ABSTRACT INTEGRAL SPACES AND MINIMAL EXTENSIONS

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1. Introduction. The development of the theory of absolute integrals derives from certain key facts. Among them are:

(I) An integral is a positive linear functional on a vector lattice, which is continuous in a certain sense.
(II) A function equal almost everywhere to a summable function is itself summable.
(III) Every measurable function is the pointwise limit of a sequence of elementary step functions.

A device that often plays an important role in measure theory, but which has not been fully exploited in the theory of abstract integrals is that of

(IV) the smallest class containing a given class and having a certain property

(such as being a σ-ring of sets). It is our purpose in this paper to examine the theory of abstract real-valued absolute integrals axiomatically, in such a way as to isolate and clarify the roles of (I) through (IV).

What emerges is a theory entirely analogous to that of Borel and Lebesgue measure in Euclidean space but which does not use (III) at all. In this more general presentation, approximation of measurable functions by L-bounded functions (Lemma 6 (iv)) plays the role that (III) plays in a measure oriented setting.

The simplest objects of consideration, related only to (I), are triples $(X, \mathscr{L}, I)$ called *integral spaces*. Briefly, X is a set, $\mathscr{L}$ is a family of extended real-valued functions on X, called a *function lattice*, which has the properties one expects of " summable " functions, and I is a positive linear functional on $\mathscr{L}$ which satisfies the " Monotone Convergence Theorem ". It is not assumed that the structure is obtained by means of an extension procedure. Integral spaces are defined and the usual " Convergence Theorems " are stated in Section 3B. Null functions and null sets are defined, and their usual properties obtained, in this generality in Section 3C. *I-complete* integral spaces (which satisfy (II)) are considered in Section 3D.

In order to get non-trivial examples of integral spaces, the Daniell extension of an elementary integral space (X, L, I) is reviewed in Section 3E. The resulting integral space is denoted by $(X, \mathscr{L}_I, I)$ and is called the *I-completion* of (X, L, I). It is characterized as the unique smallest I-complete integral space extending (X, L, I) in Section 3G (Corollary 12). This fact, together with the results of Terpe [7], makes it possible to prove that the Lebesgue integral space $(\mathbf{R}^n, L(m), \int_{-\infty}^{\infty} \cdot\, dm)$ is the only possible I-complete integral space extending the elementary Riemann integral space $(\mathbf{R}^n, C_c(\mathbf{R}^n), \int_{-\infty}^{\infty} \cdot\, dx)$ (Example 3G(1)).

In Section 3F, following Loomis [3], we describe the unique smallest integral space extending (X, L, I) (Theorem 9). It is denoted by (X, L_I, I) and is called the *Baire extension* of (X, L, I). The family $\mathscr{B}_L$ of *L-Baire functions*, defined as the smallest family containing L

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and closed under monotone pointwise limits (concept (IV)), plays the important role of measurable functions.

Since many of the results needed for the presentation are well-known, reference is often made to some recent texts (Loomis [3], Segal and Kunze [4] and Taylor [6]) rather than to primary sources of these results. This was done for the convenience of the reader, simply because the presentation given here is similar in spirit and notation to parts of these volumes, albeit in a more general setting. References of this sort should not be interpreted as reference to original results (although this sometimes is the case). The extent to which the results given here are new is discussed in Section 4.

2. Conventions and Notation. Throughout, X will denote an arbitrary non-empty set, $\mathbf{R}$ will denote the set of real numbers and $[-\infty, \infty]$ will denote the set $\mathbf{R}\cup\{-\infty, \infty\}$ of *extended real numbers.* In the sequel we shall consider various pointwise operations defined on families of extended real-valued functions, that is, functions which assume values in $[-\infty, \infty]$. We must therefore agree upon an interpretation for " indeterminate " expressions such as $0\cdot\infty$ and $\infty-\infty$ which can occur when considering $0\cdot g(x)$ and $f(x)+g(x)$ if f and g can assume the values $\pm\infty$. It will be convenient to adopt the convention $0\cdot(\pm\infty)=0$. Since there is no compelling advantage to fixing the value of $\pm(\infty-\infty)$, we will leave this expression undefined. We thus adopt the following conventions for extended real numbers. (These conventions are the same as [4] but differ from those of [5] and [6].)

Order. Define $-\infty < a < \infty$ if $a\in\mathbf{R}$, and $-\infty<\infty$. If we take

$$\inf\emptyset=\infty \quad \text{and} \quad \sup\emptyset=-\infty,$$

where $\emptyset$ denotes the empty set, then $[-\infty, \infty]$ is a totally ordered set with respect to $\leqq$, in which every subset A of $[-\infty, \infty]$ has a least upper bound (written $\sup A$) and a greatest lower bound (written $\inf A$). We also take $|-\infty|=|\infty|=\infty$. We give $[-\infty, \infty]$ the order topology.

Multiplication.

$$a(\pm\infty)=(\pm\infty)a=\begin{cases}\pm\infty, & \text{if} \quad a\in(0, \infty];\\ \mp\infty, & \text{if} \quad a\in[-\infty, 0);\\ 0, & \text{if} \quad a=0.\end{cases}$$

Addition. $a+\infty=\infty+a=\infty$, if $a\in(-\infty, \infty]$;

$a-\infty=-\infty+a=-\infty$, if $a\in[-\infty, \infty)$.

We do not define $\pm(\infty-\infty)$.

It is important to observe that addition is continuous when defined; i.e.

(1) If a_n, $b_n\in[-\infty, \infty]$, $a_n\to a$ and $b_n\to b$, where a_n+b_n and $a+b$ are defined, then $a_n+b_n\to a+b$.

