CORRESPONDENCE.

ON THE CONSIDERING THE "PAID-UP" POLICY AS THE EQUIVALENT OF THE RATIO THE PREMIUMS PAID BEAR TO THE TOTAL NUMBER PAYABLE.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In a note appended to Mr. Younger's letter in the *Journal* for January last, on the subject of "Ten Years' Nonforfeiture Policies," you suggest the examination of the question, by means of a comparison of the amount of "paid-up" policy that the surrender value of the original

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assurance would purchase, with the amount that would be given, by taking the ratio of the premiums already paid to the total number payable.

I have attempted in the following letter to carry out this suggestion; and the points to which more particularly my attention has been directed are, to find what conditions are required to make what may be called the "Empirical" method of finding the amount of "paid-up" policy true, and how far these conditions deviate from the real ones.

Thus, if a person assure at age x by n equal annual payments, and wish at age $(x+t)$ to have a "paid-up" policy payable at the same time as the original policy, he will receive by the empirical method £$\frac{t}{n}$ per £1 of original assurance.

What is the relation between this value and the true one?

Mr. Sprague, in the *Journal of the Institute*, volume vii., page 58, has shown that, on the assumption that the single payments are sold by the same table as the policy is valued by, $\left(1-\frac{\mathrm{P}_x}{\mathrm{P}_{x+t}}\right)$ represents the real amount of "paid-up" policy. Though the formula is applied there only to ordinary assurances for the whole of life, it is equally applicable to other classes of assurances, if by $\mathbb{P}_x$ we mean the pure annual premium payable under the original contract, and by $\mathbb{P}_{x+t}$ the pure annual premium that would be payable at age $(x+t)$ as the equivalent for the remaining time of the same contract.* Thus, if $\mathbb{P}_x$ be the ten years' premium at age 30 to provide £1 at death, and if $t=4$, $\mathbb{P}_{x+t}$ will be the 6 years' premium at age 34 to provide £1 at death.

Again, if $\mathbb{P}_x$ be the premium at age 30 for an Endowment Assurance payable at death or 60, $\mathbb{P}_{x+t}$ will be the premium at age 34 for an Endowment Assurance payable at death or 60. If $\mathbb{P}_x$ be the Survivorship Assurance premium, age 30 against age 70, $\mathbb{P}_{x+t}$ will be the Survivorship Assurance premium, age 34 against 74, and so on.

What we have to find, then, is, the relation between $\frac{t}{n}$ and $\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)$.

Now, since after n payments of premium, the policy-holder has discharged all *his* obligations, the Assurance Company is then liable for the total amount assured, *i.e.*, 1, while at the same period the empirical formula will give $\frac{n}{n}$, *i.e.*, 1, we find the empirical rule is correct on and after age $(x+n-1)$, (premium then due being paid). Thus the assumption

* It may not be out of place to append here another proof of Mr. Sprague's formula. Using $\mathbb{P}_x$ and $\mathbb{P}_{x+t}$ in the extended sense given to them above, and supposing $1+{}_{\overline{n-t-1}|}a_{x+t}$ to mean the annuity payable from age $(x+t)$ on to the expiry of the time named in the original contract, as also employing $\mathbb{A}_{x+t}$ to represent the single payment corresponding to the annual premium $\mathbb{P}_{x+t}$, then $\frac{\mathbb{V}_{x|t}}{\mathbb{A}_{x+t}}=$ the amount of "paid-up" policy.

Divide both numerator and denominator by $1+{}_{\overline{n-t-1}|}a_{x+t}$, and we have

$$\frac{\dfrac{\mathbb{V}_{x|t}}{1+{}_{\overline{n-t-1}|}a_{x+t}}}{\dfrac{\mathbb{A}_{x+t}}{1+{}_{\overline{n-t-1}|}a_{x+t}}}=\text{"paid-up" policy.}$$

But $\frac{\mathbb{V}_{x|t}}{1+{}_{\overline{n-t-1}|}a_{x+t}}=\mathbb{P}_{x+t}-\mathbb{P}_x$, and $\frac{\mathbb{A}_{x+t}}{1+{}_{\overline{n-t-1}|}a_{x+t}}=\mathbb{P}_{x+t}$, therefore $\frac{\mathbb{P}_{x+t}-\mathbb{P}_x}{\mathbb{P}_{x+t}}$ $=1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}=$ "paid-up" policy. *Q. E. D.*

