, b \in L$ such that $A_2 \sim_a B_2 \sim_b C_2$ with $a \leqq b'$. By assumption there exists exactly one block $D_2 \in \mathfrak{A}(L_2)$ such that $A_2 \sim_b D_2 \sim_a C_2$. This clearly implies

$$A_1 \times A_2 \sim_f A_1 \times D_2 \sim_e A_1 \times C_2,$$

i.e. that the block $D = A_1 \times D_2$ has the desired property. It is easy to see that it is the only one. We have thus shown that the property described in (4.8) is preserved under the formation of the product of two OMLs.

We now prove the general result by induction on the number n of blocks of L. If $n = 1$ the claim is trivially true. Assume $n \geqq 2$. By (1.2), inductive hypothesis and the result already proved we may assume that L is irreducible. Assume in this case $A \sim_e B \sim_f C$ and $e \leqq f'$. Since $e \neq 0 \neq f$ and $e \leqq f'$ we have $0, 1 \neq e \in A \cap B \cap C$, which implies in particular that the path is strictly proper. By the remark following (1.2) it also implies that there exists $\mathfrak{B} \in \Omega(L)$ with A, B, $C \in \mathfrak{B}$. By (1.1) $\bigcup \mathfrak{B}$ is a subalgebra of L with fewer blocks than L and hence by inductive hypothesis there exists a unique block $D \in \mathfrak{B}$ such that $A \sim_f D \sim_e C$. It thus only remains to show that no block $D \in \mathfrak{A}(L) - \mathfrak{B}$ satisfies $A \sim_f D \sim_e C$. But $D \in \mathfrak{A}(L) - \mathfrak{B}$ implies by definition of $\Omega(L)$ that $\bigcap \mathfrak{B} \nsubseteq D$ and hence that $A \cap C \nsubseteq D$. This together with $A \sim_f D \sim_e B$ would by (4.6) imply that $e = f$. Since also $e \leqq f'$ we would obtain $e = 0$, a contradiction.

5. Line-like orthomodular lattices. In this section we discuss OMLs of a special type, which we call *line-like* and which are characterized by the fact

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that its blocks admit a certain natural ordering described in the following definition.

Definition. A *line-like* ordering of the blocks of a block-finite OML L is a sequence $B_0, B_1, \ldots, B_n$ containing every block of L exactly once and satisfying:

1. If $0 \leqq i < j \leqq n$ then $\bigcup_{k=i}^{j} B_k$ is a subalgebra of L;
2. If $0 \leqq i < j < k \leqq n$ then $B_i \cap B_k \subseteq B_j$.

A line-like OML is a block-finite OML the blocks of which admit a line-like ordering.

(5.1) *Let $B_0, B_1, \ldots, B_n$ be a line-like ordering of the blocks of an* OML *L. Then:*

1. *If $0 \leqq i < j \leqq n$ then the blocks of $\bigcup_{k=i}^{j} B_k$ are exactly the blocks B_k with $i \leqq k \leqq j$;*
2. *If B_i and B_{i+1} are linked at e_i $(0 \leqq i < n)$ then $e_i \nleqq e_{i+1}'$ holds whenever $0 \leqq i \leqq n - 2$;*
3. *L is the direct product of a Boolean algebra and an irreducible* OML.

Proof. Every block B of $\bigcup_{k=i}^{j} B_k$ satisfies

$$B \subseteq (\bigcup_{k=i}^{j-1} B_k) \cup B_j.$$

The first claim follows from this and (1.4) by induction on $j - i$. If L was not the direct product of a Boolean algebra and an irreducible OML we may assume by (1.2) that it was the direct product of two OMLs L_1 and L_2 with at least two blocks each. By (4.2) there would exist an index i and blocks A, $B \in \mathfrak{A}(L_1)$, $C, D \in \mathfrak{A}(L_2)$ such that

$$B_i = A \times C, B_{i+1} = B \times C \quad \text{and} \quad B_{i+2} = B \times D.$$

But then $B_i \cup B_{i+1} \cup B_{i+2}$ would not be a subalgebra, contradicting the first condition in the definition. By (4.8) we would arrive at the same contradiction if $e_i \leqq e_{i+1}'$ would hold for some i. (5.1) is thus proved.

If $B_0, B_1, \ldots, B_n$ is a line-like ordering of the blocks of an OML we assume throughout this chapter that B_i and B_{i+1} are linked at e_i.

(5.2) *Under the assumption of* (5.1), $B_i \cap B_{i+1} = [0, e_i'] \cup [e_i, 1]$ *holds whenever* $0 \leqq i < n$.

Proof. By the definition of a link we have

$$B_i \cup B_{i+1} = ([0, e_i'] \cup [e_i, 1]) \cap (B_i \cup B_{i+1})$$

and it is by duality enough to show that $[e_i, 1] \subseteq B_i \cup B_{i+1}$, i.e. that $e_i \leqq x$ implies $x \in B_i \cup B_{i+1}$. If $e_i \leqq x$ there exists a block B_k containing both e_i and x and we may by symmetry assume that $k \leqq i$. But $e_i \in B_k \cap B_{i+1}$ implies by the definition of a line-like ordering that $e_i \in B_j$ holds for $k \leqq j$

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$\leqq i + 1$ and hence that $e_i \leqq e_j'$ or $e_j \leqq e_i$ holds for $k \leqq j \leqq i$. If $e_i \leqq e_j'$ would hold for at least one such j there would be a largest j with this property and, since $e_i \nleq e_i'$, we would have $j < i$, hence $e_i \leqq e_j'$ and $e_{j+1} \leqq e_i$, i.e. $e_j \leqq e_{j+1}'$, contradicting (5.1). It thus follows that $e_j \leqq e_i$ and hence $e_j \leqq x$ holds for $k \leqq j \leqq i$. This and $x \in B_k$ implies by induction that $x \in B_j$ holds for all j with $k \leqq j \leqq i$, in particular that $x \in B_i \subseteq B_i \cup B_{i+1}$.

From (5.2) and the second condition in the above definition we obtain:

(5.3) *Under the assumption of* (5.1)

$$(B_0 \cup \ldots B_i) \cap (B_{i+1} \cup \ldots \cup B_n) = [0, e_i'] \cup [e_i, 1]$$

holds whenever $0 \leqq i < n$; *in particular* L *can be obtained by pasting the subalgebras* $B_0 \cup \ldots \cup B_i$ *and* $B_{i+1} \cup \ldots \cup B_n$ *along a segment.*

In the following two statements (5.4) and (5.5) we assume that $B_0, B_1, \ldots, B_n$ is a line-like ordering of the blocks of L and that $b_i \in B_i - \bigcup_{j \neq i} B_j$. Such b_i exist since $B_i \subseteq \bigcup_{j \neq i} B_j$ would by (1.4) imply $B_i \subseteq \bigcup_{j<i} B_j$, or $B_i \subseteq \bigcup_{j>i} B_j$, both contradicting the first part of (5.1).