Unless otherwise specified, any function on X will tacitly be assumed extended real-valued. A function which can assume only values in $\mathbf{R}$ will be called a *real function*. If f and g are functions on X, the functions $f \vee g$, $f \wedge g$, and $|f|$ are defined at $x \in X$ by

(2) $(f \vee g)(x) = \max(f(x), g(x))$,
(3) $(f \wedge g)(x) = \min(f(x), g(x))$,
(4) $|f|(x) = |f(x)|$.

Similarly, $(cf)(x) = c \cdot f(x)$ defines the function cf on X, for any $c \in [-\infty, \infty]$. However, in light of our conventions for addition, the function $f+g$ defined by $(f+g)(x) = f(x)+g(x)$ is only defined on the set

$$D_{f+g} = \{x \in X : f(x) \text{ and } g(x) \text{ are not oppositely infinite}\},$$

which might easily be a proper subset of X. We shall say that *$f+g$ is defined* iff $D_{f+g} = X$. The relation $f \leqq g$ defined pointwise on X is taken as the partial order on $[-\infty, \infty]^X$. Then $f \vee g$ and $f \wedge g$ are respectively the supremum and infimum of the set $\{f, g\}$.

The subscript n will always run through the positive integers. Thus if $\mathscr{F}$ is a family of functions on X, we will simply say "$f_n \in \mathscr{F}$" instead of "$f_n \in \mathscr{F}, n = 1, 2, \ldots$". We write "$f_n \to f$" or "$f = \lim_n f_n$" if the sequence of functions $\{f_n\}$ converges pointwise to f. We also write "$f_n \uparrow f$" to mean "$f_1 \leqq f_2 \leqq \ldots$, and $f_n \to f$"; "$f_n \downarrow f$" is interpreted analogously. The functions $\inf_n f_n$, $\sup_n f_n$, $\underline{\lim}\, f_n$ and $\overline{\lim}\, f_n$ are likewise interpreted pointwise, where for $a_n \in [-\infty, \infty]$,

$$\underline{\lim}\, a_n = \lim_{n\to\infty} (\inf_{k \geqq n} a_k), \qquad \overline{\lim}\, a_n = \lim_{n\to\infty} (\sup_{k \geqq n} a_k).$$

The characteristic function of the set $A \subset X$ will be denoted by χ_A.

3A. Function lattices.

Definition 3A. The non-empty family $\mathscr{L}$ of extended real-valued functions on X is called a *function lattice* (*on* X) if for any $f, g \in \mathscr{L}$

[$\mathscr{L}$1] If $a, b \in \mathbf{R}$ and $af+bg$ is defined on X, then $af+bg \in \mathscr{L}$.
[$\mathscr{L}$2] $f \vee g$ and $f \wedge g$ are also in $\mathscr{L}$.

It follows from [$\mathscr{L}$1] (with $a = b = 0$) that $0 \in \mathscr{L}$. Since $\mathscr{L}$ is closed under the *lattice operations* $\vee$ and $\wedge$ defined by (2) and (3), the functions f^+ and f^- defined by

(5) $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$

are in $\mathscr{L}$ whenever $f \in \mathscr{L}$. Finally, since $|f| = f^+ \vee f^-$, we see that any function lattice is closed under the operation of forming absolute values. Note that $0 \leqq f^+ = (-f)^-$ and $0 \leqq f^- = (-f)^+$ and the identities

(6) $f = f^+ - f^-$ and $|f| = f^+ + f^-$

hold identically on X.

A function lattice which consists of real functions only (i.e. a linear space of real functions on X which is closed under the lattice operations) will be called a *real function lattice*.

Unlike the case of a real function lattice, it is not true that [$\mathscr{L}$1] and closure under the formation of absolute values implies [$\mathscr{L}$2] if $\mathscr{L}$ is a family of extended real-valued functions. For if we consider the set $X = \{a, b\}$ and take

$$\mathscr{L} = \{f: X \to [-\infty, \infty]: f = 0 \quad \text{or} \quad |f(b)| = \infty\},$$

these conditions are satisfied, but $\mathscr{L}$ is not a function lattice since the function f defined by $f(a) = 1, f(b) = -\infty$ is in $\mathscr{L}$ but $f^+ \notin \mathscr{L}$.

3B. Integral spaces.

Definition 3B. If $\mathscr{L}$ is a function lattice on X, the functional $I: \mathscr{L} \to \mathbf{R}$ is called an *integral* (*on* $\mathscr{L}$) if the following hold.

[L] (*Linearity*) For all $a, b \in \mathbf{R}$ and all $f, g \in \mathscr{L}$ such that $af+bg$ is defined on X, we have $I(af+bg) = a \cdot If + b \cdot Ig$.

[M] (*Monotonicity*) If $f, g \in \mathscr{L}$ and $f \leqq g$ then $If \leqq Ig$.

[MC] (*Monotone convergence property*) If $f_n \in \mathscr{L}, f_n \uparrow f$ and $\lim_n If_n < \infty$, then

(7) $f \in \mathscr{L}$ and $\lim_n If_n = If = I(\lim_n f_n)$.

The triple $(X, \mathscr{L}, I)$ is then called an *integral space*.

By applying [L] to the identity $0+0=0$, we see that $I(0) = 0$. Thus any integral is *positive* in the sense that $f \in \mathscr{L}$ and $f \geqq 0$ implies $If \geqq 0$. Property [MC] is an abstract formulation of the " Monotone Convergence Theorem " of the theory of Lebesgue integration.

Examples 3B. Take $X = \{a, b, c\}$, any three point set.

