4.7 MĀDHAVA'S RULE FOR FINDING ANGLE BETWEEN THE ECLIPTIC AND THE HORIZON AND ĀRYABHATA'S KNOWLEDGE OF IT

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INTRODUCTION

In Fig.1, let U be the rising point of the ecliptic *(udaya-lagna)*, T be the nonagesimal *(tribhona-lagna)* and M be the meridian-ecliptic point *(madhya-lagna)*. Since T is at a distance of one quadrant from U along the ecliptic, the complement of the zenith distance of T, that is the arc TK, will be the required angle between the ecliptic UTM and the horizon NUEKS. The equivalent problem in Indian astronomy is therefore to find what is called *'dṛkkṣepa-jyā'* or the sine of the zenith distance of the nonagesimal.

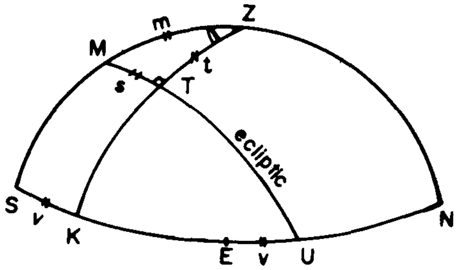

Fig: 1

In Fig.1, all angles at T and K are right angles, and it is easily seen that

arc SE = arc KU = arc TU = arc 90^0

so that

arc EU = arc SK = angle MZT = v, say.

If t and m denote the zenith distance of T and M respectively, then the required angle TUK = arc TK = 90-t.

From the spherical right-angled triangle ZMT we have

$$\sin s = \sin m . \sin v \qquad (1)$$

$$\text{and } \cos m = \cos s . \cos t \qquad (2)$$

where s denotes the arc MT.

Converting all cosines to sines in (2) and solving for sin t, we get

$$\sin t = \sqrt{(\sin^2 m - \sin^2 s) / (1-\sin^2 s)} \qquad (3)$$

while (2) directly gives

$$t = \cos^{-1} (\cos m / \cos s) \qquad (4)$$

Relation (4) may be said to represent the modern solution for finding t from known m and s by using spherical trigonometry (that is, by working on the surface of the celestial sphere). Mathematically it is equivlent to (3).

MĀDHAVA'S RULE

A mathematically correct rule for finding the *dṛkkṣepa* (= arc ZT) as given by Mādhava of Saṅgamagrāma (ca.1460-1425) has been quoted by Nīlakaṇṭha Somayājī (ca. 1500) in his commentary on the *Āryabhaṭīya* of Āryabhata I (born 476 A.D.). The rule is as follows (Pillai 1957, p.75).

Lagnaṃ tribhonaṃ dṛkkṣepalagnaṃ tanmadhyalagnayoḥ/
Vargīkṛtyantarala-jyaṃ madhyajyāvargatas-tyajet//
Trijyakṛtesca tanmule kramaso guṇahārakau/
Tābhyaṁ dṛkkṣepa-saṃsiddhih trijyāyā jāyate sadā//

"Take the square of the sine of the (arcual) distance between the nonagesimal which is three signs short (in longitude) of the orient-eliptic point and the meridian-ecliptic point. Subtract it from the square of the sine (of the zenith distance) of the meridian-ecliptic point as well as from the square of the radius. The square roots of those two results) are respectively the multiplier and divisor (of the radius). When so operated, the radius will always give the true (sine of the) zenith distance of nonagesimal".

That is, $$\sqrt{(R \sin m)^2 - (R \sin s)^2} = \text{multiplier} \qquad (5)$$

$$\sqrt{R^2 - (R \sin s)^2} = \text{divisor} \qquad (6)$$

Then

$$R. (\text{multiplier}) / (\text{divisor}) = R \sin t \qquad (7)$$

On substitution from (5) and (6) into (7), we find that Mādhava's rule is exactly equivalent to (3), the modern formula employing only sines.

Slightly earlier Nīlakaṇṭha in his *Tantra-saṅgraha* (1500 A.D.) (Sarma 1977), Verses 5-7 had given a similar rule but first computing R sin s by a formula equivalent to (1), But by quoting Mādhava by name, he has now made clear that the real credit for giving the correct rule in explicit form goes to Mādhava. In view of this, the guess of Sengupta (1934) that the correct rule was "perhaps first noticed by Raṅganātha (ca.1603)" is rendered wrong.

However, certain remarks made in the NAB (Pillai 1957) concerning Mādhava's rule show that Nīlakantha is crediting even Āryabhaṭa I for knowing it. We discuss that in the next section.

ĀRYABHAṬA'S RULE

The rule given by Āryabhaṭa I (b, 476 A.D.) in his *Āryabhaṭīya* IV (Gola), 33 for finding the sine of the zenith distance of the nonagesimal (central ecliptic point) is as follows.

Madhyajyodayajīvā-saṃvarge vyāsadalahṛte yat syāt/
Tanmadhyajyākṛtyor-viseṣamūlaṃ svadṛkkṣepaḥ//

Shukla and Sarma (1976, p.144) translated it as

"Divide the product of the *madhyajyā* and the *Udayajyā* by the radius. The square root of the difference between the squares of that (result) and the *madhyajyā* is the (Sun's or Moon's) own dṛkkṣepa".