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made by this method is, that every premium paid previously to that age secures an equal part of the sum assured, *i.e.*, that the amount of single payment policy that could be purchased by the increase in the value of the the policy, through one more payment of premium, is the same at every age. That is, the empirical method assumes

$$(1)\quad \frac{\mathbb{V}_{x|1}}{\mathbb{A}_{x+1}}-0=\frac{\mathbb{V}_{x|2}}{\mathbb{A}_{x+2}}-\frac{\mathbb{V}_{x|1}}{\mathbb{A}_{x+1}}=\frac{\mathbb{V}_{x|3}}{\mathbb{A}_{x+3}}-\frac{\mathbb{V}_{x|2}}{\mathbb{A}_{x+2}}=\ldots.$$
$$=\frac{\mathbb{V}_{x|t}}{\mathbb{A}_{x+t}}-\frac{\mathbb{V}_{x|t-1}}{\mathbb{A}_{x+t-1}}=\ldots.=\left(1-\frac{\mathbb{V}_{x|n-1}}{\mathbb{A}_{x+n-1}}\right)=\frac{1}{n}$$

But, as we have seen, $\dfrac{\mathbb{V}_{x|t}}{\mathbb{A}_{x+t}}=1-\dfrac{\mathbb{P}_x}{\mathbb{P}_{x+t}}$.

Therefore the assumption is, that

$$(2)\quad 1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+1}}=\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+2}}\right\}-\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+1}}\right\}=\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+3}}\right\}-\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+2}}\right\}=\ldots$$
$$=\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right\}-\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t-1}}\right\}=\ldots.=1-\left\{1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}}\right\}=\frac{1}{n}$$

i.e.,

$$(3)\quad 1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+1}}=\frac{\mathbb{P}_x}{\mathbb{P}_{x+1}}-\frac{\mathbb{P}_x}{\mathbb{P}_{x+2}}=\frac{\mathbb{P}_x}{\mathbb{P}_{x+2}}-\frac{\mathbb{P}_x}{\mathbb{P}_{x+3}}=\ldots=\frac{\mathbb{P}_x}{\mathbb{P}_{x+t-1}}-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}=\ldots$$
$$=\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}}=\frac{1}{n}$$

Dividing each of these terms by $\mathbb{P}_x$, our assumption is

$$(4)\quad \frac{1}{\mathbb{P}_x}-\frac{1}{\mathbb{P}_{x+1}}=\frac{1}{\mathbb{P}_{x+1}}-\frac{1}{\mathbb{P}_{x+2}}=\frac{1}{\mathbb{P}_{x+2}}-\frac{1}{\mathbb{P}_{x+3}}=\ldots.$$
$$=\frac{1}{\mathbb{P}_{x+t-1}}-\frac{1}{\mathbb{P}_{x+t}}=\ldots.=\frac{1}{\mathbb{P}_{x+n-1}}=\frac{1}{n\mathbb{P}_x}$$

That is, a series in arithmetical progression, with a common difference of $\dfrac{1}{n\mathbb{P}_x}$, is formed by the reciprocals of the premiums that would require to be paid by persons entering at every succeeding age from x to $(x+n)$ to place them in exactly the same position as that then held by the original assurer.

These premiums will, therefore, themselves form a harmonical series.

Therefore, in order that the amount of single payment policy, at any age, that could be purchased by the increase in the value of the policy, through one more payment of the premium, may be the same at every succeeding age, or, in other words, in order that the amount of "paid-up" policy at any age may be the same as the ratio the number of premiums paid bear to the total number payable, it is required that a harmonical series be formed by the premiums that would be charged at every succeeding age for a policy terminating with the original one, and under which all payments were to cease at the same time as under that contract.