(i) $\mathscr{L} = \mathbf{R}^X$; $If = f(a)+f(b)+f(c)$ (for $f \in \mathscr{L}$).
(ii) $\mathscr{L} = \{f \in \mathbf{R}^X : f(c) = 0\}$; $If = f(a)+f(b)$.
(iii) $\mathscr{L} = \{f \in [-\infty, \infty]^X : f(a) = f(b) \in \mathbf{R}, f(c) \in \{-\infty, 0, \infty\}\}$; $If = 3f(a)$.
(iv) $\mathscr{L} = \{f \in [-\infty, \infty]^X : f(a) = f(b) \in \mathbf{R}, f(c) = \infty f(a)$ if $f(a) \neq 0$ and $f(c)$ is unrestricted if $f(a) = 0\}$; $If = 3f(a)$.
(v) $\mathscr{L} = \{f \in [-\infty, \infty]^X : f(a) = f(b) \in \mathbf{R}\}$; $If = 3f(a)$.
(vi) $\mathscr{L} = \{f \in [-\infty, \infty]^X : f(a), f(b) \in \mathbf{R}\}$; $If = f(a)+2f(b)$.
(vii) $\mathscr{L} = \{f \in [-\infty, \infty]^X : f(a) \in \mathbf{R}$ and $f(b) = f(c)\}$; $If = f(a)$.

The following results, commonly known as the *convergence theorems*, are true in any integral space $(X, \mathscr{L}, I)$, for they follow from [L], [M], and [MC]. (See 6–3 IV of [6].)

Fatou's Lemma. (a) *If* $f_n \in \mathscr{L}$ *and* $\{f_n\}$ *is bounded below in* $\mathscr{L}$, *then* $\inf_n f_n \in \mathscr{L}$ *and* $I(\inf_n f_n) \leqq \inf_n If_n$.

(b) *If in addition* $\underline{\lim}\, If_n < \infty$, *then* $\underline{\lim}\, f_n \in \mathscr{L}$ *and* $I(\underline{\lim}\, f_n) \leqq \underline{\lim}\, If_n$.

Dominated Convergence Theorem. *If* $f_n \in \mathscr{L}$ *and* $|f_n| \leqq g$, *for some* $g \in \mathscr{L}$, *then*

$$f_n \to f \Rightarrow f \in \mathscr{L} \text{ and } I(\lim_n f_n) = If = \lim_n If_n.$$

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LEVI'S THEOREM. *If* $0 \leqq f_n \in \mathscr{L}$ *and* $\sum_n If_n < \infty$ *then* $\sum_n f_n \in \mathscr{L}$ *and*

$$I\left(\sum_n f_n\right) = \sum_n If_n.$$

3C. Null functions and null sets. Let $(X, \mathscr{L}, I)$ be an integral space.

DEFINITION 3C. For any function f on X, define the set K_f by

$$K_f = \{x \in X : f(x) \neq 0\}.$$

A function h is an *(I-)null function* if $h \in \mathscr{L}$ and $I(|h|) = 0$. A set $A \subset X$ is an *(I-)null set* if either $A = \emptyset$ or $A = K_h$ for some null function h. The proposition $P(x)$ about $x \in X$ is said to be true *almost everywhere* (abbreviated a.e.) if there is a null set N such that $P(x)$ is true for all x not in N.

If $A \subset X$ and $f = \chi_A$ is a null function, then A $(= K_f)$ is a null set. The converse is false, for in Example 3B(iii), the set $\{c\}$ is a null set but $\chi_{\{c\}} \notin \mathscr{L}$. However, if A is a null set, the function $\infty\chi_A$ (which assumes the value ∞ on A and the value 0 elsewhere) must be in $\mathscr{L}$, and provides a useful device for investigating null sets.

PROPOSITION 1. *Let* $(X, \mathscr{L}, I)$ *be an integral space.*

(i) *If* $N \subset X$, *then* N *is a null set* $\Leftrightarrow \infty\chi_N$ *is a null function.*
(ii) *If* $f \in \mathscr{L}$ *then* $A = \{x \in X : |f(x)| = \infty\}$ *is a null set.*
(iii) *Countable unions of null sets are also null sets.*

Proof. (i) If $f = \infty\chi_N$ is a null function, then $N = K_f$ is a null set by Definition 3C. Conversely, suppose $N = K_h$ where h is a null function. By replacing h by $|h|$ if necessary, we may assume $h \geqq 0$. Then $0 \leqq n \cdot h \uparrow \infty\chi_{K_h} = \infty\chi_N$ and $\lim_n I(n \cdot h) = \lim_n (n \cdot 0) = 0$. [MC] then asserts that $\infty\chi_N \in \mathscr{L}$ (and is a null function).

(ii) Define $f_n = |f|/n$. Then $0 \leqq f_n$ for each n and $0 \in \mathscr{L}$, so $\inf_n f_n = \infty\chi_A \in \mathscr{L}$ by Fatou's Lemma (a) of the previous section, and the result now follows from (i).

(iii) Suppose $\{N_n\}$ is a sequence of null sets. Form

$$f_n = \infty\chi_{N_1} + \ldots + \infty\chi_{N_n}.$$

Then by (i) and [L], $f_n \in \mathscr{L}$ and $If_n = 0$ for each n. Since

$$0 \leqq f_n \uparrow \infty\chi_{\cup N_n} \quad \text{and} \quad \lim_n If_n = 0,$$

we conclude from [MC] that $\infty\chi_{\cup N_n} \in \mathscr{L}$, so $\cup N_n$ is a null set by (i).

We see from (i) that the I-null sets can be obtained from an examination of $\mathscr{L}$ (cf. Examples 3B). The following result shows that an integral cannot distinguish between functions in $\mathscr{L}$ which differ on at most a null set.