(5.4) *If* $0 \leqq i < j \leqq n$ *then*

$$e_i \vee e_{i+1} \vee \ldots \vee e_{j-1} = e_i \vee e_{j-1} = \gamma(b_i, b_j) = \gamma(b_i, b_{i+1}, \ldots, b_j).$$

Proof. Since $B_i, B_{i+1}, \ldots, B_j$ is a line-like ordering of the blocks of $\bigcup_{k=i}^{j} B_k$ we may assume, without loss of generality, that $i = 0$ and $j = n$. By (5.1) L is the direct product of a Boolean algebra and an irreducible OML and it follows from this easily that we may restrict our attention to the case that L is irreducible. Since $\gamma(b_0, b_n) \leqq \gamma(b_0, b_1, \ldots, b_n)$ it is then enough to show that $e_0 \vee e_{n-1} = \gamma(b_0, b_n) = 1$. From

$$b_0 \in B_0 - \bigcup_{i=1}^{n} B_i \text{ and } b_n \in B_n - \bigcup_{i=0}^{n-1} B_i$$

it follows that

$$b_0 \vee b_n,\ b_0 \vee b_n',\ b_0' \vee b_n,\ b_0' \vee b_n' \in B_0 \cap B_n = \{0, 1\},$$

i.e. $\gamma(b_0, b_n) = 1$. From (5.1) it follows that $e_0 \vee e_{n-1} \in B_0 \cap B_n$, hence also $e_0 \vee e_{n-1} = 1$.

The following result is an immediate consequence of this.

(5.5) *If* $0 \leqq i < j \leqq n$ *then*

$$\begin{aligned} B_i \cap B_j &= [0, \gamma'(b_i, b_j)] \cup [\gamma(b_i, b_j), 1] \\ &= [0, \gamma'(b_i, b_{i+1}, \ldots, b_j)] \cup [\gamma(b_i, b_{i+1}, \ldots, b_j), 1]. \end{aligned}$$

Our next aim is to prove that if an OML L with $n + 1$ blocks contains blocks with distance n then it is line-like.

(5.6) *If an* OML *with* $n + 1$ *blocks is the direct product of two* OML*s with at least two blocks each then any two blocks* A, B *of* L *have a distance* $d(A, B) < n$.

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Proof. If the first of the factors has $k + 1$ and the second $l + 1$ $(k, l \geqq 1)$ blocks then it follows from (4.3) and (4.4) that any two blocks in L have a distance $\leqq k + l$. But

$$n + 1 = (k + 1)(l + 1) = kl + k + l + 1,$$

hence $k + l < n$.

In the next three statements (5.7), (5.8), (5.9) we assume that L is an OML with $n + 1 \geqq 1$ blocks, that B_0 and B_n are blocks of L with distance n and that $B_0 \approx B_1 \approx \ldots \approx B_n$.

(5.7) *If* $0 \leqq m \leqq n$ *then* $\bigcup_{i=0}^{m} B_i$ *is a subalgebra of* L *with* $m + 1$ *blocks and if* $m \geqq 2$ *the blocks* B_0 *and* B_m *have distance* m *in this subalgebra.*

Proof. By (1.2), (4.3) and (5.6) we may assume that L is irreducible. Assume now that for some m $(0 \leqq m < n)$, $S_m = \bigcup_{i=0}^{m} B_i$ is not a subalgebra. Then there would exist elements $a, b \in S_m$ with

$$a \vee b \in \left(\bigcup_{i=m+1}^{n} B_i \right) - S_m.$$

Define $I = \{i | 0 \leqq i \leqq n, a \in B_i\}$ and $J = \{j | 0 \leqq j \leqq n, b \in B_j\}$. Since the union of any two consecutive blocks is a subalgebra of L, it follows that $|j - i| \geqq 2$ holds whenever $i \in I$, $j \in J$ and $i, j \leqq m$. It follows from this that at least one of the sets $I \cap \{0, 1, \ldots, m\}$, $J \cap \{0, 1, \ldots, m\}$ consists of numbers $\leqq m - 2$ only and we may assume by symmetry that $I \cap \{0, 1, \ldots, m\}$ has this property, i.e. that $i \in I$ and $i \leqq m$ implies $i \leqq m - 2$. By definition of I we have

$$a \in \left(\bigcap_{i \in I} B_i \right) - \bigcup_{j \notin I} B_j,$$

which by (1.1) implies that $T = \bigcup_{i \in I} B_i$ is a subalgebra of L with $\mathfrak{A}(T) = \{B_i | i \in I\}$ and that the center $C(T) = \bigcap_{i \in I} B_i$ contains a and hence is non-trivial. Since a and $a \vee b$ are comparable there exists a block B_k containing both a and $a \vee b$ and since $a \vee b \notin S_m$, $k > m$ holds for every such k. By (4.4) any two blocks in T can be joined by a proper path in T. It would thus follow from our assumption that $\bigcup_{j=0}^{m} B_j$ was not a subalgebra that there exist indices $i, k \in I$ with $i \leqq m - 2$, $k \geqq m + 1$ and $Bi \sim B_k$. Since also $B_i \cap B_k \supseteq C(T) \supset C(L)$ it would follow that $B_0 \approx B_1 \approx \ldots \approx B_i \approx B_k \approx \ldots \approx B_n$ was a strictly proper path in L contradicting $d(B_0, B_n) = n$. We have thus shown that the sets $S_m = \bigcup_{i=0}^{m} S_i$ $(0 \leqq m \leqq n)$ are subalgebras of L. Since every block of S_{m+1} is by (1.4) either contained in S_m or in B_{m+1} it follows easily by induction on m that the blocks of S_m are exactly $B_0, B_1, \ldots, B_m$. It remains to show that the distance of B_0 and B_m in S_m is m provided that $m \geqq 2$. For every m the distance of B_0 and B_m in L is m and every strictly proper path in S_m is also a strictly proper path in L. The distance of B_0 and B_m in S_m is therefore either m or ∞ and it is thus enough to show that if it is ∞ then $m \leqq 1$. Assume that $d(B_0, B_m) = \infty$ in S_m. Then, in particular,

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$B_0 \sim B_1 \sim \ldots \sim B_m$ is not a strictly proper path in S_m and hence $C(S_m) \neq \{0, 1\}$. Furthermore, by (4.4), $B_0 \cup B_m$ is a subalgebra of L and $B_0 \cap B_m = C(S_m) \neq \{0, 1\}$. It follows that $B_0 \sim B_m \sim B_{m+1} \sim \ldots \sim B_n$ is a strictly proper path in L which, together with the assumption $d(B_0, B_n) = n$ implies $m \leqq 1$.