The general expression for any term will thus be

$$\mathbb{P}_{x+t}=\frac{2\mathbb{P}_{x+t-1}\mathbb{P}_{x+t+1}}{\mathbb{P}_{x+t-1}+\mathbb{P}_{x+t+1}}$$

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In series (3) let us examine the two terms $\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}}$ and $\frac{1}{n}$. Which is greater? The two sides of the expression are

True Method.	Empirical Method.

$$n\mathbb{P}_x \sim \mathbb{P}_{x+n-1}$$

Now, $\mathbb{P}_{x+n-1} = \frac{\mathbb{V}_{x|n-1}}{1} + \mathbb{P}_x$ (the denominator 1 of the fraction being the annuity-due at age $(x+n-1)$ for the remainder of the term $\mathbb{P}_x$ is payable). And substituting this for $\mathbb{P}_{x+n-1}$, we have

$$n\mathbb{P}_x \sim \mathbb{V}_{x|n-1} + \mathbb{P}_x$$

or

$$(n-1)\mathbb{P}_x \sim \mathbb{V}_{x|n-1}.$$

Now, *generally* speaking, the value of a policy of assurance, by annual payments not all exhausted, is less than the premiums that have been paid under it. Therefore, *generally*,

$$\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}} > \frac{1}{n}, \text{ or } \left(1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}}\right) < \left(1 - \frac{1}{n}\right), \text{ or } < \frac{n-1}{n}.$$

That is, the amount of "paid-up" policy, at the age at which the last premium is payable, is *generally* greater by the empirical method than by the true method. Again, since generally, $\frac{\mathbb{P}_x}{\mathbb{P}_{x+n-1}} > \frac{1}{n}$, and the sum of the whole series (3) $= \frac{n}{n} = 1$; since one, at least of the terms is greater than $\frac{1}{n}$, one or more of the others must be less than $\frac{1}{n}$. Let $\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}} - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}}$, or $\left(1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}}\right) - \left(1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)$, be such a term; and assume that at age $(x+t)$ the empirical and true methods give the same results.*

Then

$$1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}} - \frac{t}{n} < \frac{1}{n}$$

therefore

$$1 - \frac{\mathbb{P}_x}{\mathbb{P}_{x+t+1}} < \frac{t+1}{n}$$

* The empirical method does not necessarily give greater result than the true ones, so we are quite entitled to suppose a case accordingly. Thus, $\frac{\mathbb{V}_{x|1}}{\mathbb{A}_{x+1}} \backsim \frac{1}{n}$ represents the difference of the amounts by the two methods at end of one year; or $\mathbb{V}_{x|1} \backsim \frac{\mathbb{A}_{x+1}}{n}$ = the difference.

Now $\mathbb{V}_{x|1}$ is a maximum when the current risk is a minimum. Say, then, $l_x = l_{x+1}$. In which case $\mathbb{V}_{x|1} = \frac{\mathbb{P}_x}{v}$, and $\mathbb{A}_{x+1} = \frac{\mathbb{A}_x}{v}$, and the expression will become $\frac{\mathbb{P}_x}{v} \backsim \frac{\mathbb{A}_x}{vn}$, or $n\mathbb{P}_x \backsim \mathbb{A}_x$. But $\mathbb{A}_x = \mathbb{P}_x(1 + {}_{\overline{n-1}|}a_x)$, $\therefore$ our expression is $n\mathbb{P}_x \backsim \mathbb{P}_x(1 + {}_{\overline{n-1}|}a_x)$, or $n \backsim (1 + {}_{\overline{n-1}|}a_x)$. Now [except in the case when $l_x = l_{x+1} = l_{x+2} = \ldots . l_{x+n-1}$, and $v = 1$, or $i = 0$] n is greater than $(1 + {}_{\overline{n-1}|}a_x)$. Therefore, when the mortality is a minimum, the amount of paid-up policy at the end of one year, by the true method, will be greater than by the empirical method. *Q. E. D.*

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and the empirical method will give more than the real amount. So our only means of determining whether the "paid-up" policy, at any individual age, is greater or less by the empirical than by the real method is by actual experiment.

