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A CLOSED-FORM PRICING FORMULA FOR VARIANCE SWAPS WITH MEAN-REVERTING GAUSSIAN VOLATILITY

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Abstract

Although variance swaps have become an important financial derivative to hedge against volatility risks, closed-form formulae have been developed only recently, when the realized variance is defined on discrete sample points and no continuous approximation is adopted to alleviate the mathematical difficulties associated with dealing with the discreteness of the sample data. In this paper, a new closed-form pricing formula for the value of a discretely sampled variance swap is presented under the assumption that the underlying asset prices can be described by a mean-reverting Gaussian volatility model. With the newly found analytical formula, not only can all the hedging ratios of a variance swap be analytically derived, the numerical values of the swap price can be efficiently computed as well.

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1. Introduction

Volatility measures the standard deviation of the returns of an underlying asset; thus, it has undoubtedly become the most commonly used measure of risk. Volatility risk has drawn a wider attention in the financial markets and trading this risk has also increasingly become important to market practitioners; ranging from individuals to financial institutions and pension funds in recent years, especially after the global financial crisis. Since the mid-1990s, a new subset of derivative securities has arisen which provides the investors with the opportunity to take a direct position, not in the underlying asset, but in its volatility. Investors use volatility derivatives to trade the spread between the realized and implied volatility levels, or hedge against volatility risk of their portfolios. In practice, derivative products related to volatility and variance have been experiencing sharp increases in trading volume recently [7].

Variance and volatility swaps are the first and most fundamental products and they are now the most popular for their effective provision of volatility exposure. By nature,

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these products are forward contracts, which, at maturity, exchange the difference between a fixed strike and realized variance/volatility, scaled by a predetermined notional value. Generally, there are two types of volatility or variance swap products [10]. One is the historical volatility or variance-based product whose payoff depends on the realized volatility or variance discretely sampled at some prespecified sampling points. Most products of this type are traded over-the-counter (OTC). Another class of volatility or variance swaps is the implied-volatility-based products, such as the VIX futures traded in the Chicago Board Options Exchange (CBOE). We shall primarily focus our attention on variance swaps based on the former with the realized variance being sampled discretely.

With rapid increase of the trading volume of variance swaps recently, both market and academic interests for these products have increased and much research has been devoted to develop efficient pricing methods. The valuation approaches for variance swaps are generally classified into two types; numerical and analytical methods.

There are two subcategories of analytical methods. The most influential pioneering works were proposed by Carr and Madan [8] and Demeterfi et al. [9], who showed how to theoretically replicate a variance swap by a portfolio of standard options. Without requiring to specify the function of the volatility process, their methods are indeed very attractive. However, as pointed out by Carr and Corso [6], the replication strategy has a drawback in that the sampling time of a variance swap in their models is assumed to be continuous rather than discrete. Such an assumption implies that the results obtained from their continuous models can only be viewed as an approximation for the real cases in financial practice, in which all contacts are written with the realized variance being calculated on a set of prespecified discrete sampling points. Another drawback is that this strategy also requires options with a continuum of exercise prices, which is not actually available in marketplaces. The second type of analytical methods is the stochastic volatility models. Grunbichler and Longstaff [15] first proposed a pricing model for volatility futures based on a mean-reverting square-root volatility process. Heston and Nandi [18] derived an analytical solution for both variance and volatility swaps based on the GARCH volatility process [17]. Howison et al. [19] considered the pricing methods of variance swaps and volatility swaps under a variety of diffusion and jump-diffusion models. Elliott et al. [11] developed a method to evaluate variance swaps and volatility swaps under a continuous-time Markov-modulated version of the stochastic volatility with regime switching. However, all these stochastic volatility models assume continuous sampling of the realized variance processes, which results in a systematic bias for the price of a variance swap. As pointed out by Zhu and Lian [36], this assumption generally leads to large relative errors for variance swaps with small sampling frequencies or long tenors.

Various numerical methods were also proposed recently. Little and Pant [20] developed a finite-difference method for the valuation of the discretely sampled variance swaps in an extended Black–Scholes framework with a local volatility function. By exploring a dimension-reduction technique, their numerical approach achieved both high efficiency and accuracy. Windcliff et al. [34] improved on the pricing algorithm for the discretely sampled volatility derivatives by allowing jumps in the asset price process. Using the Monte Carlo (MC) simulation method, Broadie and Jain [5] investigated the effect of discrete sampling and asset price jumps on the fair strike prices of variance and volatility swaps under various stochastic volatility models such as the Black–Scholes model [3], the Heston stochastic volatility model [16], the Merton jump-diffusion model [22] and the Bates and Scott stochastic volatility and jump model [2, 30].

Very recently, Zhu and Lian [36] presented an approach to obtain a closed-form formula for variance swaps based on the discretely sampled realized variance under Heston's two-factor stochastic volatility model. Using the dimension-reduction technique proposed by Little and Pant [20] and the generalized Fourier transform, they found a simple formula by directly solving the governing partial differential equation (PDE) system. Their newly derived exact formula shows a substantial advantage in terms of both accuracy and efficiency over previous numerical or approximate approaches in pricing variance swaps, which is a very useful tool in trading practice in financial markets. Rujivan and Zhu [25] proposed a simplified approach which led to exactly the same results presented by Zhu and Lian [36], but without the introduction of a new state variable and the utilization of the generalized Fourier transform.

We present a new pricing formula for discretely sampled variance swaps based on the mean-reverting Gaussian volatility model. Upon applying the general asset valuation theory to obtain the associated PDE, and then solving it, we derive a closed-form exact solution for discretely sampled variance swaps with the realized variance defined as the sum of the percentage increment of the underlying asset price. We also demonstrate through analytic asymptotic analysis that the pricing formula we obtain for the discretely sampled variance swaps converges to that of its continuously monitored counterpart.

In Section 2, we provide a description of the mean-reverting Gaussian volatility model and variance swaps, followed by our analytical formula for the variance swaps. In Section 3, some numerical tests are given, demonstrating the correctness of our solution from various aspects. In the mean time, we also present some comparisons with the Heston model, discussions for the restrictions of parameter space and sensitivity tests for the key parameters. A brief summary is stated in Section 4.

2. Our model

This section briefly reviews the mean-reverting Gaussian volatility model, which we adopt to describe the dynamics of the underlying asset first. Then, we shall show our detailed approach to analytically solve the associated PDE, based on the method proposed by Rujivan and Zhu [25].

2.1. The mean-reverting Gaussian volatility model Starting with Vasicek [32], the mean-reverting Gaussian process (MRGP, also called the Ornstein–Uhlenbeck process) is among the most commonly used stochastic processes in finance. We use the MRGP to model interest rate [14, 32], spread [12], hazard rate [1], stochastic volatility

[4, 29, 31], commodity convenience yields [21, 28] and other mean-reverting financial variables. Nelson [23] has shown that the MRGP is the diffusion limit of the GARCH (1,1) model.

In the mean-reverting Gaussian volatility model, the underlying asset price S_t is modelled by the following diffusion process with a stochastic instantaneous volatility v_t:

$$dS_t = \mu S_t dt + v_t S_t dB_t^S,$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma dB_t^v,$$

where μ is the expected return of the underlying asset, θ is the long-term mean of volatility, κ is a mean-reverting speed parameter of the volatility and σ is the so-called volatility of volatility, which provides the magnitude of uncertainty in volatility. The two Wiener processes dB_t^S and dB_t^v describe the random noise in asset and volatility, respectively. They are assumed to be correlated with $(dB_t^S, dB_t^v) = \rho\, dt$.

Using the existence theorem of the equivalent martingale measure, we change the real probability measure to a risk-neutral probability measure and describe the process as

$$\begin{aligned} dS_t &= rS_t\, dt + v_t S_t\, d\tilde{B}_t^S, \\ dv_t &= \kappa^*(\theta^* - v_t)\, dt + \sigma\, d\tilde{B}_t^v, \end{aligned} \tag{2.1}$$

where r denotes the riskless interest rate; κ^* and θ^* are the risk-neutral parameters. For the rest of this paper, our analysis will be based on the risk-neutral probability measure.