(5.8) 1. *If* $0 \leqq i < j \leqq n$ *then* $\bigcup_{k=i}^{j} B_k$ *is a subalgebra with* $j - i + 1$ *blocks and if* $j - i \geqq 2$ *the blocks* B_i *and* B_j *have distance* $j - i$ *in this subalgebra.*

2. *If* $0 \leqq i < j < k \leqq n$ *then* $B_i \cap B_k \subseteq B_j$.
In particular, $B_0, B_1, \ldots, B_n$ *is a line-like ordering of* $\mathfrak{A}(L)$.

Proof. The first claim follows easily by applying (5.7) twice. The second claim we prove by induction on n. If $n = 1$ it is trivial. Assume $n \geqq 2$. If $0 < i$ or $k < n$ the claim follows from the first part of (5.8) by inductive hypothesis. We thus only have to show that $B_0 \cap B_n \subseteq B_j$ holds whenever $0 < j < n$. If $B_0 \cap B_n \nsubseteq B_j$ would hold for all such j we would obtain by (1.4) that

$$B_0 \cap B_n \nsubseteq \bigcup_{j=1}^{n-1} B_j$$

which by (1.1) would imply $B_0 \approx B_n$ contradicting $d(B_0, B_n) = n \geqq 2$. Thus there exists k, $0 < k < n$, such that $B_0 \cap B_n \subseteq B_k$, hence $B_0 \cap B_n = B_0 \cap B_k \cap B_n$. By inductive hypothesis we have $B_0 \cap B_k \subseteq B_j$ if $0 < j < k$ and $B_k \cap B_n \subseteq B_j$ if $k < j < n$, and in both cases $B_0 \cap B_n \subseteq B_j$, completing the proof.

The definition of a line-like ordering and the second statement of (5.1) completely describe how Boolean algebras have to interact in order to be the blocks of a line-like OML. This is the content of the following statement.

(5.9) *Let* $B_0, B_1, \ldots, B_n$ *be a sequence of Boolean algebras,* $\leqq_i$ *the partial ordering of* B_i, $e_i \in B_i \cap B_{i+1}$ $(0 \leqq i < n)$ *and assume that the following conditions are satisfied:*

1. $B_i \cap B_{i+1}$ *is a subalgebra of both* B_i *and* B_{i+1} $(0 \leqq i < n)$,
2. $B_i \cap B_{i+1} = [0, e_i']_i \cup [e_i, 1]_i = [0, e_i']_{i+1} \cap [e_i, 1]$ $(0 \leqq i < n)$
3. $B_i \cap B_k \subseteq B_j$ $(0 \leqq i < j < k \leqq n)$,
4. $e_i \nleqq e_{i+1}$ $(0 \leqq i < n)$.

Define $L = \bigcup_{i=0}^{n} B_i$ *and let* $\leqq$ *be the union of the partial ordering* $\leqq_i$. *Then* $\leqq$ *is a partial ordering of* L *and with this partial ordering and the obvious definition of orthocomplementation* L *is a line-like* OML *and* $B_0, B_1, \ldots, B_n$ *is a line-like ordering of its blocks.*

Proof. This follows easily from (2.2) by induction on n.

6. Orthomodular lattices with three blocks. We assume in this section that L is an OML with three blocks. By (4.4) there exist $A, B \in \mathfrak{A}(L)$ with $A \sim B$. By (4.1) this implies that, if C is the remaining block, at least one of

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the intersections $A \cap C$, $B \cap C$ equals the center of L. Thus it is not true that every two (distinct) blocks of L have distance 1 and it follows that either there exist blocks with distance 2 or there exist blocks which can not be joined by a strictly proper path. In the first case L is line-like by (5.8), and in the second case L is by (4.5) the weak horizontal sum of a Boolean algebra and an OML with two blocks, hence, as is easily seen, also line-like. We have thus proved the following statement.

(6.1) *Every* OML L *with three blocks is line-like.*

Definition. Let L be an OML with three blocks. A block $B \in \mathfrak{A}(L)$ is said to be a *middle block* of L if and only if $A \cap C \subseteq B$ holds, where A and C are the remaining blocks of L.

(6.2) *Every* OML L *with three blocks has a middle block. If B is a middle block of L and A, C are the remaining blocks then $A \cup B$ and $B \cup C$ are subalgebras of L.*

Proof. The existence of a middle block follows immediately from (6.1). If there exists a strictly proper path $A \approx B \approx C$ then B is obviously the only middle block and the second claim is obvious. If no two blocks of L have distance 2 then any two blocks have distance 1 or ∞. In both cases the union of the two blocks is a subalgebra, in the first case by definition and in the second case by (4.4).

Still another way to formulate (6.1) is the following, which follows from (5.2).

(6.3) *Every* OML L *with three blocks can be obtained by pasting a Boolean algebra and an* OML *with two blocks along a section.*

The following observation will be needed in the next two sections.

(6.4) *For an* OML *with three blocks the following are equivalent.*
1. *L has two middle blocks;*
2. *The union of any two blocks is a subalgebra of L.*

Proof. The second condition follows from the first by (6.2). To prove the converse assume that B is a middle block and A, C are the remaining blocks. If $A \cup C$ is a subalgebra, it follows from (4.1) that either $A \cap B \subseteq C$ or $B \cap C \subseteq A$, i.e. that one of A or C is another middle block.

7. Orthomodular lattices with four blocks.

(7.1) *Let L be an irreducible* OML *with four blocks B_0, B_1, B_2, B_3 satisfying*

$$B_0 \cap B_1 \not\subseteq B_2 \cup B_3,\ B_1 \approx B_2 \quad \text{and} \quad B_2 \cap B_3 \not\subseteq B_0 \cup B_1.$$

Then $B_0 \cup B_3$ is not a subalgebra.