In the foot note in which the proof of the equation $\frac{\mathbb{V}_{x|t}}{\mathbb{A}_{x+t}}=1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}$ is found, the $\mathbb{A}_{x+t}$ (besides being at the same rates as the policy is valued by) is assumed to be the *pure* single payment; that is, the assurance at age $(x+t)$ is sold at cost price, and no allowance is made for expenses, nor any provision for future bonuses; so, should the amount of "paid-up" policy be calculated by this formula,* the policyholder will not be entitled to any further share in the profit, and will only receive that which has already accrued in respect of the former payments of premium, and it will be seen from our examples that, in most of the classes of assurance to which what we have called the "empirical" method is likely to be applied, for calculating the "paid-up" policy, there is no such excess of amount by the real method as to warrant further bonuses being allocated.

The assumption then, that is made in considering the "paid-up" policy the equivalent of the ratio the premiums paid bear to the total number payable, is that the differences in the amounts of "paid-up" policy at each age are equal to each other (formula 3). If this were really the case, $\frac{1}{n}$ would necessarily represent this equal difference.

But, as experiment will show, this is not so; and the differences on the whole form a series, commencing at some number greater or less than $\frac{1}{n}$, and terminating at some number less or greater than it. The less then the difference between these differences and $\frac{1}{n}$ the nearer will the empirical method be to the actual.

The following examples will serve somewhat to show to what extent the required condition is fulfilled in various kinds of assurances. They have been calculated by the Carlisle and English Life (No. III., Males) Tables, interest being assumed at 3 per cent. Though, as was pointed out by you in your note to Mr. Younger's letter, other tables, such as the "Experience," may give less irregular results, yet, for the present object, viz., the estimating whether "paid-up" policies by this empirical mode will be, in the various kinds of assurance, greater or less than by the true method; any other table adopted will probably show the same *general* results, though the deficit or surplus in any individual case may not bear the same proportion to the correct amount at that age.

It must not, however, be understood that it is asserted that in every possible example, even by the tables adopted, the results will be of the same nature; so that in, say, the Limited Premiums, in every case, the

* Should the full value of policy not be allowed, but say, $\frac{1}{f}$ of it be deducted, and if some addition be made to the single payment, say, $\frac{1}{g}$ of it, then the amount of paid-up policy to be given would be $\frac{1-\frac{1}{f}}{1+\frac{1}{g}}\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)$.

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empirical method will give greater amounts than the real method. (We have already proved that theoretically this is not necessarily true.) On the whole, however, the results will be somewhat similar to those given, so that in, say, the "Limited Premiums" Assurance, the empirical will generally exceed the actual. This excess, however, as already shown by Mr. Younger, is not of such extent as to put any serious difficulty in the way of the adoption of the scheme, and the following examples will show that it might even more advantageously to the Offices be extended to Endowment Assurances.

Endowment—payable at 60.

Age at which Conversion takes place $(x+t)$.	AMOUNT OF PAID-UP POLICY.			EXCESS OVER EMPIRICAL OF		INCREASE IN AMOUNT FROM YEAR TO YEAR $\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)-\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t-1}}\right)=\Delta.$		$\Delta-\frac{1}{n}$.	
	Carlisle	English	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
45						·08946	·09151	+·02279	+·02484
46	08946	·09151	·06667	·02279	·02484	·08556	·08748	+·01889	+·02081
47	·17502	·17899	·13333	·04169	·04566	·08185	·08357	+·01518	+·01690
48	·25687	·26256	·20000	·05687	·06256	·07829	·07978	+·01162	+·01311
49	·33516	·34234	·36667	·06849	·07567	·07496	·07612	+·00829	+·00945
50	·41012	·41846	·33333	·07679	·08513	·07178	·07256	+·00511	+·00589
51	·48190	·49102	·40000	·08190	·09102	·06876	·06913	+·00209	+·00246
52	·55066	·56015	·46667	·08399	·09348	·06579	·06575	−·00088	−·00092
53	·61645	·62590	·53333	·08312	·09257	·06291	·36248	−·00376	−·00419
54	·67936	·68838	·60000	·07936	·08838	·06010	·05932	−·00657	−·00735
55	·73946	·74770	·66667	07279	·08103	·05735	·05624	−·00932	−·01043
56	·79681	·80394	·73333	·06348	·07061	·05469	·05326	−·01198	−·01341
57	·85150	·85720	·80000	05150	·05720	·05208	·05037	−·01459	−·01630
58	·90358	·90757	·86667	·03691	·04090	·04951	·04757	−·01716	−·01910
59	·95309	·95514	·93333	·01976	·02181	·04691	·04486	−·01976	−·02181