Notice that the MRGP allows volatility to take negative values, unlike the Heston model, and this is not admissible since volatility is positive by definition. However, as pointed out by Schöbel and Zhu [26], the probabilities that the volatility could become negative are negligibly small under the MRGP for a wide range of reasonable parameter values. Therefore, the MRGP is still among the most commonly used stochastic processes in finance for its tractability.

2.2. Variance swaps Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The long position of a variance swap pays a fixed delivery price at expiration and receives the floating amounts of annualized realized variance, while the short position is just the opposite. Thus, it can be easily used for investors to gain exposure to volatility risk.

Usually, the payoff of a variance swap at expiry can be written as $V_T = (\sigma_R^2 - K) \times L$, where T is the lifetime of the contract, σ_R^2 is the annualized realized variance over the contract life $[0, T]$, K is the strike price for the variance swap and L is the notional amount of the swap in dollars per annualized volatility point squared.

The procedure of how the realized variance should be calculated is usually clearly specified in the contract. It usually includes details about the source and the observation frequency of the price of the underlying asset, the annualization factor which is used in moving to an annualized variance and the method of calculating the

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variance. A typical formula for the measure of realized variance is

$$\sigma_R^2 = \frac{\mathrm{AF}}{N}\sum_{i=1}^{N}\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \times 100^2, \tag{2.2}$$

where S_{t_i} is the closing price of the underlying asset at the ith observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. We assume equally spaced discrete observations, so that the annualized factor is a simple expression $\mathrm{AF} = 1/\Delta t = N/T$.

Under the risk-neutral argument, the value of a variance swap at time t is the expected present value of the future payoff, $V_t = E_t^Q[e^{-r(T-t)}L(\sigma_R^2 - K)]$, where Q is the risk-neutral probability measure and $E_t^Q[\cdot] = E^Q[\cdot \mid \mathcal{F}_t]$ denotes the conditional expectation at time t; $\mathcal{F}_t$ is the filtration up to time t. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair variance delivery price is easily defined as $K = E_0^Q[\sigma_R^2]$. The variance swap valuation problem is, therefore, reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

2.3. Our approach to price variance swaps We discuss our approach to find an analytical solution for the fair delivery price of a discretely sampled realized variance swap under the mean-reverting Gaussian volatility process. Our solution approach begins with taking the expectation of σ_R^2 in (2.2). Since

$$E_0^Q[\sigma_R^2] = E_0^Q\left[\frac{AF}{N}\sum_{i=1}^{N}\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \times 100^2 = \frac{1}{T}\sum_{i=1}^{N} E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \times 100^2, \tag{2.3}$$

the problem of pricing a variance swap, therefore, reduces to calculating N expectations in the form

$$E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \tag{2.4}$$

for some fixed time period Δt and N different tenors $t_i = i\Delta t$ $(i = 1, \ldots, N)$. The rest of this section focuses on obtaining the expectation of this expression. In the process of calculating this expectation, i is regarded as a constant. Hence, both t_i and t_{i-1} are regarded as known constants.

Noticing that $\mathcal{F}_0 \subset \mathcal{F}_{t_{i-1}}$ and $S_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$-measurable, we apply the tower property [33, p. 88] to the conditional expectation in (2.4) and obtain a double conditional expectation as follows:

$$\begin{aligned} E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] &= E_0^Q\left[E_{t_{i-1}}^Q\left\{\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right\}\right] \\ &= E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}\{E_{t_{i-1}}^Q(S_{t_i}^2) - 2S_{t_{i-1}}E_{t_{i-1}}^Q(S_{t_i})\} + 1\right]. \end{aligned} \tag{2.5}$$

The expectations $E_{t_{i-1}}^Q[S_{t_i}]$ and $E_{t_{i-1}}^Q[S_{t_i}^2]$ on the right-hand side of (2.5) are easily calculated by using the following proposition with $\gamma = 1$ and $\gamma = 2$, respectively.

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Proposition 2.1. *For any given $\gamma \in \mathbb{R}\backslash\{0\}$, if S_t follows the dynamics described in (2.1) and the parameters satisfy $(\kappa^* - \rho\gamma\sigma)^2 > \sigma^2\gamma(\gamma - 1)$, then the conditional expectation of $Y_t = S_t^\gamma$ is*

$$E^Q_{t_{i-1}}[Y_t] = E^Q[Y_t \mid (Y_{t_{i-1}} = y, v_{t_{i-1}} = v)] = ye^{C(\gamma, t_i - t) + D(\gamma, t_i - t)v + E(\gamma, t_i - t)v^2} \tag{2.6}$$

for all $t \in [t_{i-1}, t_i]$ and $(y, v) \in (0, \infty) \times (-\infty, \infty)$, with

$$\begin{cases} C(\gamma, \tau) = -\dfrac{1}{2}\ln\Big[\dfrac{\{\hat{a}(\gamma) + \hat{b}(\gamma)\}e^{\hat{b}(\gamma)\tau} - \hat{a}(\gamma) + \hat{b}(\gamma)}{2\hat{b}(\gamma)}\Big] \\ \qquad + \Big[\dfrac{\hat{a}(\gamma) + \hat{b}(\gamma)}{4} + \dfrac{2\kappa^{*2}\theta^{*2}\gamma(\gamma - 1)}{\hat{b}^2(\gamma)} + \gamma r\Big]\tau \\ \qquad - \dfrac{4\kappa^{*2}\theta^{*2}\gamma(\gamma - 1)[\{2\hat{a}(\gamma) + \hat{b}(\gamma)\}e^{\hat{b}(\gamma)\tau} - 4\hat{a}(\gamma)e^{\hat{b}(\gamma)\tau/2} + 2\hat{a}(\gamma) - \hat{b}(\gamma)]}{\hat{b}^3(\gamma)((\hat{a}(\gamma) + \hat{b}(\gamma))e^{\hat{b}(\gamma)\tau} - \hat{a}(\gamma) + \hat{b}(\gamma))}, \\ D(\gamma, \tau) = \dfrac{\kappa^*\theta^*\gamma(\gamma - 1)(e^{\hat{b}(\gamma)\tau/2} - 1)^2}{\sigma^2\hat{b}(\gamma)((\hat{a}(\gamma) + \hat{b}(\gamma))e^{\hat{b}(\gamma)\tau} - \hat{a}(\gamma) + \hat{b}(\gamma))}, \\ E(\gamma, \tau) = \dfrac{\gamma(\gamma - 1)(e^{\hat{b}(\gamma)\tau} - 1)}{(\hat{a}(\gamma) + \hat{b}(\gamma))e^{\hat{b}(\gamma)\tau} - \hat{a}(\gamma) + \hat{b}(\gamma)}, \end{cases} \tag{2.7}$$

where $\hat{a}(\gamma) = 2\kappa^* - 2\rho\gamma\sigma$ and $\hat{b}(\gamma) = \sqrt{\hat{a}^2(\gamma) - 4\sigma^2\gamma(\gamma - 1)}$.

The proof of this proposition is given in Appendix A. The inequality imposed in Proposition 2.1 is a sufficient condition for a global solution. For pricing variance swaps, only two values $\gamma = 1$ and $\gamma = 2$ are needed, which have a restricted global solution with $\kappa^* > (2\rho + \sqrt{2})\,\sigma$ or $\kappa^* < (2\rho - \sqrt{2})\,\sigma$ in the parameter space. For $(2\rho - \sqrt{2})\,\sigma \le \kappa^* \le (2\rho + \sqrt{2})\,\sigma$, only local solutions exist, as shown in Appendix A.

Using Proposition 2.1 with $\gamma = 1$ and $\gamma = 2$, respectively, we compute the two conditional expectations

$$E^Q_{t_{i-1}}[S_{t_i}] = S_{t_{i-1}}e^{C(1,\Delta t) + D(1,\Delta t)v_{t_{i-1}} + E(1,\Delta t)v^2_{t_{i-1}}} = S_{t_{i-1}}e^{r\Delta t}, \tag{2.8}$$

$$E^Q_{t_{i-1}}[S^2_{t_i}] = S^2_{t_{i-1}}e^{C(2,\Delta t) + D(2,\Delta t)v_{t_{i-1}} + E(2,\Delta t)v^2_{t_{i-1}}}. \tag{2.9}$$

Therefore,

$$E^Q_0\Big[\Big(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\Big)^2\Big] = E^Q_0[e^{C(2,\Delta t) + D(2,\Delta t)v_{t_{i-1}} + E(2,\Delta t)v^2_{t_{i-1}}} - 2e^{r\Delta t} + 1], \tag{2.10}$$

obtained by substituting the expectations in (2.8) and (2.9) into (2.5).

