Appendix B

$SU(N)$ generators

$SU(N)$ is the group of special (unit determinant), unitary, $N \times N$ complex matrices.[1] By considering the various constraints on the $2N^2$ real components of the matrix owing to the special and unitary conditions, we can see that the matrix has $N^2 - 1$ independent degrees of freedom. Then, if $g \in SU(N)$, we can write

$$g = \exp(\mathrm{i}\alpha_a T^a) \tag{B.1}$$

where a sum over $a = 1, \ldots, N^2 - 1$ is implicit, α_a are real constants, and T^a are the "generators" of the group. The T^a satisfy the $SU(N)$ Lie algebra and can be represented by matrices of various dimensions. In the $N = 2$ ($SU(2)$) case, the two-dimensional representation is in terms of Pauli spin matrices, $T^a = \sigma^a/2$, or explicitly

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.2}$$

The Lie algebra is

$$[T^a, T^b] = \mathrm{i}\epsilon^{abc} T^c \tag{B.3}$$

where ϵ^{abc} is the totally antisymmetric tensor. One can also easily construct the higher dimensional representations. It is conventional to normalize the generators to satisfy

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{B.4}$$

where δ^{ab} is the Kronecker delta.

To get a set of generators for $SU(N)$, it is simplest to build on the $SU(2)$ generators in Eq. (B.2). First, one puts the Pauli spin matrices in the upper left-hand corner and obtains three $SU(N)$ generators

$$T^a = \frac{1}{2}\begin{pmatrix} \sigma^a & 0 & \cdots \\ 0 & 0 & \cdots \end{pmatrix}, \quad a = 1, 2, 3 \tag{B.5}$$

Then one puts the off-diagonal Pauli spin matrices in the off-diagonal positions. Since there are $N(N-1)/2$ off-diagonal positions of which two have already been filled by the $a = 1, 2$ generators, we can construct $N(N-1) - 2$ more generators by filling each

[1] For a review of group theory in particle physics, see [62].

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remaining position by either 1 (as in σ^1) or by $\pm$i (as in σ^2). These look like

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots \\ . & & \dots & 1_{jk} & \\ 0 & \dots & 1_{kj} & & \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & & \dots \\ 0 & 0 & \dots & \dots & \dots \\ . & & \dots & -\mathrm{i}_{jk} & \\ 0 & \dots & \mathrm{i}_{kj} & \dots & \\ \dots & \dots 0 & \dots & & \dots \end{pmatrix} \tag{B.6}$$

where the subscripts j, k denote the position in the matrix.

Finally we construct the diagonal generators. These are written by putting a series of 1s in say, n, successive diagonal positions, and then entering $-n$ in the nn entry of the matrix. This scheme ensures that the generator is traceless and the resulting matrix is

$$\mathrm{diag}(1, \dots, 1_n, -n, 0, \dots, 0) \tag{B.7}$$

where 1_n denotes 1 in the nn entry. The normalization is then fixed using the convention in Eq. (B.4) to get the generator

$$\frac{1}{\sqrt{2n(n+1)}}\mathrm{diag}(1, \dots, 1_n, -n, 0, \dots, 0) \tag{B.8}$$

In this way we construct $N-1$ diagonal generators, one for each value of n. The third Pauli matrix is already included as the $a = 3$ generator.

As a check, we find that the total number of generators constructed is $3 + (N(N-1) - 2) + (N-2) = N^2 - 1$ and this agrees with the degrees of freedom in $SU(N)$.

In the $SU(5)$ Grand Unified model discussed in Chapter 2 an alternate set of diagonal generators is useful.

$$\begin{aligned} \lambda_3 &= \frac{1}{2}\mathrm{diag}(1, -1, 0, 0, 0) \\ \lambda_8 &= \frac{1}{2\sqrt{3}}\mathrm{diag}(1, 1, -2, 0, 0) \\ \tau_3 &= \frac{1}{2}\mathrm{diag}(0, 0, 0, 1, -1) \\ Y &= \frac{1}{2\sqrt{15}}\mathrm{diag}(2, 2, 2, -3, -3) \end{aligned}$$

After the $SU(5)$ symmetry is broken by the canonical vacuum expectation value of Φ (Eq. (2.6)), λ_3 and λ_8 are generators of the unbroken $SU(3)$, τ_3 of $SU(2)$, and Y of $U(1)$.

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