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On general transformations and variational Principles of an ideal three-dimensional compressible

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Abstract. In this paper we have used the general theory of Arnold (1965,1966) and Vladimirov *et al.* (1997) to obtain sufficient conditions for linear stability of steady MHD flows of an ideal three-dimensional compressible gravitating flows.

Keywords. magnetohydrodynamics;gravitating fluid;stability.

1. Introduction

In this paper, we consider the stability of steady magnetohydrodynamic (MHD) flows of an ideal three-dimensional compressible gravitating flows. The stability of such steady states is considered by an appropriate generalization of Arnold energy techniques. It is known that in the convection zone of the Sun,the thermal buoyancy vigorously drives strong turbulence, under the influence of rotation, magnetic field, and strong density stratification.

2. Basic equations and their invariants.

We consider an compressible, inhomogeneous (with density ρ), perfectly conducting MHD flow contained in a domain D with fixed boundary ∂D. We shall in general suppose that D is bounded, but the theory may be easily modified to deal with the case of an unbounded domain. Let $u(r,t)$ be the velocity field, $h(r,t)$ the magnetic field (in Alfvén velocity units), $p(r,t)$ the gas pressure, $\ell(r,t)$ entropy field, $T(r,t)$ temperature, $J = \nabla \times h$ the current density in the fluid, $\omega = \nabla \times u$ the vorticity field, and the mass force field $F(r,t)$ with a potential $\Phi(r,t)$ such that $F = -\nabla\Phi$. For self-gravitation we have $\Delta\Phi = 4\pi G\rho$. Then the governing equations in the Boussinesq approximation are Vladimirov, V. A. (1986) , Phillips, O. M. (1980) and Chandrasekhar, S. (1987).

$$u_t = u \times \omega + j \times h - \nabla(p + \frac{1}{2}u^2) - \rho\nabla\Phi, \qquad D\rho + \rho\nabla \cdot u = 0, \qquad D\ell = 0, \tag{2.1}$$

$$Lh = \frac{\partial h}{\partial t} - \nabla \times (u \times h) = 0, \qquad e = e(\rho, \ell), \quad de = Td\ell - pd(\frac{1}{\rho}), \tag{2.2}$$

$$\nabla \cdot u = \nabla \cdot h = 0, \tag{2.3}$$

Here L is a form of Lie derivative and e is the internal energy.

3. Sufficient conditions to three-dimensional gravitating flows stability.

Let us now consider in more details the expression for $\delta^2 E$,

$$\delta^2 E = \frac{1}{2}\int_D \{h^2 + h\cdot(J\wedge\zeta) + (A\Phi^{''}(A) + 2\Phi^{'}(A) + 2\mathcal{E}_A + A\mathcal{E}_{AA}))\rho^2) \\ + 2(\mathcal{E}_{\mathcal{L}} + A\mathcal{E}_{A\mathcal{L}})\delta\ell\delta\rho + A\mathcal{E}_{\mathcal{L}\mathcal{L}}(\delta\ell)^2\}\, dV, \tag{3.1}$$

i) Parallel flow and field: Let Ω be an infinite cylinder of arbitrary cross-section with axis parallel to oz, and suppose that

$$H = H_0(x,y)e_z \tag{3.2}$$

after some algebra, (3.1) reduces to

$$\delta^2 E = \frac{1}{2}\int_D \{H_0^2((e_z\cdot\nabla)\zeta)^2 + (A\Phi^{''}(A) + 2\Phi^{'}(A) + 2\mathcal{E}_A + A\mathcal{E}_{AA}))\rho^2) \\ + 2(\mathcal{E}_{\mathcal{L}} + A\mathcal{E}_{A\mathcal{L}})\delta\ell\delta\rho + A\mathcal{E}_{\mathcal{L}\mathcal{L}}(\delta\ell)^2\}\, dV. \tag{3.3}$$

Proposition 3.1 The state (3.2) is linearly stable to isomagnetovortical (imv) perturbations provided

$$A\Phi^{''}(A) + 2\Phi^{'}(A) + 2\mathcal{E}_A + A\mathcal{E}_{AA} \geqslant 0 \,\text{in}\, D. \tag{3.4}$$

ii) Annular basic state: Let D be an annular region between two cylinders C_1, C_2 of arbitrary cross-sections, and let

$$H = -e_z\wedge\nabla B, \tag{3.5}$$

where $B(x,y)$ is the flux-function of H. We shall suppose that $|\nabla B| \neq 0$ in D, i.e H has no neutral points in D.

The expression for $\delta^2 E$ may be reduced to the form

$$\delta^2 E = \frac{1}{2}\int_D [((h + (\zeta\cdot\nu)(J\wedge\nu))^2 + \nabla^2 B(-\nabla^2 B + \frac{\nabla B\cdot\nabla(H^2)}{2H^2}))(\zeta\cdot\nu)^2 \\ + (A\Phi^{''}(A) + 2\Phi^{'}(A) + 2\mathcal{E}_A + A\mathcal{E}_{AA})\rho^2 + 2(\mathcal{E}_{\mathcal{L}} + A\mathcal{E}_{A\mathcal{L}})\delta\ell\delta\rho \\ + A\mathcal{E}_{\mathcal{L}\mathcal{L}}(\delta\ell)^2]dV. \tag{3.6}$$

Proposition 3.2 The state (3.5) is linearly stable for the isentropic currents ($\mathcal{L}' \equiv 0$, $\ell \equiv 0$) to imv perturpations provide

$$\nabla^2 B \geqslant 0\,,\; -\nabla^2 B + \frac{\nabla B\cdot\nabla(H^2)}{2H^2} \geqslant 0\,,\quad A\Phi^{''}(A) + 2\Phi^{'}(A) + 2\mathcal{E}_A + A\mathcal{E}_{AA} \geqslant 0. \tag{3.7}$$

or

$$\nabla^2 B \leqslant 0\,,\; -\nabla^2 B + \frac{\nabla B\cdot\nabla(H^2)}{2H^2} \leqslant 0\,,\quad A\Phi^{''}(A) + 2\Phi^{'}(A) + 2\mathcal{E}_A + A\mathcal{E}_{AA} \geqslant 0. \tag{3.8}$$

4. Conclusion

In this paper, we have given the sufficient conditions for linear stability of steady three-dimensional gravitating flows.The stability of such steady states is considered by an appropriate generalization of (Arnold) energy techniques.

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