For the case $i = 1$, the time $t_{i-1} = 0$ and $v_{t_{i-1}} = v_0$ is $\mathcal{F}_0$-measurable, so the expectation in (2.10) can be reduced to

$$E^Q_0\Big[\Big(\frac{S_{t_1} - S_0}{S_0}\Big)^2\Big] = e^{C(2,\Delta t) + D(2,\Delta t)v_0 + E(2,\Delta t)v_0^2} - 2e^{r\Delta t} + 1. \tag{2.11}$$

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For any other cases with $i > 1$, notice that the expectation in (2.10) is not related to the process S_t, since the right-hand side of (2.10) is independent of S_t. So,

$$E_0^Q\Big[\Big(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\Big)^2\Big] = E_0^Q[e^{C(2,\Delta t)+D(2,\Delta t)v_{t_{i-1}}+E(2,\Delta t)v_{t_{i-1}}^2} - 2e^{r\Delta t} + 1]$$
$$= \int_{-\infty}^{\infty} f(v_{t_{i-1}})p(v_{t_{i-1}}|v_0)\,dv_{t_{i-1}}, \tag{2.12}$$

where $f(v) = e^{C(2,\Delta t)+D(2,\Delta t)v+E(2,\Delta t)v^2} - 2e^{r\Delta t} + 1$ and $p(v_{t_{i-1}} \mid v_0)$ is the transition density of the MRGP from state $(v_0, 0)$ to state $(v_{t_{i-1}}, t_{i-1})$: that is, if we solve the MRGP with $v(0) = v_0$, then the random variable $v(t_{i-1})$ has density $p(v_{t_{i-1}} \mid v_0)$ in the variable $v_{t_{i-1}}$.

We derive the transition density $p(v_{t_{i-1}} \mid v_0)$ by solving the MRGP with $v(0) = v_0$. Applying Itô's formula [24, p. 44] to $e^{\kappa^* t}v(t)$ gives

$$de^{\kappa^* t}v(t) = \kappa^* e^{\kappa^* t}v(t)\,dt + e^{\kappa^* t}[\kappa^*(\theta^* - v(t))\,dt + \sigma\,d\tilde{B}_t^v]$$
$$= \kappa^*\theta^* e^{\kappa^* t}\,dt + \sigma e^{\kappa^* t}\,d\tilde{B}_t^v. \tag{2.13}$$

Integration of both sides of (2.13) yields

$$v(t) = e^{-\kappa^* t}v_0 + \theta^*(1 - e^{-\kappa^* t}) + \sigma e^{-\kappa^* t}\int_0^t e^{\kappa^* s}\,d\tilde{B}_s^v. \tag{2.14}$$

Note that the random variable $\int_0^t e^{\kappa^* s}\,d\tilde{B}_s^v$ appearing on the right-hand side of (2.14) is normally distributed with mean zero and variance

$$\int_0^t e^{2\kappa^* s}\,ds = \frac{1}{2\kappa^*}(e^{2\kappa^* t} - 1).$$

Therefore, $v(t)$ is normally distributed with mean $e^{-\kappa^* t}v_0 + \theta^*(1 - e^{-\kappa^* t})$ and variance $(\sigma^2/2\kappa^*)(1 - e^{-2\kappa^* t})$. For simplicity of notation, we denote $\hat{\mu}(t) = e^{-\kappa^* t}v_0 + \theta^*(1 - e^{-\kappa^* t})$ and $\hat{\sigma}^2(t) = (\sigma^2/2\kappa^*)(1 - e^{-2\kappa^* t})$. Then, the transition density of the MRGP is

$$p(v_{t_{i-1}} \mid v_0) = \frac{1}{\sqrt{2\pi}\hat{\sigma}(t_{i-1})}e^{-[v_{t_{i-1}}-\hat{\mu}(t_{i-1})]^2/2\hat{\sigma}^2(t_{i-1})}. \tag{2.15}$$

After a careful calculation, we have successfully carried out the integration in equation (2.12) analytically and obtained a fully closed-form solution as our final solution for the price of a variance swap with the realized variance defined by (2.2). We have the following solution: when the parameters satisfy $[1/2\hat{\sigma}^2(t_{i-1})] - E(2, \Delta t) > 0$,

$$E_0^Q\Big[\Big(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\Big)^2\Big]$$
$$= (1 - 2\hat{\sigma}^2(t_{i-1})E(2, \Delta t))^{-1/2}e^{[(\hat{\sigma}^2(t_{i-1})D(2,\Delta t)+\hat{\mu}(t_{i-1}))^2/2\hat{\sigma}^2(t_{i-1})(1-2\hat{\sigma}^2(t_{i-1})E(2,\Delta t))]+L(t)}$$
$$- 2e^{r\Delta t} + 1, \tag{2.16}$$

where $L(t) = C(2, \Delta t) - \hat{\mu}^2(t_{i-1})/2\hat{\sigma}^2(t_{i-1})$. The details of analytically carrying out the integration in equation (2.12) are provided in Appendix B.

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Utilizing equations (2.11) and (2.16), the summations in equation (2.3) are carried out. When the parameters satisfy $[1/2\hat{\sigma}^2(t_{i-1})] - E(2, \Delta t) > 0$, for any $2 \le i \le N$, we obtain the fair strike price for the variance swap as

$$K = E_0^Q[\sigma_R^2] = \frac{1}{T}\Big[f(v_0) + \sum_{i=2}^{N} f_i(v_0)\Big] \times 100^2, \tag{2.17}$$

where

$$f_i(v_0) = (1 - 2\hat{\sigma}^2(t_{i-1})E(2, \Delta t))^{-1/2} e^{[(\hat{\sigma}^2(t_{i-1})D(2,\Delta t)+\hat{\mu}(t_{i-1}))^2/2\hat{\sigma}^2(t_{i-1})(1-2\hat{\sigma}^2(t_{i-1})E(2,\Delta t))]+L(t)} \\ - 2e^{r\Delta t} + 1$$

and $f(v_0) = e^{C(2,\Delta t)+D(2,\Delta t)v_0+E(2,\Delta t)v_0^2} - 2e^{r\Delta t} + 1$.

Note that the restrictions for any $2 \le i \le N$, $[1/2\hat{\sigma}^2(t_{i-1})] - E(2, \Delta t) > 0$ are reduced, since $1/2\hat{\sigma}^2(t) = \kappa^*/\sigma^2(1 - e^{-2\kappa^* t})$ is decreasing as a function of t, and $0 = t_0 < t_1 < \cdots < t_N = T$. So, the above restrictions are reduced for $i = N$, $[1/2\hat{\sigma}^2(t_{N-1})] - E(2, \Delta t) > 0$, namely,

$$\frac{\kappa^*}{\sigma^2(1 - e^{-2\kappa^*(N-1)\Delta t})} - E(2, \Delta t) > 0. \tag{2.18}$$

Further, $\kappa^*/[\sigma^2(1 - e^{-2\kappa^*(N-1)\Delta t})] - E(2, \Delta t)$ is decreasing as a function of Δt, because its derivative function is always negative. So, if Δt is sufficiently small, that is, $\Delta t < \Delta t^*$, condition (2.18) is fulfilled, where

$$\Delta t^* = \min_{\Delta t > 0}\Big[\frac{\kappa^*}{\{\sigma^2(1 - e^{-2\kappa^*(N-1)\Delta t})\}} - E(2, \Delta t) = 0\Big].$$

Contracts with log returns are usually priced in a similar way as well. But the pricing formula in this case is much more complicated and the corresponding derivation process is also quite involved. We plan a forthcoming paper to price log returns in a similar way.