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Proof. Pick $u \in (B_0 \cap B_1) - (B_2 \cup B_3)$, $v \in (B_2 \cap B_3) - (B_0 \cup B_1)$ and put $e = \gamma(u, v)$. Since $B_1 \cup B_2$ is a subalgebra we have by (3.2) that $B_1 \sim_e B_2$. If $B_0 \cup B_3$ was also a subalgebra we would by the same argument have $B_0 \sim_e B_3$, hence

$$e \in B_0 \cap B_1 \cap B_2 \cap B_3 = C(L) = \{0, 1\},$$

hence $e = 0$ or $e = 1$. But $B_1 \sim_e B_2$ implies $e \neq 0$ by an earlier remark and $e = 1$ would imply $B_1 \cap B_2 = \{0, 1\}$, contradicting $B_1 \approx B_2$.

(7.2) *If L is an irreducible* OML *with four blocks B_0, B_1, B_2, B_3 and $B_0 \approx B_1 \approx B_2$ then $B_0 \cup B_1 \cup B_2$ is a subalgebra* (*which clearly has three blocks only*).

Proof. Assume $B_0 \cup B_1 \cup B_2$ is not a subalgebra. Then there would exist

$$a \in B_0 - (B_1 \cup B_2), b \in B_2 - (B_0 \cup B_1)$$

such that $a \vee b \in B_3 - (B_0 \cup B_1 \cup B_2)$. Since a and $a \vee b$ are comparable they both belong to some block and hence $a \in (B_0 \cap B_3) - (B_1 \cup B_2)$, which implies $B_0 \cap B_3 \nsubseteq B_1 \cup B_2$. By symmetry we obtain $B_2 \cap B_3 \nsubseteq B_0 \cup B_1$. But $B_0 \cap B_1 \nsubseteq B_2 \cup B_3$ would contradict (7.1) and we thus have either $B_0 \cap B_1 \subseteq B_2$ or $B_0 \cap B_1 \subseteq B_3$. The first of these conditions would imply $B_0 \cap B_1 = B_0 \cap B_1 \cap B_2$. Since $B_0 \cup B_1 \cup B_2$ is not a subalgebra we have by (1.1) that $B_0 \cap B_1 \cap B_2 \subseteq B_3$ and we would obtain $B_0 \cap B_1 = \{0, 1\}$, contradicting $B_0 \approx B_1$. Thus $B_0 \cap B_1 \subseteq B_2$ is impossible and hence we have $B_0 \cap B_1 \subseteq B_3$. Since $B_1 \cup B_2$ is a subalgebra we have by (4.1) either $B_1 \cap B_3 \subseteq B_2$ or $B_2 \cap B_3 \subseteq B_1$. The second of these conditions contradicts $B_2 \cap B_3 \nsubseteq B_0 \cup B_1$. We would thus obtain

$$B_0 \cap B_1 = B_0 \cap B_1 \cap B_3 = B_0 \cap B_1 \cap B_2 \cap B_3 = \{0, 1\},$$

contradicting $B_0 \approx B_1$. (7.2) is thus proved.

Definition. Let L be an OML with four blocks. A block B of L is said to be a *middle block* if and only if whenever A and C are two of the remaining blocks then $A \cup B \cup C$ is a subalgebra with three blocks and middle block B.

Definition. The *valence* of a block B of a block-finite OML L is the number of blocks A satisfying $A \approx B$.

(7.3) *Let L be an* OML *with four blocks B_0, B_1, B_2, B_3 and assume that the block B_1 has valence* 3. *Then either B_1 is a middle block of L or L is line-like.*

Proof. Since no block in the direct product of two OMLs with two blocks each has valance 3 we may by (1.2) assume that L is irreducible. By (7.2) each of $B_0 \cup B_1 \cup B_2$, $B_0 \cup B_1 \cup B_3$ and $B_2 \cup B_1 \cup B_3$ is a subalgebra with three blocks. If B_1 is a middle block of each of these, it is a middle block of L and there is nothing left to prove. If this is not the case we may assume without loss of generality that B_1 is not a middle block of $B_0 \cup B_1 \cup B_2$. It then follows from (6.2) that $B_0 \cup B_2$ is a subalgebra and that $B_0 \cap B_1 \subseteq B_2$ and

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$B_1 \cap B_2 \subseteq B_0$. Since $B_0 \cup B_2$ is a subalgebra we obtain from (4.1) that either $B_0 \cap B_3 \subseteq B_2$ or $B_2 \cap B_3 \subseteq B_0$ and we may by symmetry assume that $B_0 \cap B_3 \subseteq B_2$. Since $B_1 \cup B_3$ is a subalgebra we obtain by the same argument that either $B_0 \cap B_3 \subseteq B_1$ or $B_0 \cap B_1 \subseteq B_3$. The second of these conditions would imply $B_0 \cap B_1 = \{0, 1\}$, contradicting $B_0 \approx B_1$. We thus have $B_0 \cap B_3 \subseteq B_1$ and hence $B_0 \cap B_3 = \{0, 1\}$. We claim that $B_0 \approx B_2 \approx B_1 \approx B_3$ is a line-like ordering in this case. The only thing left to prove to establish this is $B_2 \cap B_3 \subseteq B_1$. But $B_2 \cap B_3 \not\subseteq B_1$ would as before imply $B_1 \cap B_2 \subseteq B_3$, hence $B_1 \cap B_2 = \{0, 1\}$, contradicting $B_1 \approx B_2$.

(7.4) *Let L be an* OML *with four blocks B_0, B_1, B_2, B_3 satisfying $B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_0$ and having no other strong links. Then L is isomorphic with the direct product of two* OMLs *with two blocks each.*

Proof. Assume that L was not a direct product of the described kind. We may then assume that L was irreducible. By (7.2), $B_0 \cup B_1 \cup B_2$ would be a subalgebra and since $B_0 \not\approx B_2$ we would have $B_0 \cap B_2 \subseteq B_1$ and, by symmetry, $B_0 \cap B_2 \subseteq B_3$ and hence $B_0 \cap B_2 = \{0, 1\}$. Again by symmetry we would obtain $B_1 \cap B_3 = \{0, 1\}$. We would thus obtain $B_i \cap B_{i+1} \not\subseteq B_{i+2} \cup B_{i+3}$ ($i = 0, 1, 2, 3$, indices modulo 4), contradicting (7.1).