PROPOSITION 2. *Let* $(X, \mathscr{L}, I)$ *be an integral space.*

(i) *If* $f(x) \leqq g(x)$ *a.e. then* $If \leqq Ig$.
(ii) *If* $f(x) = g(x)$ *a.e. then* $If = Ig$.
(iii) *If* $f \in \mathscr{L}$ *then* f *is a null function* $\Leftrightarrow K_f$ *is a null set* $\Leftrightarrow f(x) = 0$ *a.e.*

Proof. (i) We first suppose $f \geqq 0$ and $g \geqq 0$. Let N be a null set such that $f(x) \leqq g(x)$ whenever $x \notin N$. Proposition 1(i) assures us that $\infty\chi_N \in \mathscr{L}$ and $I(\infty\chi_N) = 0$. But $f \leqq g + \infty\chi_N$ (which is defined on all of X), so by [M] and [L], $If \leqq I(g + \infty\chi_N) = Ig$. In the general case,

$$f(x) \leqq g(x) \text{ a.e. implies } f^+(x) \leqq g^+(x) \text{ a.e. and } f^-(x) \geqq g^-(x) \text{ a.e.}$$

Applying [L] to (6) we get $If = If^+ - If^- \leqq Ig^+ - Ig^- = Ig$, where the inequality follows from the first case considered.

(ii) follows from (i) since $f(x) = g(x)$ a.e. implies that $f(x) \leqq g(x)$ a.e. and $g(x) \leqq f(x)$ a.e.

(iii) is a direct consequence of the definitions and (ii).

3D. *I*-complete integral spaces. We know from Proposition 2(ii) that two functions in $\mathscr{L}$ which differ on at most a null set must have the same integral. This does not mean that we are free to arbitrarily adjust the values of a function in $\mathscr{L}$ on a null set, for there need not be a guarantee that the new function formed will be in $\mathscr{L}$. The following discussion should make this clear.

DEFINITION 3D. An integral space $(X, \mathscr{L}, I)$ is called *I-complete* if

(8) for any function g, if $f \in \mathscr{L}$ and $f(x) = g(x)$ a.e. then $g \in \mathscr{L}$.

Taking $f = 0$ in (8) gives

(9) $g \in [-\infty, \infty]^X$ and $g(x) = 0$ a.e. $\Rightarrow g$ is a null function.

As an immediate consequence of (9) we have

(10) $N \subset X$ is a null set $\Leftrightarrow \chi_N$ is a null function.

(11) Every subset of any null set is also a null set.

Thus (8) $\Rightarrow$ (9), (9) $\Rightarrow$ (10) and (9) $\Rightarrow$ (11). The following examples show that the converses of these implications are false.

EXAMPLES 3D. Consider Examples 3B. (i) and (ii) are trivially I-complete since $\emptyset$ is the only null set. (v) and (vi) are also I-complete and each has $\emptyset$ and $\{c\}$ as null sets. In (iii), (11) holds but (9) and (10) fail. In (vii), (10) holds but (9) and (11) fail. (iv) satisfies (9)–(11) but is not I-complete, for if $f(a) = f(b) = 1$ and $f(c) = \infty$, $f \in \mathscr{L}$ and $f(x) - 1$ a.e. but $1 \notin \mathscr{L}$. There are integral spaces for which all of (9)–(11) fail.

If $f, g, h \in [-\infty, \infty]^X$, we say that h *extends* $f + g$ if

(12) $h(x) = f(x) + g(x)$ whenever $f(x) + g(x)$ is defined.

PROPOSITION 3. *The integral space* $(X, \mathscr{L}, I)$ *is I-complete iff*

(13) $f, g \in \mathscr{L}$ *and* h *extends* $f + g \Rightarrow h \in \mathscr{L}$ *and* $Ih = If + Ig$.

Proof. Suppose $(X, \mathscr{L}, I)$ is I-complete, $f, g \in \mathscr{L}$ and h extends $f + g$. The set N on which f and g assume oppositely infinite values is null by Proposition 1(ii) and (11), so if we define f'

and g' by $f'(x) = f(x)$ and $g'(x) = g(x)$ if $x \notin N$, $f'(x) = g'(x) = 0$ if $x \in N$, we have $f', g' \in \mathscr{L}$ by (8). Thus $f' + g'$, which is defined on X, must be in $\mathscr{L}$ by [$\mathscr{L}$1]. But $h(x) = (f+g)(x) = (f'+g')(x)$ for $x \notin N$, so $h \in \mathscr{L}$ by (8). Furthermore $Ih = I(f'+g') = If' + Ig' = If + Ig$, by Proposition 2(ii) and [L], so (13) holds.

Conversely, suppose (13) holds, $f \in \mathscr{L}$ and g is any function on X satisfying $g(x) = f(x)$ a.e. We may assume that $f \geqq 0$ and $g \geqq 0$. If N is a null set such that $f(x) = g(x)$ if $x \notin N$, then

$$g \text{ extends } (f + \infty\chi_N) + (-\infty\chi_N)$$

so that $g \in \mathscr{L}$ by (13), and the proof is complete.

I-complete integral spaces have the desirable property that the assumption of pointwise convergence in the convergence theorems can be replaced by the even weaker assumption of convergence a.e. For example, [MC] takes the following form.

Suppose $(X, \mathscr{L}, I)$ is an I-complete integral space. If f is a function on X and there exist $f_n \in \mathscr{L}$ which satisfy $f_n(x) \uparrow f(x)$ a.e. and $\lim_n If_n < \infty$ then $f \in \mathscr{L}$ and $If_n \uparrow If$.

3E. Elementary integral spaces, the Daniell extension. In this section we review the problem of extending an elementary integral (such as the Riemann integral) on a given function lattice to an integral (satisfying [MC)] on a larger function lattice.

DEFINITION 3E–1. If $\mathscr{L}$ and $\hat{\mathscr{L}}$ are families of real or extended real valued functions on X, and I and $\hat{I}$ are functionals on $\mathscr{L}$ and $\hat{\mathscr{L}}$ respectively, we say $(X, \hat{\mathscr{L}}, \hat{I})$ *extends* $(X, \mathscr{L}, I)$ and write $(X, \mathscr{L}, I) \leqq (X, \hat{\mathscr{L}}, \hat{I})$ if $\mathscr{L} \subset \hat{\mathscr{L}}$, and I is the restriction to $\mathscr{L}$ of $\hat{I}$. We write $(X, \mathscr{L}, I) = (X, \hat{\mathscr{L}}, \hat{I})$ if $\mathscr{L} = \hat{\mathscr{L}}$ and $I = \hat{I}$.