A couple of more points should be noted at the end of this subsection. Firstly, with the newly found analytical solution, all the hedging ratios of a variance swap are also analytically obtained by taking partial derivatives against various parameters in the model. Since the partial derivatives are readily calculated with some symbolic calculation packages, these are omitted here. Secondly, using formula (2.7) in Proposition 2.1, one easily derives an explicit formula for every conditional moment of the underlying asset price, $E_{t_{i-1}}^Q[S_t^\gamma]$, under the mean-reverting Gaussian volatility model (2.1). These explicit formulae are used in a similar way to what has been presented in this paper to price derivatives based on higher moments such as skewness swaps and kurtosis swaps discussed by Schoutens [27].

2.4. Fair strike price for continuously monitored variance swaps As mentioned in Section 1, most of the existing pricing models of variance derivative products assume continuous sampling of the realized variance processes. For the convenience

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of calculation, in the continuous models, the realized variance (2.2) is usually approximated by

$$\sigma_R^2 = \frac{1}{T}\int_0^T v_t^2\,dt \times 100^2. \tag{2.19}$$

The expectation of this continuous integral is easily obtained by utilizing the second stochastic process defined in equation (2.1). Indeed, noticing that $v(t)$ is normally distributed with mean $\hat{\mu}(t) = e^{-\kappa^* t} v_0 + \theta^*(1 - e^{-\kappa^* t})$ and variance $\hat{\sigma}^2(t) = \sigma^2(1 - e^{-2\kappa^* t})/2\kappa^*$, we have $E_0^Q[v_t^2] = \mathrm{Var}[v_t] + E^2[v_t] = \hat{\sigma}^2(t) + \hat{\mu}^2(t)$, where $\mathrm{Var}[\cdot]$ denotes the variance of a random variable. Therefore, the fair strike price of the continuously sampled variance swaps is

$$\begin{aligned} K_\infty &= E_0^Q[\sigma_R^2] = \frac{1}{T}\int_0^T E_0^Q[v_t^2]\,dt \times 100^2 = \frac{1}{T}\int_0^T (\hat{\sigma}^2(t) + \hat{\mu}^2(t))dt \times 100^2 \\ &= \left[\theta^{*2} + \frac{\sigma^2}{2\kappa^*} + \frac{2\theta^*(v_0 - \theta^*)}{\kappa^* T}(1 - e^{-\kappa^* T}) + \frac{(v_0 - \theta^*)^2 - \sigma^2/2\kappa^*}{2\kappa^* T}(1 - e^{-2\kappa^* T})\right] \times 100^2. \end{aligned} \tag{2.20}$$

It is proved that our solution (2.17) approaches the equation (2.20) by taking the asymptotic limit of vanishing sampling time interval, that is,

$$\begin{aligned} K_\infty &= \lim_{\Delta t \to 0} \frac{1}{T}\left[f(v_0) + \sum_{i=2}^{N} f_i(v_0)\right] \times 100^2 \\ &= \left[\theta^{*2} + \frac{\sigma^2}{2\kappa^*} + \frac{2\theta^*(v_0 - \theta^*)}{\kappa^* T}(1 - e^{-\kappa^* T}) + \frac{(v_0 - \theta^*)^2 - \sigma^2/2\kappa^*}{2\kappa^* T}(1 - e^{-2\kappa^* T})\right] \times 100^2. \end{aligned} \tag{2.21}$$

The details of the proof of this limit are presented in Appendix C. This limit implies that the continuous sampling case is mostly viewed as a special case of our solution for the discrete sampling variance swaps with the sampling period shrinking down to zero.

3. Numerical tests and discussion

In this section, we show some numerical tests for illustration purposes. Some comparisons with the Monte Carlo (MC) simulations give readers a sense of verification for our solution. In addition, comparisons with the continuous sampling model will also help readers understand the improvement in accuracy with our exact solution. We also discuss the connection between the Heston model and ours, the restrictions of parameter space and their influence on the strike price and show the sensitivity of the fair strike price of discretely sampled variance swaps to the change of the key parameters in the model.

In our numerical examples, we adopt the following nondimensional parameters (unless otherwise stated): $v_0 = 0.2$, $\kappa^* = 4$, $\theta^* = 0.2$, $\rho = -0.64$, $\sigma = 0.1$, $r = 0.0953$,

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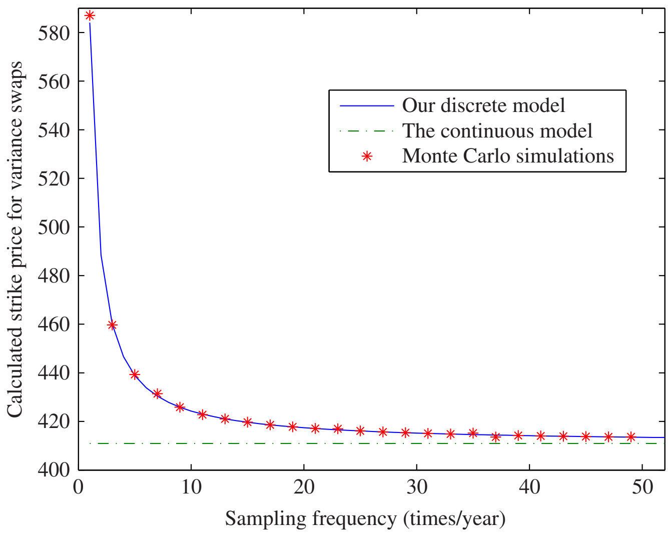

FIGURE 1. A comparison of fair strike values based on the discrete model, continuous model and MC simulations.

$T = 1$. This set of parameters for the mean-reverting Gaussian volatility model was also adopted by Stein and Stein [31] and Schöbel and Zhu [26]. We took the asset price $S_0 = 1$ and the number of the paths $N = 200\,000$ for all the MC simulations presented here.

3.1. Monte Carlo simulations The aim in our MC simulations is primarily to obtain some benchmark values for our solution equation (2.17), so we do not focus our attention on the use of some variance-reduction techniques that can further enhance the computational efficiency. We employ the simple Euler–Maruyama discretization for the mean-reverting Gaussian volatility model in our MC simulations:

$$\begin{cases} S_t = S_{t-1} + rS_{t-1}\Delta t + v_{t-1}S_{t-1}\sqrt{\Delta t}\,(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2), \\ v_t = v_{t-1} + \kappa^*(\theta^* - v_{t-1})\Delta t + \sigma\sqrt{\Delta t}W_t^1, \end{cases} \tag{3.1}$$

where W_t^1 and W_t^2 are two independent standard normal random variables.

We have shown in Figure 1 and Table 1 that there are three sets of data for the strike price of variance swaps obtained from equation (2.17), those from MC simulations (3.1) (the numbers in the parentheses are the standard errors for those simulation results) and the numerical results obtained from the continuously monitored realized variance equation (2.20). The results from our exact solution perfectly match the results from the MC simulations, which illustrates that our exact solution is correct.

From Figure 1, the values of our discrete model asymptotically approach the values of the continuous approximation model when the sampling frequency increases and the variance defined in (2.19) appears to be the limit of the realized variance defined in

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TABLE 1. The numerical results of the discrete model, continuous model and MC simulations.

Sampling frequency	Discrete model	Continuous model	MC simulations
Quarterly ($N = 4$)	446.6086	410.9380	446.5703 (315.9506)*
Monthly ($N = 12$)	421.9536	410.9380	421.8617 (187.1228)
Fortnightly ($N = 26$)	415.8955	410.9380	415.6284 (139.8798)
Weekly ($N = 52$)	413.3882	410.9380	413.6953 (113.9293)
Daily ($N = 252$)	411.4388	410.9380	411.4958 (88.2443)

*The reason why the standard errors in our MC simulations seem to be a bit large is that the realized variance defined by (2.2) has been magnified by 10 000 times.

equation (2.2) as $\Delta t \to 0$. This is in line with our claim in (2.21) and once again verified the correctness of our solution for the discrete sampling cases, taking the continuous sampling case as a special case with the sampling period shrinking down to zero.