(7.5) *Every* OML *with four blocks satisfies one of the following conditions:*

1. *It is the direct product of a Boolean algebra and two irreducible* OMLs *with two blocks each;*
2. *It is line-like;*
3. *It has a middle block.*

Proof. From (4.5) and the structure theorems for OMLs with at most three blocks it follows easily that L is line-like if there exist blocks which can not be joined by a strictly proper path. Hence we may assume that L is connected. If it has a block of valence 3 the claim follows from (7.3). If every block has valence at most two then it either satisfies the assumption (7.4) and hence the first condition of (7.5) or, with suitable enumeration of the blocks, we have $B_0 \approx B_1 \approx B_2 \approx B_3$ and these are the only strong links. But then $d(B_0, B_3) = 3$ and, by (5.8), L is line-like.

(7.6) *Every* OML *L with four blocks is either the direct product of two* OMLs *with two blocks each or can be obtained by pasting a Boolean algebra and an* OML *with three blocks along a segment.*

Proof. By (5.2) and (7.5) it is enough to show that every OML L with four blocks which has a middle block can be obtained by pasting in the described way. Let B_0, B_1, B_2, B_3 be the blocks of L, assume that B_1 is a middle block and $B_0 \sim_e B_1$. We then have by (5.1) and (6.1):

$$\begin{aligned} B_0 \cap (B_1 \cup B_2 \cup B_3) &= B_0 \cap B_1 = ([0, e'] \cup [e, 1]) \cap (B_0 \cup B_1 \cup B_2) \\ &= ([0, e'] \cup [e, 1]) \cap (B_0 \cup B_1 \cup B_3) = ([0, e'] \cup [e, 1]) \\ &\qquad \cap (B_0 \cup B_1 \cup B_2 \cup B_3) = [0, e'] \cup [e, 1], \end{aligned}$$

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i.e. L is obtained by pasting B_0 and $B_1 \cup B_2 \cup B_3$ along the segment $[0, e'] \cup [e, 1]$.

8. Orthomodular lattices with five blocks. Whereas all OMLs with up to four blocks could be obtained by either taking direct products or pasting OMLs with fewer blocks, a new phenomenon appears if the OML has five blocks, namely the existence of "loops". We deal with this case first.

(8.1) *Let L be an* OML *with five blocks B_0, B_1, B_2, B_3, B_4 such that $B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_4 \approx B_0$ holds and such that there are no other strong links. Then*

$$B_i \cap B_{i+1} \nsubseteq B_{i+2}, B_{i+1} \cap B_{i+2} \nsubseteq B_i \text{ and } B_i \cap B_{i+2} \subseteq B_{i+1}$$

holds for all i and for no i is $B_i \cup B_{i+2}$ a subalgebra. (Indices modulo 5).

Proof. By (1.2) we may assume that L is irreducible. We show first that for no i is $B_i \cup B_{i+2}$ a subalgebra. If it were a subalgebra for some i then by (6.4) $B_i \cup B_{i+1} \cup B_{i+2}$ would be a subalgebra with at least two middle blocks and hence one of

$$B_i \cap B_{i+1} \subseteq B_{i+2}, B_{i+1} \cap B_{i+2} \subseteq B_i$$

would hold. Since $B_i \not\approx B_{i+2}$ we would also have $B_i \cap B_{i+2} = \{0, 1\}$ and we would obtain either $B_i \cap B_{i+1} = \{0, 1\}$ or $B_{i+1} \cap B_{i+2} = \{0, 1\}$, both contradicting the assumptions. We have thus proved that none of the unions $B_i \cup B_{i+2}$ is a subalgebra. To prove the rest of the claim it is by symmetry enough to assume $i = 0$. If $B_0 \cup B_1 \cup B_2$ is a subalgebra, then, as we have seen, B_1 is the only middle block of it and the claim follows trivially. We may thus assume for the rest of the proof that $B_0 \cup B_1 \cup B_2$ is not a subalgebra. From (1.1) then follows that

$$B_0 \cap B_1 \cap B_2 \subseteq B_3 \cup B_4,$$

which by (1.4) gives either

$$B_0 \cap B_1 \cap B_2 \subseteq B_3 \text{ or } B_0 \cap B_1 \cap B_2 \subseteq B_4.$$

By symmetry we may assume that $B_0 \cap B_1 \cap B_2 \subseteq B_3$. Assume now that

$$B_0 \cap B_1 \cap B_2 \cap B_3 \nsubseteq B_4.$$

Then, by (1.1), $B_0 \cup B_1 \cup B_2 \cup B_3$ would be a subalgebra with four blocks. Since $B_0 \not\approx B_3$ it can not be the product of two OMLs with two blocks each. Thus, by (7.5), it would be either line-like or have a middle block. Since both $B_0 \cup B_2$ and $B_1 \cup B_3$ are not subalgebras, it can not have a middle block. Since none of $B_0 \cup B_2$, $B_0 \cup B_3$, $B_1 \cup B_3$, $B_0 \cup B_1 \cup B_2$ is a subalgebra it cannot be line-like either. It thus follows that

$$B_0 \cap B_1 \cap B_2 \cap B_3 \subseteq B_4$$

https://doi.org/10.4153/CJM-1979-090-6 Published online by Cambridge University Press

and hence

$$B_0 \cap B_1 \cap B_2 = \{0, 1\}.$$

With this $B_0 \cap B_1 \subseteq B_2$ would imply $B_0 \cap B_1 = \{0, 1\}$ and $B_1 \cap B_2 \subseteq B_0$ would imply $B_1 \cap B_2 = \{0, 1\}$, both contradictions. We thus have $B_0 \cap B_1 \not\subseteq B_2$ and $B_1 \cap B_2 \not\subseteq B_0$. Since $B_1 \cup B_2$ is a subalgebra, the first of these inequalities implies $B_0 \cap B_2 \subseteq B_1$ by (4.1), proving (8.1).

(8.2) *Under the assumption of* (8.1) *the following statements hold.*

1. *The union of three or four-blocks of L is never a subalgebra;*
2. *The only unions of two blocks which are subalgebras are the unions $B_i \cup B_{i+1}$ (indices* mod 5);
3. $B_i \cap B_{i+1} \not\subseteq B_{i+2} \cup B_{i+3} \cup B_{i+4}$ *holds for all i (indices* mod 5);
4. $B_i \cap B_{i+2} = C(L)$ *hold for all i (indices* mod 5).