The premiums for endowments, as the time draws nearer for their enjoyment, increase very rapidly, so that $\mathbb{P}_{x+t}$ is large in proportion to $\mathbb{P}_x$, and the empirical paid-up endowment is less than the true. At age $(x+n-1)$, since the value of an endowment is greater than the premiums paid under it, in the inequality $(n-1)\mathbb{P}_x \sim \mathbb{V}_{x|n-1}$ on page 300, we have $\mathbb{V}_{x|n-1}$ the greater, and therefore the amount of paid-up endowment at the age immediately preceding the termination of the original contract is always less by the empirical than by the true method. It generally, though not necessarily, happens, whichever method in most assurance schemes gives the greater amount at age $(x+n-1)$ will give the greater amount at all other ages as well. Were this universally true, we should have "paid-up" endowments by the empirical method always less than by the correct one; and, indeed, this will generally be found to be the case.

In the foregoing examples $\left(\Delta \sim \frac{1}{n}\right)$ is always less by the Carlisle than by the English table, and therefore it follows that for endowments payable at 60, the conversion ages ranging from 45 to 59, the empirical method of calculating "paid-up" endowments will give results nearer the Carlisle than the English table.

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Temporary Assurances—till Age 60.

Age at which Conversion takes place $(x+t)$.	Amount of Paid-up Policy.			Deficit under Empirical of		Increase in Amount from Year to Year $\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)-\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t-1}}\right)=\Delta$.		$\Delta-\frac{1}{n}$.	
	Carlisle.	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
45						·01029	·02531	−·05638	−·04136
46	·01029	·02531	·06667	·05638	·04136	·01163	·02514	−·05504	−·04153
47	02192	·05045	·13333	·11141	·08288	·01460	·02499	−·05207	−·04168
48	·03652	·07544	·20000	·16348	·12456	·02101	·02489	−·04566	−·04178
49	05753	·10033	·26667	·20914	·16634	·02579	·02489	−·04088	−·04178
50	08332	·12522	·33333	·25001	·20811	·03155	·02502	−·03512	−·04165
51	·11487	·15024	·40000	·28513	·24976	·03087	·02249	−·03580	−·04418
52	·14574	·17273	·46667	·32093	·29394	·03051	·02192	−·03616	−·04475
53	17625	·19465	·53333	·35708	·33868	·03063	·02154	−·03604	−·04513
54	·20688	·21619	·60000	·39312	·38381	·03344	·02123	−·03323	−·04544
55	·24032	·23742	·66667	·42635	·42925	·03633	·02102	−·03034	−·04565
56	27665	·25844	·73333	·45668	·47489	·04265	·02089	−·02402	−·04578
57	·31930	·27933	·80000	·48070	·52067	·04821	·02105	−·01846	−·04562
58	·36751	·30038	·86667	·49916	·56629	·04673	·02026	−·01994	−·04641
59	·41424	·32064	·93333	·51909	·61269	·58576	·67936	+·51909	+·61269

The premiums for Temporary Assurances increase very slowly, so $\mathbb{P}_{x+t}$ is very little larger than $\mathbb{P}_x$, and consequently the empirical "paid-up" temporary assurances are greatly in excess of the true amounts —so much so, indeed, as to make this method of calculating the "paid-up" policy quite inapplicable for this class of assurance. In the foregoing $\left(\Delta\sim\frac{1}{n}\right)$ is at first much smaller by the English than by the Carlisle tables, but latterly it is the reverse, so the empirical method of calculating "paid-up" temporary assurances for ages between 45 and 59, and terminating at 60, will give towards the beginning results nearer the English Life, and towards the end, nearer the Carlisle, table.