Compared with MC simulations, computational efficiency is enormously enhanced in our exact solution in terms of computational time. The MC simulations take a much longer time than our analytical solution does; for example, for weekly sampling variance swaps when the number of paths is 500 000 in MC simulations, computational time reaches 2 168.4 s, while implementing equation (2.17) just took 0.012 seconds. This is not surprising at all since time consumption is a well-known drawback of MC simulations. The difference is even more significant when the sampling frequency is increased.

3.2. Comparison with Heston model We compare our formula (2.17) with the fair strike price obtained by Rujivan and Zhu [25]. As pointed out by Schöbel and Zhu [26], it is relatively difficult to exhibit the inherent connection between the Heston model and our model since he models variance instead of volatility. If the volatility follows a MRGP as in (2.1), from Itô's formula, then the process for the squared volatility $y(t) = v^2(t)$ is

$$dy_t = [\sigma^2 + 2\kappa^*\theta^*\sqrt{y_t} - 2\kappa^* y_t]\,dt + 2\sigma\sqrt{y_t}\,d\tilde{B}_t^v. \tag{3.2}$$

This is a mean-reverting double square-root process with the additional drift term $2\kappa^*\theta^*\sqrt{y_t}$. For the special case $\theta^* = 0$, (3.2) is reduced to the Heston model with parameters

$$\kappa_h = 2\kappa^*, \quad \theta_h = \frac{\sigma^2}{2\kappa^*}, \quad \sigma_h = 2\sigma. \tag{3.3}$$

Indeed, Heston assumed that volatility followed an Ornstein–Uhlenbeck process with a mean-reversion level equal to zero in Heston [16], that is,

$$dv_t = -\beta v_t\,dt + \delta\,dB_t. \tag{3.4}$$

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TABLE 2. The numerical results of our formula and the formula in Rujivan and Zhu [25].

Sampling frequency	Our formula	Formula in Rujivan and Zhu [25]
Quarterly ($N = 4$)	85.9348	85.9348
Monthly ($N = 12$)	69.0009	69.0009
Weekly ($N = 52$)	62.7607	62.7607
Daily ($N = 252$)	61.2996	61.2996

Then, the variance of instantaneous stock returns $y(t) = v^2(t)$ follows a square-root process

$$dy_t = \kappa_h(\theta_h - y_t)\,dt + \sigma_h\sqrt{y_t}\,dB_t \tag{3.5}$$

with

$$\kappa_h = 2\beta, \quad \theta_h = \frac{\delta^2}{\kappa_h}, \quad \sigma_h = 2\delta. \tag{3.6}$$

The only difference between equation (2.1) and equation (3.4) is the mean-reversion parameter θ^*, which in (2.1) generally differs from zero, whereas in (3.4) it is always nil. Since θ^* gives the level of volatility in the long run, the process (3.4) is not very reasonable. But the Heston model is based on the process (3.5) not (3.4). Note that the parameters in (3.5) are overdetermined by (3.6). Hence, for a wide range of values for κ_h, θ_h and σ_h, process (3.5) cannot be derived from (3.4). Therefore, the two processes (3.4) and (3.5) are not mutually consistent for many parameter values.

Table 2 gives the numerical results of our formula and the fair strike price obtained in Rujivan and Zhu [25] with the parameters $\theta^* = 0$, $\kappa_h = 2\kappa^*$, $\theta_h = \sigma^2/2\kappa^*$ and $\sigma_h = 2\sigma$. The values obtained from the two models are the same in this special case. The theoretical proof of this consistency is in Appendix D.

3.3. Restrictions of parameter space We discuss the restrictions of parameter space under which our formula (2.17) is financially meaningful, that is, the strike price obtained from (2.17) is a nonnegative finite real number. Proposition 2.1 has shown that for the parameters satisfying $\kappa^* > (2\rho + \sqrt{2})\,\sigma$ or $\kappa^* < (2\rho - \sqrt{2})\,\sigma$, equation (2.6) has a global solution, while for those satisfying $(2\rho - \sqrt{2})\,\sigma \le \kappa^* \le (2\rho + \sqrt{2})\,\sigma$, only local solutions of (2.6) exist. Indeed, for the parameters $(2\rho - \sqrt{2})\,\sigma \le \kappa^* \le (2\rho + \sqrt{2})\,\sigma$, the numerical implementations of (2.17) resulted in some complex numbers, which is apparently unreasonable since they are the strike price of some variance swaps.

Table 3 presents the numerical results of fair strike values obtained from our solution (2.17) and MC simulations with the parameters on the dividing line in the parameter space, namely, we take $\kappa^* = (2\rho + \sqrt{2})\,\sigma = 0.0134$ in the calculation. This table shows that with the parameters on the dividing line in the parameter space, our exact solution (2.17) has produced some complex numbers; thus, it is not suitable any more, while the strike price obtained from MC simulations is still a finite real number. We have the same results for those parameters satisfying $(2\rho - \sqrt{2})\,\sigma < \kappa^* <$

TABLE 3. The numerical results of our formula and MC simulations with $\kappa^* = 0.0134$.

Sampling frequency	Our formula	MC simulations
Quarterly ($N = 4$)	4.8180e+002 + 2.0722e−005i	483.9658 (440.7224)
Monthly ($N = 12$)	4.5853e+002 + 2.0003e−005i	461.2613 (314.1468)
Weekly ($N = 52$)	4.5742e+002 + 1.9593e−005i	452.1572 (255.5134)
Daily ($N = 252$)	4.5208e+002 + 1.9594e−005i	450.7324 (241.3925)

TABLE 4. The numerical results of our formula and MC simulations with $\kappa^* = 0.005$.

Sampling frequency	Our formula	MC simulations
Quarterly ($N = 4$)	4.8390e+002 + 1.9780e−012i	483.7322 (435.5521)
Monthly ($N = 12$)	4.6103e+002 − 2.2727e−012i	460.3388 (313.7185)
Weekly ($N = 52$)	4.5240e+002 − 8.9277e−013i	452.4789 (256.2517)
Daily ($N = 252$)	4.5036e+002 − 6.6077e−011i	450.5263 (241.9733)

$(2\rho + \sqrt{2})\sigma$ and Table 4 is an example when $\kappa^* = 0.005$. Fortunately, for a wide range of reasonable parameter values (that is, those adopted by Stein and Stein [31] and Schöbel and Zhu [26]), the inequalities $\kappa^* > (2\rho + \sqrt{2})\sigma$ or $\kappa^* < (2\rho - \sqrt{2})\sigma$ (at the same time (2.18)) is satisfied and our exact solution (2.17) is used safely.

We provide a plot of the fair strike price against mean-reverting speed parameter κ^* in Figure 2 to illustrate the influence on the strike price when κ^* varies across the dividing line in the parameter space. Since for the parameters presented previously in this section, those κ^* satisfying the inequality $(2\rho - \sqrt{2})\sigma \le \kappa^* \le (2\rho + \sqrt{2})\sigma$ are very small, that is, $0 < \kappa^* \le 0.0134$ (note that κ^* needs to be positive), we change the parameters with $\rho = -0.3$ and $\sigma = 0.8$ in Figure 2 in order to see the plot clearly, and the other parameters remain the same as before. From Figure 2, we see that for those $(2\rho - \sqrt{2})\sigma \le \kappa^* \le (2\rho + \sqrt{2})\sigma$, the strike price obtained from MC simulations is still a finite real number, which means that for those parameters, the calculated fair strike price of discretely sampled variance swaps under the mean-reverting Gaussian volatility model remains financially meaningful, but in those situations our solution (2.17) is not suitable any more. This is because we assumed the solution with some particular form when solving the governing PDE system, whereas for those

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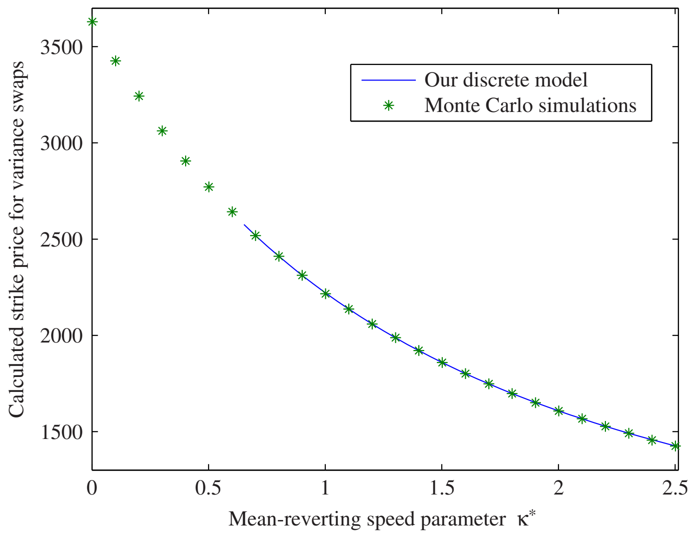

FIGURE 2. The fair strike price against mean-reverting speed parameter κ^* (weekly sampling).