DEFINITION 3E–2. If L is a real function lattice on X, the functional $I : L \to \mathbf{R}$ is called an *elementary integral on* L and the triple (X, L, I) is called an *elementary integral space*, if for any $f, g, f_n \in L$ and any $a, b \in \mathbf{R}$ the following hold:

[L] $I(af + bg) = aIf + bIg$;
[M] if $f \leqq g$ then $If \leqq Ig$;
[E] if $f_n \downarrow 0$ then $If_n \downarrow 0$.

Conditions [L] and [M] simply assert that I is a positive linear functional on the real function lattice L. Clearly, condition [E] may be replaced by the condition

[E′] If $f_n \uparrow f$ where $f \in L$, then $If_n \uparrow If$.

It follows from [MC] that [E]′, hence [E], is a necessary condition for there to be an integral extending the positive linear functional I. The importance of the *Daniell extension* to be reviewed next, is that it shows (Theorem 5) that [L], [M] and [E] are actually sufficient to guarantee the existence of an integral space extending (X, L, I).

The construction of an integral from an elementary integral was first carried out by Young [9], Daniell [2] and others at the beginning of this century, and further investigated by Stone [5]. The presentation summarized below is essentially that given in Taylor [6].

For the remainder of this section, (X, L, I) will denote a fixed elementary integral space. We first define the family

(14) $L^u = \{\varphi : X \to (-\infty, \infty]$: there exist $f_n \in L$ such that $f_n \uparrow \varphi\}$.

Clearly $L \subset L^u$ and, for any $\varphi, \psi \in L^u$ and any $c \in [0, \infty)$, $\varphi \vee \psi$, $\varphi \wedge \psi$ and $\varphi + c\psi$ are also in L^u. Note that $\varphi + c\psi$ is defined on all of X because $\varphi(x)$ and $\psi(x)$ cannot be $-\infty$. If $\varphi \in L^u$, define

(15) $\bar{I}\varphi = \lim_n If_n$, where $f_n \in L$ and $f_n \uparrow \varphi$.

This definition doesn't depend on the sequence $\{f_n\}$ in L which converges to φ ([6], Lemma 6–2II), so $\bar{I}f = If$ whenever $f \in L$. Moreover, for any $\varphi, \psi \in L^u$ and all $c \in [0, \infty)$,

(16) $\varphi \leqq \psi \Rightarrow \bar{I}\varphi \leqq \bar{I}\psi$, and $\bar{I}(\varphi + c\psi) = \bar{I}\varphi + c\bar{I}\psi$.

Finally, we use $\bar{I}$ of (15) to define two functionals on the family of all functions on X. If $f: X \to [-\infty, \infty]$, define the *upper* and *lower* integrals of f by

(17) $\bar{I}f = \inf\{\bar{I}\varphi : \varphi \in L^u \text{ and } \varphi \geqq f\}$,
(18) $\underline{I}f = -\bar{I}(-f)$,

respectively. (The conventions $\sup\emptyset = -\infty$, $\inf\emptyset = \infty$ are needed here.) It is easy to see that (15) and (17) yield the same value $\bar{I}\varphi$ when $\varphi \in L^u$.

DEFINITION 3E–3. Define the family $\mathscr{L}_I$ of *I-summable functions* by

(19) $\mathscr{L}_I = \{f \in [-\infty, \infty]^X : \underline{I}f = \bar{I}f \in \mathbf{R}\}$.

For any I-summable function f, we define If to be

(20) $If = \underline{I}f = \bar{I}f$.

The triple $(X, \mathscr{L}_I, I)$ is called the *I-completion* of (X, L, I).

THEOREM 5. *$(X, \mathscr{L}_I, I)$ is an I-complete integral space which extends the elementary integral space (X, L, I).*

Proof. Theorems 6–3II and III of [6] combine to prove that $(X, \mathscr{L}_I, I)$ is an integral space. To see that it is I-complete, suppose that $f, g \in \mathscr{L}_I$ and that h extends $f+g$. Then for any $\varphi, \psi \in L^u$ such that $\varphi \geqq f$ and $\psi \geqq g$, we have $\varphi + \psi \in L^u$ and $\varphi + \psi \geqq f+g$, so $\bar{I}h \leqq If + Ig$ by (17), (16), and (20). Similarly, $-h$ extends $-f + -g$, and we conclude $\underline{I}h \geqq If + Ig \geqq \bar{I}h$. But we always have $\underline{I}h \leqq \bar{I}h$ (see Lemma 3.3.2 of [4]), so we see by (19) that $h \in \mathscr{L}_I$ and $Ih = If + Ig$. $(X, \mathscr{L}_I, I)$ is therefore I-complete by Proposition 3.

3F. The Baire extension of (X, L, I). It is well known that there exist larger integral spaces than $(X, \mathscr{L}_I, I)$ which extend (X, L, I) (see Scholium 3.10 of [4]), and maximal ones have been investigated† (see Terpe [8]). It is our purpose in this section to construct the unique smallest integral space, denoted by (X, L_I, I), which extends (X, L, I). To do this, we consider small families from which the functions in L_I must come. The approach is much like that of Loomis (Section 12 of [3]), where a less extensive discussion is carried out for real functions only.

† This was kindly brought to my attention by Professor A. C. Zaanen.