Endowment Assurance—payable at 60 *or Death.*

Age at which Conversion takes place $(x+t)$.	Amount of Paid-up Policy.			Excess over Empirical of		Increase in Amount from Year to Year $\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)-\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t-1}}\right)=\Delta$.		$\Delta-\frac{1}{n}$.	
	Carlisle.	English.	Empirical	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
45						·06954	·07101	+·00287	+·00434
46	·06954	·07101	·06667	·00287	·00434	·06943	·07031	+·00276	+·00364
47	·13897	·14132	·13333	·00564	·00799	·06957	·06940	+·00290	+·00273
48	·20854	·21072	·20000	·00854	·01072	·06985	·06889	+·00318	+·00222
49	·27839	27961	·26667	·01172	·01294	·06970	·06808	+·00303	+·00141
50	34809	·34769	·33333	·01476	·01436	·06936	·06738	+·00269	+·00071
51	·41745	·41507	·40000	·01745	·01507	·06819	·06653	+·00152	−·00014
52	·48564	·48160	·46667	·01897	·01493	·06700	·06590	+·00033	−·00077
53	·55264	·54750	·53333	·01931	·01417	·06613	·06543	−·00054	−·00124
54	·61877	·61293	·60000	·01877	·01293	·06531	·06506	−·00136	−·00161
55	·68408	·67799	·66667	·01741	·01132	·06451	·06471	−·00216	−·00196
56	·74859	·74270	·73333	·01526	·00937	·06380	·06448	−·00287	−·00219
57	·81239	·80718	·80000	·01239	·00718	·06306	·06431	−·00361	−·00236
58	·87545	·87149	·86667	·00878	·00482	·06240	·06424	−·00427	−·00243
59	·93785	·93573	·93333	·00452	·00240	·06215	·06427	−·00452	−·00240

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In Endowment Assurance premiums (they being made up of endowment premiums, which, as we have seen, give much greater "paid-up" policies by direct calculation than by the empirical method, and of temporary assurance premiums which give much less by the former than by the latter) the amounts will lie between these two extremes, and, as the examples will show, are, on the whole, more favourable to the Insurance Company adopting this new method of finding the "paid-up" policy than are those in the case of limited premium assurances.

Assurances for the Whole of Life—by equal Annual Premiums, payable till Death occur.

NOTE.—n in this case will be the difference between the limiting age in the Table, or the year which the last survivor enters upon, but fails to complete, and the age at entry.

$n=\omega+1-x=$ in Carlisle Table $105-x$, and in English (Males) $=108-x$.

Age at which Conversion takes place $(x+t)$.	Amount of Paid-up Policy.				Excess over Empirical of		Increase in Amount from Year to Year $\left(1-\frac{P_x}{P_{x+t}}\right)-\left(1-\frac{P_x}{P_{x+t-1}}\right)=\Delta$.		$\Delta-\frac{1}{n}$.	
	Carlisle.		English.							
	True.	Empirical	True.	Empirical	Carlisle.	English.	Carlisle.	English.	Carlisle	English.
45							·03195	·03540	+·01528	+·01953
46	03195	·01667	03540	·01587	·01528	·01953	·03233	·03479	+·01566	+·01892
47	·06428	·03333	·07019	·03175	·03095	·03844	·03308	·03415	+·01641	+·01828
48	·09736	·05000	·10434	·04762	·04736	·05672	·03448	·03355	+·01781	+·01768
49	·13184	·06667	·13789	06349	·06517	·07440	·03513	·03294	+·01846	+·01707
50	·16697	·08333	·17083	·07937	·08364	·09146	·03570	·03234	+·01903	+·01647
51	·20267	·10000	·20317	·09524	·10267	·10793	·03467	·03105	+·01800	+·01518
52	·23734	·11667	·23422	·11111	·12067	·12311	·03371	·03034	+·01704	+·01447
53	·27105	·13333	·26456	·12698	·13772	·13758	·03281	·02970	+·01614	+·01383
54	·30386	·15000	·29426	·14286	·15386	·15140	·03225	·02908	+·01558	+·01321
55	·33611	·16667	·32334	·15873	·16944	·16461	·03145	·02845	+·01478	+·01258
56	·36756	·18333	·35179	·17460	·18423	·17719	·03072	·02784	+·01405	+·01197
57	·39828	·20000	·37963	·19048	·19828	·18915	·02935	·02720	+·01268	+·01133
58	·42763	·21667	·40683	·20635	·21096	·20048	·02694	·02653	+·01027	+·01066
59	·45457	·23333	·43336	·22222	·22124	·21114	·02422	·02583	+·00755	+·00996
60	·47879	·25000	·45919	·23810	·22879	·22109	......	......	.....	......
....	....	....	....	....	......	......	......	......	......	......
....	....	....	....	....	.....	......	......	......	.. ..	......
104	·96892	·98333	....	....	−·01441	......	·03108	......	+·01441	......
....			....	....		......		......		......
....			....	....		.....		......		......
107			·96567	·98413		−·01846		·03433		+·01846