TABLE 5. The sensitivity of strike price of variance swap (daily sampling).

Model parameter	Value	Sensitivity (%)
κ^*	4	−0.0227
θ^*	0.2	1.47
σ	0.1	0.0529
v_0	0.2	0.48

$(2\rho - \sqrt{2})\,\sigma \le \kappa^* \le (2\rho + \sqrt{2})\,\sigma$, the PDE system may have other forms of real solutions. We postpone this discussion to future works. Indeed, many closed-form formulae for discretely sampled variance swaps proposed recently (that is, [25, 35, 36]) also have some restrictions in the parameter space. In other words, there is a subspace in which their solution is valid, and guarantees a nonnegative finite real fair delivery price, although they do not explicitly mention this issue in their papers.

We also performed some sensitivity tests in this subsection to demonstrate how sensitive the strike price is to the change of the key parameters in the model. The results of the percentage change of the strike price are shown in Table 5, when a model parameter changes by 1% from its base value used in the example presented in this section. The strike price of a variance swap appears to be the most sensible to the long-term mean volatility θ^* for the case studied. The spot volatility v_0 also has a significant influence in terms of the sensitivity of the strike price and the least sensible parameter is the mean-reverting speed parameter κ^*.

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4. Conclusion

In this paper, a new closed-form pricing formula for the value of discretely sampled variance swaps is presented under the assumption that the underlying assets can be described by a mean-reverting Gaussian volatility model. One of the greatest advantages to use closed-form analytical formulae, rather than any numerical solution approach, is the computational efficiency when the numerical value of the fair strike price of a discretely sampled variance swap needs to be computed by market practitioners. We carried out some numerical tests and demonstrated that not only the results obtained from our pricing formula match perfectly with those obtained from MC simulation, it is also far more efficient to compute price as well as all hedging ratios of a variance swap from the newly derived formula. We also discussed the connection between the Heston model and ours, the restrictions of parameter space and showed their influence on the strike price.

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Appendix A

Here we give a brief proof of Proposition 2.1. First, we show that the transformation $Y_t = S_t^\gamma$ is well defined for any $\gamma \in \mathbb{R}\backslash\{0\}$. From (2.1), for any $t \in [t_{i-1}, t_i]$ one can obtain by Itô's formula that $S_t = S_{t_{i-1}} \exp(\int_{t_{i-1}}^t (r - \frac{1}{2}v_s^2)\, ds + \int_{t_{i-1}}^t v_s\, d\tilde{B}_s^S)$. This implies that if $S_{t_{i-1}} > 0$, then $S_t > 0$ with probability 1; therefore, $Y_t > 0$ is a real-valued stochastic process for any $\gamma \in \mathbb{R}\backslash\{0\}$. Next, we show the derivation of Formula (2.7). Applying Itô's formula to the above transformation gives

$$dY_t = (\gamma r + \tfrac{1}{2}\gamma(\gamma - 1)v_t^2)Y_t\, dt + \gamma v_t Y_t\, d\tilde{B}_t^S.$$

We now consider a contingent claim $U_i^{(\gamma)}(y, v, t) = E^Q[Y_t \mid (Y_{t_{i-1}} = y, v_{t_{i-1}} = v)]$, whose payoff at expiry t_i is Y_{t_i}. Following the general asset valuation theory by Garman [13], $U_i^{(\gamma)}$ satisfies

$$\frac{\partial U_i^{(\gamma)}}{\partial t} + \frac{1}{2}\gamma^2 v^2 y^2 \frac{\partial^2 U_i^{(\gamma)}}{\partial y^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 U_i^{(\gamma)}}{\partial v^2} + \rho\gamma\sigma v y \frac{\partial^2 U_i^{(\gamma)}}{\partial y \partial v} + \left[\gamma r + \frac{1}{2}\gamma(\gamma - 1)v^2\right] y \frac{\partial U_i^{(\gamma)}}{\partial y} + \kappa^*(\theta^* - v)\frac{\partial U_i^{(\gamma)}}{\partial v} = 0, \tag{A.1}$$

subject to the terminal condition

$$U_i^{(\gamma)}(y, v, t_i) = y$$

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for all $t \in [t_{i-1}, t_i)$ and $(y, v) \in (0, \infty) \times (-\infty, \infty)$. Let $\tau = t_i - t$. We assume the solution of the above PDE with the form

$$U_i^{(\gamma)}(y, v, t) = ye^{C(\gamma, t_i - t) + D(\gamma, t_i - t)v + E(\gamma, t_i - t)v^2}. \tag{A.2}$$

Substituting (A.2) into the PDE (A.1) yields a set of ordinary differential equations

$$\frac{dE}{d\tau} = 2\sigma^2 E^2 + (2\rho\gamma\sigma - 2\kappa^*)E + \frac{1}{2}\gamma(\gamma - 1), \tag{A.3}$$

$$\frac{dD}{d\tau} = 2\sigma^2 DE + \rho\gamma\sigma D + 2\kappa^*\theta^* E - \kappa^* D, \tag{A.4}$$

$$\frac{dC}{d\tau} = \sigma^2 E + \frac{1}{2}\sigma^2 D^2 + \gamma r + \kappa^*\theta^* D, \tag{A.5}$$

subject to the initial conditions

$$E(\gamma, 0) = 0, \tag{A.6}$$

$$D(\gamma, 0) = 0, \tag{A.7}$$

$$C(\gamma, 0) = 0. \tag{A.8}$$

There are three cases in the parameter space, which need to be considered separately, depending on the values of $\hat{c}(\gamma) = (2\kappa^* - 2\rho\gamma\sigma)^2 - 4\sigma^2\gamma(\gamma - 1)$ and $\hat{a}(\gamma) = 2\kappa^* - 2\rho\gamma\sigma$.

Case 1. $\hat{c}(\gamma) > 0$. In this case, the explicit form of $E(\gamma, \tau)$, as a global solution, is given in (2.7).

Case 2. $\hat{c}(\gamma) = 0$, which needs to be further divided into two subcases: Case 2.1: $\hat{c}(\gamma) = 0$ and $\hat{a}(\gamma) \geq 0$ and Case 2.2: $\hat{c}(\gamma) = 0$ and $\hat{a}(\gamma) < 0$. In Case 2.1, a global solution is found as $E_1(\gamma, \tau) = (\gamma(\gamma - 1)\tau)/(2 + \hat{a}(\gamma)\tau)$ for all $\tau \in [0, \infty)$. Applying L'Hôpital's rule, we can show that $\lim_{\hat{b}(\gamma)\to 0^+} E(\gamma, \tau) = \lim_{\hat{c}(\gamma)\to 0^+} E(\gamma, \tau) = E_1(\gamma, \tau)$ for all $\tau \in [0, \infty)$. This implies that Case 2.1 is a special case of Case 1, in which $\hat{b}(\gamma)$ approaches zero from above. In Case 2.2, on the other hand, we get the same $E_1(\gamma, \tau)$ as a local solution only for all $\tau \in [0, -2/\hat{a}(\gamma))$.

Case 3. $\hat{c}(\gamma) < 0$, which also yields a local solution as

$$E_2(\gamma, \tau) = \frac{1}{4\sigma^2}\left[\sqrt{-\hat{c}(\gamma)} \tan\left(\frac{\sqrt{-\hat{c}(\gamma)}}{2}\tau + \varphi(\gamma)\right) + \hat{a}(\gamma)\right]$$

for all $\tau \in [0, \zeta(\gamma))$, where $\zeta(\gamma) = (\pi - 2\varphi(\gamma))/\sqrt{-\hat{c}(\gamma)}$ and $\varphi(\gamma) = \arctan(-\hat{a}(\gamma)/\sqrt{-\hat{c}(\gamma)})$.