Proof. By (8.1) we have $B_i \cap B_{i+1} \not\subseteq B_{i+2}, B_{i+4}$. But $B_i \cap B_{i+1} \subseteq B_{i+3}$ would by (8.1) imply

$$B_i \cap B_{i+1} = B_i \cap B_{i+1} \cap B_{i+3} \subseteq B_{i+4},$$

a contradiction. We thus have that $B_i \cap B_{i+1}$ is contained in neither of $B_{i+2}, B_{i+3}, B_{i+4}$, hence by (1.4),

$$B_i \cap B_{i+1} \not\subseteq B_{i+2} \cup B_{i+3} \cup B_{i+4},$$

proving the third claim. If the union of two blocks with non-consecutive indices (mod 5) were a subalgebra it would be of the form $B_i \cup B_{i+2}$. But this would make $B_i \cup B_{i+1} \cup B_{i+2}$ a subalgebra with three blocks and two middle blocks, contradicting (8.1) and proving the second claim. We show next that none of the unions $B_i \cup B_{i+1} \cup B_{i+2}$ is a subalgebra. By (1.2) we may assume that L is irreducible. By symmetry it is enough to show that $B_0 \cup B_1 \cup B_2$ is not a subalgebra. If it were it would clearly have three blocks and by (8.1) B_1 would be the only middle block. By condition 3 we may pick

$$a \in (B_0 \cap B_4) - (B_1 \cup B_2 \cup B_3) \text{ and } b \in (B_2 \cap B_3) - (B_0 \cup B_1 \cup B_4).$$

Since $B_3 \cup B_4$ is a subalgebra we have $a \vee b \in B_3 \cap B_4$. Since by assumption $B_0 \cup B_1 \cup B_2$ is a subalgebra we have $a \vee b \in B_0 \cap B_1 \cap B_2$, hence $a \vee b \in C(L) = \{0, 1\}$, hence $a \vee b = 1$. By symmetry we also obtain

$$a \vee b' = a' \vee b = a' \vee b' = 1,$$

hence, if we put $e = \gamma(a, b)$ we have $e = 1$. But by (3.2) we have $B_3 \sim_e B_4$. We would thus obtain $B_3 \cap B_4 = \{0, 1\}$ contradicting $B_3 \approx B_4$. If a union $B_i \cup B_{i+1} \cup B_{i+3}$ were a subalgebra it would clearly have three blocks, and it would follow that at least one of $B_i \cup B_{i+3}$ or $B_{i+1} \cup B_{i+3}$ was a subalgebra, contrary to what we have already shown. Thus the union of no three blocks of L is a subalgebra. Every union of four blocks is of the form $B_i \cup B_{i+1}$

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$\cup B_{i+2} \cup B_{i+3}$. If it were a subalgebra it would clearly have four blocks. If it were line-like or had a middle block the union of three blocks would be a subalgebra, which is not the case as we have already seen. If it were a subalgebra, it would by (7.5) be the direct product of two OMLs with two blocks, which again is impossible because of a missing strong link. We have thus proved the first claim. Since $B_i \cup B_{i+1} \cup B_{i+2}$ is not a subalgebra we obtain from (1.1) and (8.1) that

$$B_i \cap B_{i+2} = B_i \cap B_{i+1} \cap B_{i+2} \subseteq B_{i+3} \cup B_{i+4}$$

and we may by (1.4) and symmetry assume that

$$B_i \cap B_{i+1} \cap B_{i+2} \subseteq B_{i+3}.$$

Since $B_i \cup B_{i+1} \cup B_{i+2} \cup \mathrm{B}_{i+3}$ is not a subalgebra we have

$$B_i \cap B_{i+1} \cap B_{i+2} \cap B_{i+3} \subseteq B_{i+4}$$

hence

$$B_i \cap B_{i+2} = \cap_{j=0}{}^4 B_j = C(L),$$

proving the last claim.

(8.3) *Let L be an* OML *with five blocks B_i* $(0 \leqq i \leqq 4)$ *satisfying*

$$B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_0,\ B_0 \cap B_1 \nsubseteq B_2 \cup B_3 \text{ and } B_1 \cap B_2 \nsubseteq B_3.$$

Then $B_0 \cap B_1 \cap B_2 \cap B_3 \nsubseteq B_4$ holds, and in particular $B_0 \cup B_1 \cup B_2 \cup B_3$ is a subalgebra with four blocks.

Proof. By (1.2) we may assume that L is irreducible. If the claim were not true we would then have $B_0 \cap B_1 \cap B_2 \cap B_3 = \{0, 1\}$. We show that this leads to a contradiction. The assumptions

$$B_0 \cap B_1 \nsubseteq B_3,\ B_0 \sim B_3,\ B_1 \cap B_2 \nsubseteq B_3 \text{ and } B_2 \sim B_3$$

imply by (4.1) that $B_1 \cap B_3 \subseteq B_0$ and $B_1 \cap B_3 \subseteq B_2$, hence $B_1 \cap B_3 = \{0, 1\}$. Since $B_0 \cap B_1 \nsubseteq B_2$, $B_1 \sim B_2$ and $B_2 \sim B_3$ we obtain by the same argument that $B_0 \cap B_2 \subseteq B_1$ and one of $B_0 \cap B_2 \subseteq B_3$ or $B_0 \cap B_3 \subseteq B_2$ holds. The second of these inclusions would imply

$$B_0 \cap B_3 = B_0 \cap B_2 \cap B_3 \subseteq B_1,$$

hence $B_0 \cap B_3 = \{0, 1\}$, contradicting $B_0 \approx B_3$. We thus have $B_0 \cap B_2 \subseteq B_3$ and we obtain:

(*) $\quad B_0 \cap B_2 = B_1 \cap B_3 = \{0, 1\}.$

Choose a, b, d such that $B_0 \sim_a B_1 \sim_b B_2$, $B_0 \sim_d B_3$. Since $a \vee b \geqq a, b$ and $a \vee b \in B_1$ we obtain $a \vee b \in B_0 \cap B_2$, hence, by (*), $a \vee b = 1$ and $a' \leqq b$. By the same argument we obtain $a' \leqq d$. Since $b \neq 0, 1$ and $b \in B_1$

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$\cap B_2$ we obtain from (*) that

$$b \in (B_1 \cap B_2) - (B_0 \cup B_3).$$

By the same argument we obtain

$$d \in (B_0 \cap B_3) - (B_1 \cup B_2).$$

Since $B_0 \sim B_1$ and $B_2 \sim B_3$ this implies

$$a' \leqq b \wedge d \in (B_0 \cap B_1) \cap (B_2 \cap B_3) = \{0, 1\},$$

hence $a' = 0$, contradicting $B_0 \approx B_1$.