DEFINITION 3F–1. Suppose $\mathscr{F}$ and L are arbitrary families of functions on X. $\mathscr{F}$ is called a *monotone* family if

$$f_n \in \mathscr{F} \quad \text{and either} \quad f_n \uparrow f \quad \text{or} \quad f_n \downarrow f \quad \text{implies} \quad f \in \mathscr{F};$$

i.e. $\mathscr{F}$ is closed under monotone pointwise limits. A function $f \in \mathscr{F}$ is *L-bounded* if $|f| \leqq h$ for some $h \in L$. The smallest monotone family of functions which contains $\mathscr{F}$ will be denoted by $\mathscr{B}_{\mathscr{F}}$. Functions in $\mathscr{B}_{\mathscr{F}}$ will be called *($\mathscr{F}$-)Baire functions.* It will be convenient to let $\mathscr{F}^+$ denote the family of non-negative functions in $\mathscr{F}$, i.e.

$$\mathscr{F}^+ = \{f \in \mathscr{F} : f \geqq 0\}.$$

We observe that the smallest monotone family containing $\mathscr{F}$ exists, for it is given by the non-void intersection

$$\mathscr{B}_{\mathscr{F}} = \cap \{\mathscr{M} : \mathscr{M} \text{ is a monotone family and } \mathscr{M} \supset \mathscr{F}\}.$$

LEMMA 6. *Let L be a family of functions on X, which is closed under $\vee$ and $\wedge$ and which contains the function* 0.

(i) *$\mathscr{B}_L$ is closed under $\vee$ and $\wedge$.*
(ii) *$\mathscr{B}_L$ is closed under arbitrary pointwise limits.*
(iii) *$(\mathscr{B}_L)^+ = \mathscr{B}_{L^+}$, i.e. $(\mathscr{B}_L)^+$ is the smallest monotone family containing L^+.*
(iv) *For any $f \in \mathscr{B}_L^+$ there exist L-bounded functions $f_n \in \mathscr{B}_L^+$ such that $f_n \uparrow f$.*
(v) *If $L^+ \subset \mathscr{F} \subset \mathscr{B}_L^+$ and $\mathscr{F}$ satisfies the condition*

(21) *$f_n \in \mathscr{F}$, f_n L-bounded and $f_n \uparrow f$ or $f_n \downarrow f$ implies $f \in \mathscr{F}$, then $\mathscr{F} = \mathscr{B}_L^+$.*

Proof. See Section 12H and Section 12I of [3]. The relevant proofs are valid when the functions in L are extended real-valued.

Theorem 12H of [3], proved there for real functions, generalizes as follows.

THEOREM 7. *If L is a real function lattice, then $\mathscr{B}_L$ is a function lattice satisfying the condition*

(22) *if $f, g \in \mathscr{B}_L$, there exists an $h \in \mathscr{B}_L$ which extends $f+g$.*

Proof. $\mathscr{B}_L$ is closed under the lattice operations by Lemma 6(i). Also, for any $c \in \mathbf{R}$, the family $\{f \in \mathscr{B}_L : cf \in \mathscr{B}_L\}$ is monotone and contains L, so $\mathscr{B}_L$ is closed under multiplication by real scalars. To complete the proof we must show that

$$\mathscr{M}_f = \{g \in \mathscr{B}_L : \text{ there is an } h \in \mathscr{B}_L \text{ extending } f+g\}$$

is all of $\mathscr{B}_L$, for any $f \in \mathscr{B}_L$. An immediate consequence of (1) is

$$f + (\lim_n g_n) = \lim_n (f + g_n)$$

whenever all the sum functions involved are defined on X, and provided that $\lim_n g_n$ exists. It follows that if f is real-valued (so that $f+g$ is defined on X for any function g) then $\mathscr{M}_f$ is a monotone family. If $f \in L$ then $L \subset \mathscr{M}_f$ and so $\mathscr{M}_f = \mathscr{B}_L$; that is,

(*) $f+g \in \mathscr{B}_L$ whenever $f \in L$ and $g \in \mathscr{B}_L$.

Next, if $g \in \mathscr{B}_L$ and is L-bounded, then g is real-valued; hence $\mathscr{M}_g$ is monotone and contains L by (*), so $\mathscr{M}_g = \mathscr{B}_L$ in this case too, whence

(**) $f+g \in \mathscr{B}_L$ whenever $f, g \in \mathscr{B}_L$ and g is L-bounded.

Finally, for arbitrary $f, g \in \mathscr{B}_L$, we use Lemma 6(iv) to get L-bounded functions $g_n^+, g_n^- \in \mathscr{B}_L$ such that $0 \leqq g_n^+ \uparrow g^+$ and $0 \leqq g_n^- \uparrow g^-$. If we define $g_n = g_n^+ - g_n^-$, then the functions g_n are in $\mathscr{B}_L$ and are L-bounded. Also, $g_n \to g$, whence by (**),

(***) $f+g_n \in \mathscr{B}_L$ and $h = \lim_n (f+g_n)$ exists.

Also, by (1), $h(x) = f(x)+g(x)$ whenever $(f+g)(x)$ is defined; that is, h extends $f+g$. But (***) and Lemma 6(ii) imply that $h \in \mathscr{B}_L$, proving that $g \in \mathscr{M}_f$ and completing the proof.

(The conclusion of the above theorem is also true with the weaker hypothesis that L is an arbitrary function lattice satisfying the condition (22). However the proof is rather more technical and we shall not need it, so we omit it. It is not known whether the condition of (22) is true for every function lattice.)

We can now describe the smallest integral space extending the given elementary integral space (X, L, I).

DEFINITION 3F–2. The family L_I of *Baire summable functions* is defined by

$$L_I = \mathscr{L} \cap_I \mathscr{B}_L,$$

where $(X, \mathscr{L}_I, I)$ is the I-completion of (X, L, I). If I also denotes the restriction to L_I of the integral on $\mathscr{L}_I$ (Definition 3E–3), the triple (X, L_I, I) is called the *Baire extension* of (X, L, I).