In the formula $\left(1-\frac{P_x}{P_{x+t}}\right)\sim\frac{t}{n}$, should $(n-t)$ be a very large quantity, as the probability of the life reaching the older ages has very little effect on the quantity $\frac{P_x}{P_{x+t}}$, which would remain at nearly the same value were the oldest age in the tables much less than it is, while, on the other hand, as the probability of reaching every age is assumed the same in the quantity $\frac{t}{n}$, if n be very large compared with t, $\frac{t}{n}$ will be a very small fraction, and much less than $\left(1-\frac{P_x}{P_{x+t}}\right)$, and consequently at the younger ages, and indeed at all the ages likely to occur in practice, the "paid-up" policy by the empirical method, to an assurer on this system, will be very much less than the correct amount.

In course of time, however the excess diminishes, and latterly turns the other way. Thus, at age $(x+n-1)$, or the oldest age in the table, the expression is $\left(1-\frac{P_x}{v}\right)\sim\frac{n-1}{n}$ or $v\sim nP_x$, of which the latter term, which corresponds to the result by the empirical method, is the greater.

The quantity $\left(\Delta-\frac{1}{n}\right)$ is in the above examples at first less and afterwards greater by the Carlisle than by the English table, and therefore assurers for the whole of life, by equal annual premiums, will, should they between ages 45 and 59 change their policies into "paid-up" ones, get by the empirical method, at first, results nearer the Carlisle, and, afterwards, nearer the English tables.

Assurances for the Whole of Life—by 5 Payments.

Age at which Conversion takes place $(x+t)$.	Amount of Paid-up Policy.			Deficit under Empirical of		Increase in Amount from Year to Year $\left(1-\frac{P_x}{P_{x+t}}\right)-\left(1-\frac{P_x}{P_{x+t-1}}\right)=\Delta$.		$\Delta-\frac{1}{n}$.	
	Carlisle	English.	Empirical.	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
30						·19657	·19745	−·00343	−·00255
31	·19657	·19745	·20000	·00343	·00255	·19827	·19862	−·00173	−·00138
32	·39484	·39607	·40000	·00516	·00393	·24015	·19988	+·00015	−·00012
33	·59499	·59595	·60000	·00501	·00405	·20180	·20127	+·00180	+·00127
34	·79679	79722	·80000	·00321	·00278	·20321	·20278	+·00321	+·00278

Assurances for the Whole of Life—by 10 Payments.

Age at which Conversion takes place $(x+t)$.	Amount of Paid-up Policy.			Deficit under Empirical of		Increase in Amount from Year to Year $\left(1-\frac{P_x}{P_{x+t}}\right)-\left(1-\frac{P_x}{P_{x+t-1}}\right)=\Delta$.		$\Delta-\frac{1}{n}$.	
	Carlisle.	English.	Empirical	Carlisle.	English.	Carlisle.	English.	Carlisle.	English.
30						·09677	·09792	−·00323	−·00208
31	09677	·09792	·10000	·00323	·00208	·09748	·09826	−·00252	−·00174
32	·19425	·19618	·20000	·00575	·00382	·09850	·09863	−·00150	−·00137
33	·29275	·29481	·30000	·00725	·00519	·09941	·09904	−·00059	−·00096
34	·39216	·39385	·40000	·00784	·00615	·09998	·09949	−·00002	−·00051
35	·49214	·49334	·50000	·00786	·00666	·10054	·10001	+·00054	+·00001
36	59268	·59335	·60000	·00732	·00665	·10097	·10059	+·00097	+·00059
37	·69365	·69394	·70000	·00635	·00606	·10147	·10124	+·00147	+·00124
38	·79512	·79518	·80000	·00488	00482	·10209	·10199	+·00209	+·00199
39	·89721	·89717	·90000	·00279	·00283	·10279	·10283	+·00279	+·00283