Substituting $E(\gamma, \tau)$ into (A.4), we obtain a first-order linear equation with respect to $D(\gamma, \tau)$. In Case 1, a global solution can be easily obtained as $D(\gamma, \tau)$ expressed in (2.7). The two $D_1(\gamma, \tau)$ solutions corresponding to $E_1(\gamma, \tau)$ for both Case 2.1 and Case 2.2 are of the same form $D_1(\gamma, \tau) = \kappa^*\theta^*\gamma(\gamma - 1)\tau^2/(\hat{a}(\gamma)\tau + 2)$. However, for Case 2.1, this is a global solution with the domain $\tau \in [0, \infty)$, while it is only a local solution for

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Case 2.2 with the domain $\tau \in [0, -2/\hat{a}(\gamma))$. In Case 3, a local solution corresponding to $E_2(\gamma, \tau)$ can be found as

$$D_2(\gamma, \tau) = \frac{\kappa^* \theta^* [1 - \cos(\tau \sqrt{-\hat{c}(\gamma)}/2)]}{\sigma^2 \cos(\tau \sqrt{-\hat{c}(\gamma)}/2 + \varphi(\gamma)) \cos \varphi(\gamma)}$$

for all $\tau \in [0, \zeta(\gamma))$.

Once $E(\gamma, \tau), D(\gamma, \tau)$ are found, $C(\gamma, \tau)$ is easily obtained by integrating (A.5) subject to (A.8), that is, $C(\gamma, \tau) = \int_0^\tau (\sigma^2 E(\gamma, s) + \frac{1}{2}\sigma^2 D^2(\gamma, s) + \gamma r + \kappa^* \theta^* D(\gamma, s)) ds$. In Case 1, a global solution of $C(\gamma, \tau)$ is given in (2.7). For both Case 2.1 and Case 2.2, the two solutions $C_1(\gamma, \tau)$ are of the same form:

$$C_1(\gamma, \tau) = -\frac{1}{2} \ln\Big(\frac{\hat{a}(\gamma)}{2}\tau + 1\Big) + \frac{\kappa^{*^2} \theta^{*^2} \gamma(\gamma - 1)}{24}\tau^3 + \frac{\kappa^{*^2} \theta^{*^2} \hat{a}(\gamma)}{16\sigma^2}\tau^2 + \Big(\frac{\hat{a}(\gamma)}{4} - \frac{\kappa^{*^2} \theta^{*^2}}{8\sigma^2} + \frac{1}{4(\hat{a}(\gamma)\tau + 2)} + \gamma r\Big)\tau.$$

This is a global solution for all $\tau \in [0, \infty)$ in Case 2.1, and only a local solution for all $\tau \in [0, -2/\hat{a}(\gamma))$ in Case 2.2, similar to $E_1(\gamma, \tau)$ and $D_1(\gamma, \tau)$, respectively. We obtain a local solution

$$C_2(\gamma, \tau) = -\frac{1}{2} \ln \frac{\cos(\tau \sqrt{-\hat{c}(\gamma)}/2 + \varphi(\gamma))}{\cos \varphi(\gamma)} + \Big(\frac{1}{4}\hat{a}(\gamma) + \gamma r + \frac{2\kappa^{*^2} \theta^{*^2} \gamma(\gamma - 1)}{\hat{c}(\gamma)}\Big)\tau + \frac{\kappa^{*^2} \theta^{*^2} \hat{a}(\gamma)}{\sigma^2 \hat{c}(\gamma)} + \frac{\kappa^{*^2} \theta^{*^2} [(2\hat{a}^2(\gamma) - \hat{c}(\gamma)) \sin(\tau \sqrt{-\hat{c}(\gamma)}/2 + \varphi(\gamma)) + 8\sigma^2 \gamma(\gamma - 1)\hat{a}(\gamma)]}{\sigma^2 (-\hat{c}(\gamma))^{3/2} \cos(\tau \sqrt{-\hat{c}(\gamma)}/2 + \varphi(\gamma))}$$

for all $\tau \in [0, \zeta(\gamma))$ in Case 3.

Appendix B

A substitution of (2.15) into (2.12) yields

$$\begin{aligned} E_0^Q\Big[\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2\Big] &= \int_{-\infty}^{\infty} f(v_{t_{i-1}}) p(v_{t_{i-1}} \mid v_0)\, dv_{t_{i-1}} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\hat{\sigma}(t_{i-1})} e^{(-1/2\hat{\sigma}^2(t_{i-1}))(v_{t_{i-1}} - \hat{\mu}(t_{i-1}))^2 + C(2,\Delta t) + D(2,\Delta t) v_{t_{i-1}} + E(2,\Delta t) v_{t_{i-1}}^2}\, dv_{t_{i-1}} - K(t) \\ &= \frac{1}{\sqrt{2\pi}\hat{\sigma}(t_{i-1})} \int_{-\infty}^{\infty} e^{(E(2,\Delta t) - 1/2\hat{\sigma}^2(t_{i-1})) v_{t_{i-1}}^2 + (D(2,\Delta t) + \hat{\mu}(t_{i-1})/\hat{\sigma}^2(t_{i-1})) v_{t_{i-1}} + L(t)}\, dv_{t_{i-1}} - K(t) \\ &= \frac{1}{\sqrt{2\pi}\hat{\sigma}(t_{i-1})} \int_{-\infty}^{\infty} e^{-(1/2\hat{\sigma}^2(t_{i-1}) - E(2,\Delta t)) M(t) + N(t) + L(t)}\, dv_{t_{i-1}} - K(t), \end{aligned} \tag{B.1}$$

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where $K(t) = 2e^{r\Delta t} - 1$, $L(t) = C(2,\Delta t) - \hat{\mu}^2(t_{i-1})/2\hat{\sigma}^2(t_{i-1})$,

$$M(t) = \left[v_{t_{i-1}} - \frac{D(2,\Delta t) + \hat{\mu}(t_{i-1})/\hat{\sigma}^2(t_{i-1})}{2(1/2\hat{\sigma}^2(t_{i-1}) - E(2,\Delta t))}\right]^2$$

and

$$N(t) = \frac{(D(2,\Delta t) + \hat{\mu}(t_{i-1})/\hat{\sigma}^2(t_{i-1}))^2}{4(1/2\hat{\sigma}^2(t_{i-1}) - E(2,\Delta t))}.$$

To calculate this integration, we claim that for any $\alpha > 0$, $\mu \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\alpha(x-\mu)^2}\, dx = \sqrt{\frac{\pi}{\alpha}},$$

while, for any $\alpha \le 0$,

$$\int_{-\infty}^{\infty} e^{-\alpha(x-\mu)^2}\, dx = \infty.$$

Indeed, for $\alpha > 0$, $\mu \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\alpha(x-\mu)^2}\, dx = \sqrt{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2[\sqrt{2\alpha}(x-\mu)]^2}\, dx = \sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = \sqrt{\frac{\pi}{\alpha}},$$

where we have made the change of dummy variable $y = \sqrt{2\alpha}(x-\mu)$ and used that the integral of the standard normal density is equal to one. For $\alpha \le 0$,

$$\int_{-\infty}^{\infty} e^{-\alpha(x-\mu)^2} dx = \int_{-\infty}^{\infty} e^{-\alpha y^2} dy = 2\int_{0}^{\infty} e^{-\alpha y^2} dy \ge 2\int_{0}^{\infty} 1\, dy = \infty.$$

Using the fact that the parameters satisfy $[1/2\hat{\sigma}^2(t_{i-1})] - E(2,\Delta t) > 0$, the integration in equation (B.1) is

$$\begin{aligned} E_0^Q&\left[\left(\frac{S_i - S_{i-1}}{S_{i-1}}\right)^2\right] \\ &= \frac{1}{\sqrt{2\pi}\hat{\sigma}(t_{i-1})}\sqrt{\frac{\pi}{[1/2\hat{\sigma}^2(t_{i-1})] - E(2,\Delta t)}} \times e^{N(t)+L(t)} - K(t) \\ &= (1 - 2\hat{\sigma}^2(t_{i-1})E(2,\Delta t))^{-1/2} e^{[(\hat{\sigma}^2(t_{i-1})D(2,\Delta t)+\hat{\mu}(t_{i-1}))^2/2\hat{\sigma}^2(t_{i-1})(1-2\hat{\sigma}^2(t_{i-1})E(2,\Delta t))]+L(t)} - K(t) \end{aligned}$$

and, when the parameters satisfy $1/[2\hat{\sigma}^2(t_{i-1})] - E(2,\Delta t) \le 0$, the integral in equation (B.1) is infinite.