Clearly L_I, being the intersection of two function lattices (Theorem 7 and Theorem 5), is itself a function lattice. Also, since $\mathscr{B}_L$ is monotone, [MC] holds; that is, I is an integral on L_I. The Baire extension (X, L_I, I) is thus an integral space satisfying $(X, L, I) \leqq (X, L_I, I) \leqq (X, \mathscr{L}_I, I)$. The following strong uniqueness result implies that (X, L_I, I) is the minimal extension of (X, L, I) we are looking for.

THEOREM 8. *If $(X, \mathscr{L}, I)$ and $(X, \mathscr{Q}, \hat{I})$ are integral spaces, $L \subset \mathscr{L} \cap \mathscr{Q}$, where L is a lattice containing* 0, *and $I|_L = \hat{I}|_L$, then for any $f \in \mathscr{B}_L^+$*

(23) *either $f \in \mathscr{L} \cap \mathscr{Q}$ and $If = \hat{I}f$, or there exist $f_n \in \mathscr{L} \cap \mathscr{Q}$ such that $0 \leqq f_n \uparrow f$ and $\lim_n If_n = \lim_n \hat{I}f_n = \infty$.*

Proof. Let $\mathscr{F} = \{f \in \mathscr{B}_L^+ : (23) \text{ holds}\}$. Clearly $L^+ \subset \mathscr{F}$. Suppose that $f_n \in \mathscr{F}$ and are L-bounded, say $0 \leqq f_n \leqq h_n \in L$. Since by hypothesis $Ih_n = \hat{I}h_n < \infty$ and $h_n \geqq f_n \in \mathscr{F}$, we must have $f_n \in \mathscr{L} \cap \mathscr{Q}$ and $If_n = \hat{I}f_n$ by (23). If $f_n \uparrow f$, then an application of [MC] implies that f satisfies (23). On the other hand, if $f_n \downarrow f$ then $f \in \mathscr{L} \cap \mathscr{Q}$ and $If = \hat{I}f$ by the Dominated Convergence Theorem of 3B, so $f \in \mathscr{F}$ in this case too, proving that $\mathscr{F}$ satisfies (21). We conclude from Lemma 6(v) that $\mathscr{F} = \mathscr{B}_L^+$, and this completes the proof.

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If in Theorem 8 we take $\mathscr{L} = L_I$ (Definition 3F–2), we get the next result.

THEOREM 9. *If* $(X, \hat{\mathscr{L}}, \hat{I})$ *is any integral space extending the elementary integral space* (X, L, I), *then for all* $f \in \mathscr{B}_L^+$

$$\textit{either} \quad \underline{I}f = \bar{I}f = \infty \quad \textit{or} \quad f \in L_I \cap \hat{\mathscr{L}} \quad \textit{and} \quad If = \hat{I}f,$$

whence $(X, L_I, I) \leqq (X, \hat{\mathscr{L}}, \hat{I})$. (X, L_I, I) *is thus the unique minimal integral space extending* (X, L, I).

3G. Uniqueness of $(X, \mathscr{L}_I, I)$. Here again (X, L, I) denotes a fixed elementary integral space.

LEMMA 10. *Suppose* $(X, \mathscr{L}, I)$ *is an I-complete integral space, and that f is a function on X for which there exist* $h, h' \in \mathscr{L}$, *such that* $h' \leqq f \leqq h$ *and* $Ih' = Ih$. *Then* $f \in \mathscr{L}$ *and* $f(x) = h(x)$ *a.e.*

Proof. Define g on X by

$$g(x) = \begin{cases} h(x) - h'(x), & \text{if defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \geqq 0$ and g extends $h - h'$, so by Proposition 3

$$g \in \mathscr{L} \quad \text{and} \quad 0 \leqq Ig = Ih - Ih' = 0.$$

g is thus an I-null function, so by Definition 3C,

$$K_g = \{x \in X : g(x) \neq 0\} = \{x \in X : h(x) \neq h'(x)\}$$

is an I-null set. This implies that $h(x) = h'(x)$ a.e. But (11), together with the fact that $h' \leqq f \leqq h$, implies that $f(x) = h(x)$ a.e., so that $f \in \mathscr{L}$ by (8).

THEOREM 11. (*Approximating functions in* $\mathscr{L}_I$ *by L-Baire functions.*) $f \in \mathscr{L}_I$ *if and only if there exist* $h, h' \in L_I$ *such that*

$$h' \leqq f \leqq h \quad \textit{and} \quad Ih' = Ih.$$

Proof. $(X, \mathscr{L}_I, I)$ is I-complete so the sufficiency follows from Lemma 10. The converse is Corollary 3.5.2 of [4].

(One can actually show that in the above result $\{x \in X : h(x) \neq h'(x)\}$ is a null set in (X, L_I, I) but we will not need this stronger result here.)

COROLLARY 12. (*Uniqueness of* $(X, \mathscr{L}_I, I)$. *If* $(X, \hat{\mathscr{L}}_I, \hat{I})$ *is any I-complete integral space extending* (X, L, I), *then* $(X, \hat{\mathscr{L}}, \hat{I})$ *extends* $(X, \mathscr{L}_I, I)$. *Thus* $(X, \mathscr{L}_I, I)$ *is the unique smallest I-complete integral space extending* (X, L, I).