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Assurances for the Whole of Life—by 15 *Payments.*

Age at which Conversion takes place $(x+t)$.	Amount of Paid-up Policy.			Deficit under Empirical of		Increase in Amount from Year to Year $\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t}}\right)-\left(1-\frac{\mathbb{P}_x}{\mathbb{P}_{x+t-1}}\right)=\Delta$.		$\Delta-\frac{1}{n}$.	
	Carlisle	English	Empirical.	Carlisle	English.	Carlisle.	English.	Carlisle.	English.
30						·06416	·06539	−·00251	−·00128
31	·06416	06539	·06667	00251	·00128	·06456	·06546	−·00211	−·00121
32	·12872	·13085	·13333	00461	·00248	·06529	·06553	−·00138	−·00114
33	19401	·19638	·20000	·00599	·00362	·06595	·06562	−·00072	−·00105
34	·25996	26200	·26667	00671	·00467	·06625	·06574	−·00042	−·00093
35	·32621	·32774	·33333	·00712	·00559	·06655	·06588	−·00012	−·00079
36	·39276	·39362	·40000	00724	·00638	·06660	·06603	—·00007	−·00064
37	·45936	·45965	·46667	·00731	·00702	·06671	·06624	+·00004	−·00043
38	·52607	·52589	·53333	·00726	·00744	·06688	·06649	+·00021	−·00018
39	·59295	·59238	·60000	00705	·00762	·06683	·06678	+·00016	+·00011
40	·65978	·65916	·66667	·00689	·00751	·06674	·06715	+·00007	+·00048
41	·72652	·72631	·73333	·00681	·00702	·06716	·06756	+·00049	+·00089
42	·79368	·79387	·80000	·00632	·00613	·06785	·06808	+·00118	+·00141
43	·86153	86195	·86667	·00514	·00472	·06877	·06867	+·00210	+·00200
44	·93030	93062	·93333	·00303	·00271	·06970	·06938	+·00303	+·00271

For the foregoing examples, then, the "paid-up" policy in this class of assurance is a little greater by the empirical than by the correct method—and other examples would have shown that, generally speaking, this will be the case.

The difference, however, is so small as not to render it at all hazardous for the Insurance Companies to grant "paid-up" policies calculated in this manner.

It will be observed from the column $\left(\Delta-\frac{1}{n}\right)$, in the preceding cases, that the "English" table gives results, on the whole, nearer those by the empirical method than does the "Carlisle" for these ages.

Finally, it has been shown that, in order that the ratio of premiums paid to the total number payable may express the correct amount of "paid-up" policy on the original status, it is necessary that the premiums that would require to be paid by persons entering at every succeeding age from x to $(x+n)$, to place them in exactly the same position as that then held by the original assurer, form a series in harmonical progression, and that, as $\mathbb{P}_x$ and $\mathbb{P}_{x+t}$ may be said to be independent of each other, it is not possible to prove in a general form whether the "paid-up" policy will be greater or less by using this mode of calculation than the correct amount; but that, on the whole, by this method, the "paid-up" policy granted would be much too large in the case of temporary assurances, and much too small for assurances by premiums payable till death and for endowments, the variation being so great as to render it inapplicable for any of these classes; and that for endowment assurances and policies by limited premiums, should the number payable, after change in the policy, not be very great, it will give results very close to the truth; in the first case, perhaps a little favourable to the Company; in the second, perhaps a little against it; but that, under all ordinary circumstances, this difference is so small as to permit Offices to adopt the system with perfect safety.

I am, Sir, your obedient servant,

JAMES R. MACFADYEN.

City of Glasgow Life Assurance Company,
Glasgow, 10*th July,* 1869.

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