Appendix C

Now, we prove equation (2.21). From the definition of $C(\gamma,\tau)$, $D(\gamma,\tau)$ and $E(\gamma,\tau)$, we verify that

$$\lim_{\Delta t\to 0} C(2,\Delta t) = 0, \qquad \lim_{\Delta t\to 0} D(2,\Delta t) = 0, \qquad \lim_{\Delta t\to 0} E(2,\Delta t) = 0$$

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and

$$\lim_{\Delta t\to 0} f(v_0) = 0, \qquad \lim_{\Delta t\to 0} f_i(v_0) = 0.$$

Using L'Hôpital's rule, we prove that

$$\lim_{\Delta t\to 0} \frac{f(v_0)}{\Delta t} = v_0^2, \qquad \lim_{\Delta t\to 0} \frac{f_i(v_0)}{\Delta t} = \hat{\sigma}^2(t_{i-1}) + \hat{\mu}^2(t_{i-1}).$$

Therefore,

$$\begin{aligned}
K_\infty &= \lim_{\Delta t\to 0} \frac{1}{T}\Big[f(v_0) + \sum_{i=2}^{N} f_i(v_0)\Big] \times 100^2 \\
&= \frac{1}{T} \lim_{\Delta t\to 0} \sum_{i=2}^{N} \Delta t\Big(v_0^2 + \frac{f_i(v_0)}{\Delta t}\Big) \times 100^2 \\
&= \frac{1}{T} \lim_{\Delta t\to 0} \sum_{i=1}^{N} \Delta t[\hat{\sigma}^2(t_{i-1}) + \hat{\mu}^2(t_{i-1})] \times 100^2 \\
&= \frac{1}{T} \int_0^T \Big[\frac{\sigma^2}{2\kappa^*}(1 - e^{-2\kappa^* t}) + \{e^{-\kappa^* t} v_0 + \theta^*(1 - e^{-\kappa^* t})\}^2\Big] dt \times 100^2 \\
&= \Big[\theta^{*2} + \frac{\sigma^2}{2\kappa^*} + \frac{2\theta^*(v_0 - \theta^*)}{\kappa^* T}(1 - e^{-\kappa^* T}) + \frac{(v_0 - \theta^*)^2 - (\sigma^2/2\kappa^*)}{2\kappa^* T}(1 - e^{-2\kappa^* T})\Big] \times 100^2.
\end{aligned}$$

Appendix D

The fair strike price formula obtained by Rujivan and Zhu [25] is

$$K_{var} = \frac{e^{r\Delta t}}{T}\Big[f^h(v_0) + \sum_{i=2}^{N} f_i^h(v_0)\Big] \times 100^2$$

with

$$\begin{aligned}
f^h(v_0) &= e^{\tilde{C}(\Delta t) + \tilde{D}(\Delta t) v_0} + e^{-r\Delta t} - 2, \\
f_i^h(v_0) &= e^{\tilde{C}(\Delta t) + [c_i e^{-\kappa_h t_{i-1}}]/[c_i - \tilde{D}(\Delta t)]\tilde{D}(\Delta t) v_0} \Big[\frac{c_i}{c_i - \tilde{D}(\Delta t)}\Big]^{2\kappa_h\theta_h/\sigma_h^2} + e^{-r\Delta t} - 2, \\
\tilde{C}(\Delta t) &= r\Delta t + \frac{\kappa_h \theta_h}{\sigma_h^2}\Big[(\tilde{a} + \tilde{b})\Delta t - 2\ln\Big(\frac{1 - \tilde{g} e^{\tilde{b}\Delta t}}{1 - \tilde{g}}\Big)\Big], \\
\tilde{D}(\Delta t) &= \frac{\tilde{a} + \tilde{b}}{\sigma_h^2}\Big(\frac{1 - e^{\tilde{b}\Delta t}}{1 - \tilde{g} e^{\tilde{b}\Delta t}}\Big), \\
c_i &= \frac{2\kappa_h}{\sigma_h^2(1 - e^{-\kappa_h t_{i-1}})}, \\
\tilde{a} &= \kappa_h - 2\rho\sigma_h, \qquad \tilde{b} = \sqrt{\tilde{a}^2 - 2\sigma_h^2}, \qquad \tilde{g} = \frac{\tilde{a} + \tilde{b}}{\tilde{a} - \tilde{b}},
\end{aligned} \tag{D.1}$$

where we rewrite $f(v_0)$, $f_i(v_0)$, κ^*, θ^* and σ_v in Rujivan and Zhu [25] as $f^h(v_0)$, $f_i^h(v_0)$, κ_h, θ_h and σ_h, respectively, to resolve the ambiguity. On the other hand, for the case

$\theta^* = 0$, our formula (2.17) becomes

$$K = \frac{1}{T}\Big[f(v_0) + \sum_{i=2}^{N} f_i(v_0)\Big] \times 100^2$$

with

$$\begin{aligned}
f(v_0) &= e^{C(2,\Delta t)+E(2,\Delta t)v_0^2} - 2e^{r\Delta t} + 1,\\
f_i(v_0) &= [1 - 2\hat{\sigma}^2(t_{i-1})E(2,\Delta t)]^{-1/2} e^{[E(2,\Delta t)\hat{\mu}^2(t_{i-1})]/[1-2\hat{\sigma}^2(t_{i-1})E(2,\Delta t)]+C(2,\Delta t)} - 2e^{r\Delta t} + 1,\\
C(2,\Delta t) &= -\frac{1}{2}\ln\Big[\frac{(\hat{a}(2)+\hat{b}(2))e^{\hat{b}(2)\Delta t} - \hat{a}(2) + \hat{b}(2)}{2\hat{b}(2)}\Big] + \Big[\frac{\hat{a}(2)+\hat{b}(2)}{4} + 2r\Big]\Delta t,\\
E(2,\Delta t) &= \frac{2(e^{\hat{b}(2)\Delta t} - 1)}{(\hat{a}(2)+\hat{b}(2))e^{\hat{b}(2)\Delta t} - \hat{a}(2) + \hat{b}(2)}.
\end{aligned}$$

It is verified that with the parameters $\kappa_h = 2\kappa^*$, $\theta_h = \sigma^2/2\kappa^*$ and $\sigma_h = 2\sigma$, the functions $\tilde{C}(\Delta t)$, $\tilde{D}(\Delta t)$ and c_i in (D.1) are equal to $C(2,\Delta t) - r\Delta t$, $E(2,\Delta t)$ and $1/[2\hat{\sigma}^2(t_{i-1})]$, respectively. Hence, $e^{r\Delta t} f^h(v_0^2) = f(v_0)$, and

$$\begin{aligned}
e^{r\Delta t} f_i^h(v_0^2) = [1 - 2\hat{\sigma}^2(t_{i-1})E(2,\Delta t)]^{-1/2} e^{[E(2,\Delta t)e^{-2\kappa^* t_{i-1}} v_0^2]/[1-2\hat{\sigma}^2(t_{i-1})E(2,\Delta t)]+C(2,\Delta t)}\\
- 2e^{r\Delta t} + 1 = f_i(v_0),
\end{aligned} \tag{D.2}$$

where we used the fact that $\hat{\mu}(t_{i-1}) = e^{-\kappa^* t_{i-1}} v_0$ in the last step of (D.2). That is, the two formulae are the same in this special case.

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