Proof. Use Theorem 11, Theorem 9 and Lemma 10 to show that

$$f \in \mathscr{L}_I \quad \text{implies} \quad f \in \hat{\mathscr{L}}, \quad \text{and} \quad If = \hat{I}f.$$

Corollary 12 justifies the use of the name I-completion. A slight refinement of the above argument gives the following results:

COROLLARY 13. *(Uniqueness of I on $\mathscr{L}_I$.) If $(X, \hat{\mathscr{L}}, \hat{I})$ is any integral space extending (X, L, I) and $f \in \mathscr{L}_I \cap \hat{\mathscr{L}}$, then $If = \hat{I}f$.*

COROLLARY 14. *(Null functions of $(X, \mathscr{L}_I, I)$.) f is a null function of $(X, \mathscr{L}_I, I)$ iff there is a null Baire function $h \in L_I$ such that $0 \leqq |f| \leqq h$.*

COROLLARY 15. *(Null sets of $(X, \mathscr{L}_I, I)$.) The null sets of $(X, \mathscr{L}_I, I)$ are precisely the subsets of null sets of (X, L_I, I).*

EXAMPLES 3G.

(1) *The Lebesgue integral in Euclidean space.* Take $X = \mathbf{R}$, $L = C_c(\mathbf{R})$, the real function lattice of continuous functions with compact support, and $I = \int_{-\infty}^{\infty} \cdot\, dx$, the (improper) Riemann integral which is clearly a positive linear functional on L. That it satisfies [E] (see Example 4, p. 284 of [6]) is a consequence of Dini's Theorem.

It is clear that the family $\mathscr{B}_L$ of Definition 3F–1 coincides with the family of *Baire functions on* $\mathbf{R}$, i.e. the smallest family containing the continuous real functions on $\mathbf{R}$ and closed under pointwise limits. But these in turn coincide with the Borel measurable functions on $\mathbf{R}$.

If $(\mathbf{R}, \mathscr{L}_I, I)$ denotes the I-completion of the elementary integral space $(\mathbf{R}, C_c(\mathbf{R}), \int_{-\infty}^{\infty} \cdot\, dx)$ then it is easy to see that $(\mathbf{R}, \mathscr{L}_I, I) = (\mathbf{R}, L(m), \int \cdot\, dm)$, where m denotes Lebesgue measure on $\mathbf{R}$, $L(m)$ is the function lattice of Lebesgue summable functions on $\mathbf{R}$, and $\int \cdot\, dm$ is the usual Lebesgue integral on $L(m)$. Indeed, since $(\mathbf{R}, L(m), \int \cdot\, dm)$ is well known to be I-complete, we have $(\mathbf{R}, \mathscr{L}_I, I) \leqq (\mathbf{R}, L(m), \int \cdot\, dm)$, by Corollary 12. But every Lebesgue summable function is equal a.e. to a summable Borel measurable function (i.e. a function in $L_I = \mathscr{L}_I \cap \mathscr{B}_L$) so it must be in $\mathscr{L}_I$ by (8). Combining this fact with the maximality of $(X, L(m), \int \cdot\, dm)$ proved by Terpe (Beispiel of Section 2 and Satz 1 of [9]), we see that $(\mathbf{R}, L(m), \int \cdot\, dm)$ *is the only possible I-complete integral space extending the elementary Riemann integral space* $(\mathbf{R}, C_c(\mathbf{R}), \int_{-\infty}^{\infty} \cdot\, dx)$. A similar discussion applies to n-dimensional Euclidean space.

(2) *The abstract Lebesgue integral.* Let $(X, \mathscr{S}, \mu)$ be any σ-finite measure space. (That is, μ is a σ-finite, and countably additive non-negative measure on the σ-ring $\mathscr{S}$ of subsets of X.) If L denotes the real function lattice of *summable step functions* s of the form

$$s = \sum_{i=1}^{n} c_i \chi_{A_i}, \quad \text{where} \quad c_i \in \mathbf{R}, \quad A_i \in \mathscr{S} \quad \text{and} \quad \mu(A_i) < \infty,$$

and we take

$$Is = \sum_{i=1}^{n} c_i \mu(A_i) = \int s\, d\mu$$

then I is well defined and (X, L, I) is an elementary integral space. In this case, $\mathscr{B}_L$ coincides with the $\mathscr{S}$-measurable functions. For a non-negative $\mathscr{S}$-measurable function f on X, take

$$\int f\, d\mu = \sup \{\int s\, d\mu : 0 \leqq s \leqq f\},$$

where s varies over all such summable step functions. Finally, $L(\mu)$ denotes the family of *μ-summable functions*, i.e. those $\mathscr{S}$-measurable functions for which $\int |f|\, d\mu < \infty$. (These definitions are more or less standard.) In this case the integral space $(X, L(\mu), \int \cdot\, d\mu)$ coincides with (X, L_I, I) and, letting $\bar{\mu}$ denote the completion of μ, we have $(X, L(\bar{\mu}), \int \cdot\, d\bar{\mu}) = (X, \mathscr{L}_I, I)$.

4. Discussion of related results. The abstract integral space postulated in Section 3B (using [$\mathscr{L}1$]) seem to be new. They are certainly more general than any extension of some elementary integral space; Examples 3B(iii) and (v) illustrate this. Taylor ([**6**], Section 6–4) obtains the results of Section 3C by methods similar to ours but only for I-completions and using a more restrictive definition of null sets.

Loomis (Section 12 of [**3**]) constructed what we call the Baire extension (X, L_I, I) of the elementary integral space (X, L, I), considering real functions only. The minimality characterization of (X, L_I, I) (Theorem 9) seems to be new as does the characterization of its I-completion as the unique, smallest, I-complete extension of (X, L, I) (Corollary 12). Bogdanowicz [**1**] describes the minimal extension of (X, L, I), but his argument depends heavily upon the fact that L satisfies the hypothesis: $f \in L \Rightarrow f \wedge 1 \in L$ (due to Stone).

Corollary 12 (or Theorem 8) renders unnecessary a direct verification that two given extensions of a smaller integral structure coincide. The uniqueness of the Lebesgue integral on $\mathbf{R}^n$ asserted in Example 3G(i) illustrates this.

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