According to their explanation (Shukla & Sarma 1976 , p.145), the first part of the above rule gives

$$R \sin s = (R \sin m).(R \sin v)/R \qquad (8)$$

and the second part then gives

$$R \operatorname{Sin} t = \sqrt{(R \sin m)^2 - (R \sin s)^2} \qquad (9)$$

which, they further say, "is obtained by treating the triangle formed by the Sines of the sides of the spherical triangle ZTM as plane right-angled triangle (which assumption is however incorrect)". Since(9) is only as approximate formula, the above interpretation or translation discredits Āryabhaṭa I for not knowing the correct or exact rule which should be equivalent to (3).

However, the remarks made by Nīlakaṇṭha in his *NAB* about the above rule and the corresponding rule of Mādhava (see Section 2 above) show that Āryabhaṭa knew the correct rule. Just after quoting Mādhava's rule, Nīlakaṇṭha says (Pillai 1957, p.75).

Atra yā dṛkkṣepalagna-madhyalagnāntarālajīvā saiva madhyajyodayajīvā-saṃvargad vyasadalaptā,Itah paramubhayatrapi samānaṃ karma.

"Here (in Mādhava's rule) what is called R sin s is the same as the product of R sin m and R sin v divided by R (in the Āryabhaṭa's rule). After this both the procedures are the same (*samānam*)".

That is, the difference between the two rules is only an initial one in the sense that Mādhava takes directly the quantity R sin s which itself is first calculated by Āryabhaṭa by using (8). Then there is no difference. *NAB* (Pillai 1957, p.75) continues and states that what is called "*svadṛkkṣepa*" by Āryabhaṭa is verily the same (*saiva*) as

$$\sqrt{(R \sin m)^2 - (R \sin s)^2}$$

explicitly taken by Mādhava.

In other words, the second part of Āryabhata's rule does not represent the final *dṛkkṣepajya* (= R sin t) as Shukla and Sarma (1976) think but gives only

$$svadṛkkṣepa = \sqrt{(R \sin m)^2 - (R \sin s)^2} \qquad (10)$$

instead of (9)

The *NAB* (Pillai 1957, pp.75-76) attaches a special significance to the prefixing word *sva* (literally "own") and explains how to obtain the actual or final *(param) dṛkkṣepa* from the *svadṛkkṣepa* given by (10). Nīlakaṇṭha says (Pillai 1957, p.76).

Yat punariha svasabdena sūcitaṃ trairāsikaṁ tadeva mādhavena vispaṣṭaṃ pradarsitam

"The Rule of Three which is indicated here by the word *sva* (in Āryabhaṭa's rule), the same has been explicitly given by Mādhava(in the last part of his rule)".

That is (with some more details avilable in NAB), Āryabhaṭa's *svadṛkkṣepa* represents only an intermediary step as the sine-chord in a circle of radius R cos s from which the true sine of the zenith distance of the nonagesimal is to be obtained by adjusting the value to the standard circle of radius R by the Rule of Three thereby giving

$$R \sin t = (svadṛkkṣepa) . R/R \cos s \qquad (11)$$

On substitution from (10) into (11) we see that Āryabhaṭa's rule will yield the correct value which is same as more clearly expressed by Mādhava. In fact the exposition in *NAB* (Pillai 1957) gives the impression that Mādhava is only elaborating Āryabhaṭa's rule but in more explicit form. If this interpretation is accepted Āryabhaṭa is to be credited for knowing the correct rule and the modern translations of his text in *Āryabhaṭīya*, IV (Gola), 33 are to be modified.

RATIONALE AND CONCLUDING REMARKS

That the rule discussed above was correctly known to Āryabhaṭa is also shown by his knowledge of the correct solution of a mathematically similar problem of finding the right ascension from a given longitude and obliquity (and declination which itself depends on the longitude and obliquity). With reference to Fig.2,

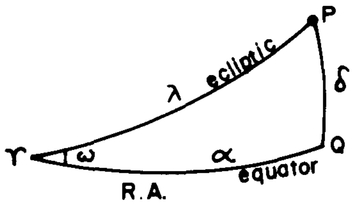

Fig. 2

the solution given in *Āryabhaṭīya*, IV, 25 is equivalent to (Shukla & Sarma 1976, pp.133-134)

$$R \sin\alpha = (R \sin \lambda). \ (R \cos w) \ / \ R \cos \delta \qquad (12)$$

If we apply this formula to the solution of the analogous spherical right-angled triangle ZMT in Fig.1, we get

$$R \sin t = (R \sin m).(R \cos v) \ / \ R \cos s \ (13)$$

which can be easily seen to be equivalent to

$$R \sin t = \left[\sqrt{(R \sin m)^2 - \left(\frac{(R \sin m).(R \sin v)}{R}\right)^2}\right] (R/R \cos s)$$

$$= \left[\sqrt{(R \sin m)^2 - (R \sin s)^2}\right] (R/R \cos s)$$

by (8). Thus we get the desired rule.

Once we note that the problem of finding *dṛkkṣepa* is exactly analogous to that of finding the right ascension, the derivation of the rule for the former will be similar to that of latter. And the Indian derivation of (12) by working inside the celestial sphere depends on applying the the *trairasika* (Rule of Three) twice. Details of this simple rationale are already known (Gupta 1974, Shukla & Sarma 1976, p.134).

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