(8.4) *If at least one block of an* OML L *with five blocks has valence at least 3 then there exist four blocks in* L *the union of which is a subalgebra of* L *with four blocks.*

Proof. We may assume that L is irreducible and that the blocks are enumerated in such a way that $B_0 \approx B_1 \approx B_2$ and $B_1 \approx B_3$ hold and that B_4 is the remaining block. Assume first that $B_0 \cup B_1 \cup B_2 \cup B_3$ is not a subalgebra. Then by symmetry, we may assume that there exist elements $a \in B_0 - (B_1 \cup B_2)$ and $b \in B_2 - (B_0 \cup B_1)$ such that

$$a \vee b \in B_4 - (B_0 \cup B_1 \cup B_2 \cup B_3).$$

It follows from this that a and b are both in B_4. Since not both of them are in B_3 it can either happen that none of them or one of them, say b, is in B_3. It follows that either

$$B_0 \cap B_4 \not\subseteq B_1 \cup B_2 \cup B_3 \text{ and } B_2 \cap B_4 \not\subseteq B_0 \cup B_1 \cup B_3$$

or

$$B_0 \cap B_4 \not\subseteq B_1 \cup B_2 \cup B_3 \text{ and } B_2 \cap B_3 \cap B_4 \not\subseteq B_0 \cup B_1$$

holds. In the first case the assumptions of (8.3) are satisfied (with suitable permutation of the indices) and the claim is proved. In the second case $B_2 \cup B_3 \cup B_4$ is a subalgebra with three blocks and hence one of $B_2 \sim B_4$ or $B_3 \sim B_4$ holds. Since we have $B_2 \cap B_4 \not\subseteq B_1$ and $B_3 \cap B_4 \not\subseteq B_1$ the assumptions of (8.3) are satisfied in both cases and there is nothing left to prove. We may thus assume that $B_0 \cup B_1 \cup B_2 \cup B_3$ is a subalgebra. If $B_0 \cup B_1 \cup B_2$ is a subalgebra then $B_0 \cup B_1 \cup B_2 \cup B_3$ has by (1.4) four blocks and the proof is again complete. If not there exist

$$a \in B_0 - (B_1 \cup B_2) \text{ and } b \in B_2 - (B_0 \cup B_1)$$

with $a \vee b \in (B_3 \cup B_4) - (B_0 \cup B_1 \cup B_2)$ and we obtain by the usual argument that

$$B_0 \cap (B_3 \cup B_4) \not\subseteq B_1 \cup B_2 \text{ and } B_2 \cap (B_3 \cup B_4) \not\subseteq B_0 \cup B_1$$

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hold. This implies that one of the conditions

$$B_0 \cap B_3 \nsubseteq B_1 \cup B_2 \cup B_4,\ B_0 \cap B_4 \nsubseteq B_1 \cup B_2 \cup B_3,\qquad B_0 \cap B_3 \cap B_4 \nsubseteq B_1 \cup B_2$$

and one of the conditions

$$B_2 \cap B_3 \nsubseteq B_0 \cup B_1 \cup B_4,\ B_2 \cap B_4 \nsubseteq B_0 \cup B_1 \cup B_3,\qquad B_2 \cap B_3 \cap B_4 \nsubseteq B_0 \cup B_1$$

is satisfied. If $B_0 \cap B_3 \nsubseteq B_1 \cup B_2 \cup B_4$ then $B_0 \cup B_3$ is a subalgebra and $B_0 \cup B_1 \cup B_2 \cup B_3$ has four blocks. The same conclusion is obtained if $B_2 \cap B_3 \nsubseteq B_0 \cup B_1 \cup B_4$. If $B_0 \cap B_4 \nsubseteq B_1 \cup B_2 \cup B_3$ and $B_2 \cap B_4 \nsubseteq B_0 \cup B_1 \cup B_3$ the desired conclusion follows again from (8.3). Using symmetry it is thus enough to consider the cases

$$B_0 \cap B_4 \nsubseteq B_1 \cup B_2 \cup B_3 \text{ and } B_2 \cap B_3 \cap B_4 \nsubseteq B_0 \cup B_1$$

or

$$B_0 \cap B_3 \cap B_4 \nsubseteq B_1 \cup B_2 \text{ and } B_2 \cap B_3 \cap B_4 \nsubseteq B_0 \cup B_1.$$

In the first of these cases we have

$$B_2 \cap B_4 \nsubseteq B_1,\ B_3 \cap B_4 \nsubseteq B_1$$

and one of $B_2 \sim B_4$ or $B_3 \sim B_4$, so that we may apply (8.3) again. If in the second case either $B_0 \sim B_3$ or $B_2 \sim B_3$ the subalgebra $B_0 \cup B_1 \cup B_2 \cup B_3$ has four blocks. In the remaining case we have

$$B_0 \approx B_4,\ B_2 \approx B_4,\ B_0 \cap B_4 \nsubseteq B_1 \cup B_2 \text{ and } B_2 \cap B_4 \nsubseteq B_0 \cup B_1$$

and (8.3) applies again, completing the proof.

(8.5) *Let L be an OML with five blocks B_0, B_1, B_2, B_3, B_4 and assume that $B_0 \cup B_1 \cup B_2 \cup B_3$ is a subalgebra with four blocks. Then L is obtained by pasting $B_0 \cup B_1 \cup B_2 \cup B_3$ and B_4 along a segment.*

Proof. We may assume without loss of generality that L is irreducible. Since

$$(B_0 \cup B_1 \cup B_2 \cup B_3) \cap B_4 \subseteq B_0 \cup B_1 \cup B_2 \cup B_4$$

it follows from (1.4) and (7.5) that there exists an index i, $0 \leqq i \leqq 3$ satisfying $(B_0 \cup B_1 \cup B_2 \cup B_3) \cap B_4 \subseteq B_i$. It is easy to see that then $B_i \cup B_4$ is a subalgebra of L and, if $B_i \sim B_j$ holds for some $j \neq 4$, then $B_j \cup B_i \cup B_4$ is a subalgebra of L with three blocks and middle block B_i. Furthermore, there exists $e \in B_i \cap B_4$ such that

$$(B_0 \cup B_1 \cup B_2 \cup B_3) \cap B_4 = B_i \cap B_4 = ([0, e'] \cup [e, 1]) \cap (B_i \cup B_4).$$

We now have to distinguish various cases. If $B_0 \cup B_1 \cup B_2 \cup B_3$ is line-like we may assume that $B_0 \sim B_1 \sim B_2 \sim B_3$ is a line-like ordering of the blocks

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and it is by symmetry enough to consider the cases $i = 3$ and $i = 2$. If $i = 3$ it is easy to see that $B_0 \sim B_1 \sim B_2 \sim B_3 \sim B_4$ is a line-like ordering of the blocks of L and the claim follows from (5.2). If $i = 2$ it is easy to see that $B_0 \cup B_1 \cup B_2 \cup B_4$ is a subalgebra of L and that $B_0 \sim B_1 \sim B_2 \sim B_4$ is a line-like ordering of its blocks. It follows from (5.2) that

$$\begin{aligned} B_2 \cap B_4 &= ([0, e'] \cup [e, 1]) \cap (B_0 \cup B_1 \cup B_2 \cup B_4) \\ &= ([0, e'] \cup [e, 1]) \cap (B_4 \cup B_2 \cup B_3) = [0, e'] \cup [e, 1], \end{aligned}$$

which again proves the claim. If $B_0 \cup B_1 \cup B_2 \cup B_3$ has a middle block we may assume that B_1 is a middle block and it is by symmetry enough to consider the cases $i = 2$ and $i = 1$. If $i = 2$, $B_3 \cup B_1 \cup B_2 \cup B_4$ and $B_0 \cup B_1 \cup B_2 \cup B_4$ are line-like subalgebras and the given orderings are line-like orderings of their blocks. The claim then follows from (5.2) as before. If $i = 1$, $B_0 \cup B_1 \cup B_2$, $B_0 \cup B_1 \cup B_3$ and $B_0 \cup B_1 \cup B_4$ are subalgebras with three blocks and middle block B_1 and the claim follows as before using (6.1). It remains the case that $B_0 \cup B_1 \cup B_2 \cup B_3$ is isomorphic with the direct product of two OMLs with two blocks each and we may by symmetry assume that

$$B_0 \sim B_1 \sim B_2 \sim B_3 \text{ and } (B_1 \cup B_2 \cup B_3) \cap B_4 \subseteq B_0,$$

i.e. $i = 0$. In this case $B_4 \cup B_0 \cup B_1$ and $B_4 \cup B_0 \cup B_2$ are subalgebras with middle block B_0 and we obtain as before

$$B_0 \cap B_4 = ([0, e'] \cup [e, 1]) \cap (B_0 \cup B_1 \cup B_3 \cup B_4).$$

But $e \leqq x \in B_2 - (B_0 \cup B_1 \cup B_3 \cup B_4)$ would imply

$$e \in B_0 \cap B_2 = B_0 \cap B_1 \cap B_2 \cap B_3 \cap B_4 = \{0, 1\},$$

i.e. $e = 1$, in which case the claim is trivially true. If $e \neq 1$ we thus obtain

$$B_0 \cap B_4 = [0, e'] \cup [e, 1]$$

and the claim is again proved.

We are now in a position to describe all OMLs with five blocks completely.

(8.6) *Every* OML L *with five blocks either satisfies the assumption of* (8.1) *or can be obtained by pasting an* OML *with four blocks and a Boolean algebra along a segment.*

Proof. If at least one block of L has valence at least three the claim follows from (8.4) and (8.5). If there exist blocks which can not be joined by a strictly proper path the claim follows easily from (4.5), (6.1) and (7.5). We may thus assume that any two blocks can be joined by a strictly proper path and that every block has valence at most two. It is then easy to see that with suitable enumeration of the blocks the only strong links are either

$$B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_4 \text{ or } B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_4 \approx B_0.$$

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In the second case the assumptions of (8.1) are satisfied. In the first case we have $d(B_0, B_4) = 4$ and the claim follows from (5.8) and (5.2). The theorem is thus proved.

Since it is not difficult to prove that any five Boolean algebras satisfying the conditions of (8.1) and (8.2) can be amalgamated to give an OML with five blocks, the last result describes all OMLs with five blocks completely.

9. Two related results. The two results of this chapter are not directly related to the methods developed in this paper; they are both consequences of (1.5). But since both of them concern block-finite OMLs we present them here.

Let MOn $(n \geqq 2)$ be the modular OL consisting of $2n$ pairwise incomparable elements and bounds 0, 1. It is well known that the only finite irreducible modular OLs are the lattices MOn. (See [**2**], proof of (4.4). The result with a different proof was known much earlier among the lattice theorists at the University of Massachusetts, where I learned of it in 1970). We show here that the result remains true for block-finite modular OLs.

(9.1) *The only block-finite, irreducible, modular* OL*s are the lattices* MOn *and* 2. *The variety of all modular* OL*s is not generated by the block-finite members. The equation* $\gamma(x, \gamma(y, z)) = 0$ *holds for all block-finite modular* OL*s but does not hold in all modular* OL*s.*

Proof. Let L be a block-finite, irreducible modular OL. To prove the first result it is obviously enough to show that for all $a, b \in L$, $a < b$ implies $a = 0$ or $b = 1$. Assume that $a < b$. Let $B_0, B_1, \ldots, B_n$ be the blocks of L and let M be a finite subset of L which contains a, b and an element of each of the differences $B_i - B_j$ $(i \neq j,\ 0 \leqq i, j \leqq n)$. Let S be the subalgebra of L generated by M. By (1.5), S is finite. Since S contains an element of each of the differences $B_i - B_j$ the blocks of S are exactly the sets $S \cap B_i$ $(0 \leqq i \leqq n)$. Since L is irreducible S is also irreducible. By what is known it follows that S is $MO(n + 1)$, i.e. that $a = 0$ or $b = 1$, proving the first part. The rest is a consequence of (4.4) of [**2**].

The second application of (1.5) concerns varieties of OMLs. In [**4**] it was shown that every finite OML L which does not belong to the variety $[MO2]$ generated by $MO2$ contains one of the lattices of figures 2 to 5 of [**4**] as a homomorphic image of a subalgebra. This can be generalized as follows:

(9.2) *Every block-finite* OML L *which is not in* $[MO2]$ *contains one of the* OML*s of figures 2 to 5 of* [**4**] *as a homomorphic image of a subalgebra.*

This follows from the fact that if a block-finite OML L does not belong to $[MO2]$ then a finitely generated subalgebra S of L does not belong to $[MO2]$. Since by (1.5) every such S is finite we may apply the quoted result of [**4**] to obtain (9.2).

https://doi.org/10.4153/CJM-1979-090-6 Published online by Cambridge University Press

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https://doi.org/10.4153/CJM-1979-090-6 Published online by Cambridge University Press