Dear Editor,

Some comments on 'Sequential random packing in the plane' by H. J. Weiner

Rényi (1958) showed that the asymptotic mean proportion of coverage resulting from sequential random packing of unit lengths on a long line segment is $\eta \simeq 0.7476$. Palásti (1960) conjectured that the corresponding quantity for rectangular cars with sides parallel to rectangular boundaries is simply η^2. Subsequently (Blaisdell and Solomon (1970)), the conjecture was broadened to η^n in $\mathbb{R}^n$. Proof of this generalized conjecture would mean that analytical solutions were available for sequential random packing in $\mathbb{R}^n$. We wish to point out that a recently published proof of the conjecture (Weiner (1978)) rests upon an invalid assumption.

For convenience, we reproduce the introduction to, and statement of, Weiner's Lemma 2.

'Consider the $a \times b$ rectangle with coordinates $(0,0)$, $(0, b)$, $(a, 0)$, (a, b), and rectangular $\alpha \times \beta$ cars, $\alpha, \beta \ll a, b$. Let l denote the line segment $(0, b - \beta)$ to $(a, b - \beta)$. A key lemma is the following.

Lemma 2. The $\alpha \times \beta$ cars parked in the $a \times b$ rectangle according to Model I intersect line segment l in segments (of length α) in accord with a one-dimensional law of Model I for cars of length α parked on a segment of length a.'

The proof of this lemma depends on the claim that 'the x, y-coordinates which determine the placement of a car to be parked are chosen i.i.d. uniformly.' This is false. It is true, of course, that the coordinates of *attempted* placements are chosen in this way, but the success or failure of the attempt depends on the positions of rectangles already in place. Thus Weiner's claim that 'the horizontal placement and parking of cars on l is independent of all other parked cars and depends only on the x-coordinate,' which is essential to his proof, is clearly untrue. It is conceivable that the effects of particular configurations on the mean density may average out to produce η^2, but Weiner has not proved this. He has simply assumed the latter independence and shown that the Palásti conjecture then follows.

A simple example illustrates the nature of this effect. Figure 1 shows segments of the boundary and the line l together with three rectangles which are already

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in place. The hatched region is that available for the centres of rectangles which will intersect l. Though equal segments of l have equal probability of being chosen for trial placements, acceptable values of y are much more likely in region B. Thus, the corresponding segment of l is much more heavily weighted for successful placement.

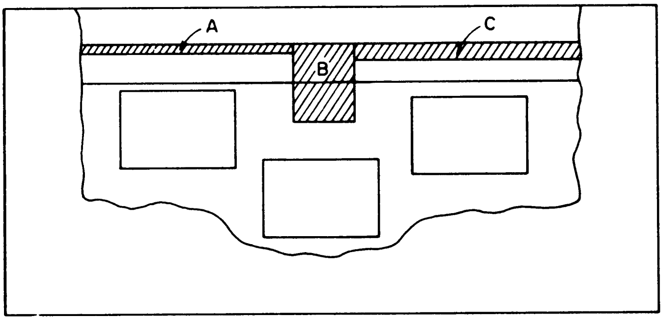

Figure 1
Effect of predecessors on the placement of rectangles in the top row of a rectangular region.

Nor can we remove this disparity by postulating that the top row is laid down first. A given configuration of n rectangles can be achieved by $n!$ different sequences, but the likelihood of this configuration is not the same under these permutations. That is, the random variables are simply not exchangeable.

With labels based on position, the dependence of the location of one rectangle on that of the others is completely indecipherable. If, on the other hand, $\boldsymbol{x}_i$ is the position of the centre of the ith rectangle added, all conditional p.d.f.'s are known and this has been used in an efficient Monte Carlo method (Jodrey and Tory (1979)). Figure 2 shows the effect of order on the joint p.d.f. for the

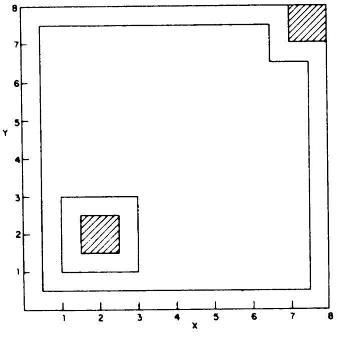

Figure 2
Disparate effects of two unit squares on their joint p.d.f. for placement in an 8×8 square.

placement of two unit squares in an 8×8 square. If the probability density for successful placement is uniform over the area still available for centres,

$$f(\boldsymbol{x}_1) = 1/49.$$

If the unit square in the upper right-hand corner is placed first, then

$$f(\boldsymbol{x}_2 \mid \boldsymbol{x}_1 = (7.5,7.5)) = 1/48.$$

If the other unit square is first,

$$f(\boldsymbol{x}_2 \mid \boldsymbol{x}_1 = (2.2)) = 1/45.$$

Thus

$$f(\boldsymbol{x}_1, \boldsymbol{x}_2) = f(\boldsymbol{x}_1) f(\boldsymbol{x}_2 \mid \boldsymbol{x}_1)$$

depends on the particular position which is occupied first.

We can imagine a different parking problem in which cars begin parking from the upper left-hand corner in a way which insures that no subsequent car can park directly above or directly to the left. Let each probability density be uniform over the region allowed for a given centre $\boldsymbol{x}_i$. Then x_i and y_i are independent in the top row, but not in subsequent rows. Figure 3 shows the first two rectangles of the top row and the region available (in this scheme) for the centre of the first rectangle of the second row. Since this region is not rectangular, the x, y-coordinates of the rectangle to be centred there are not independent. Though Weiner's Lemma 2 would hold for the top row, his Lemma 3, which applies to subsequent rows and requires the same independence, would fail. This packing could be simulated by a method similar to that used by Jodrey and Tory (1979) for the usual packing problem (Weiner's Model I); it might be

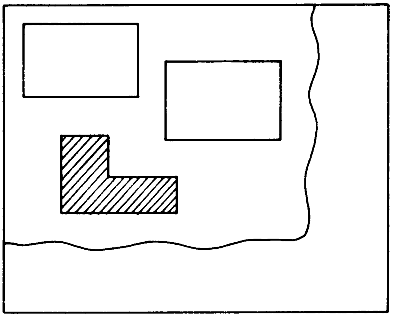

Figure 3

Interdependence of x, y-coordinates in a left-to-right, top-to-bottom sequential packing scheme. Hatched region shows area available for the centre of the first rectangle of the second row.

interesting to compare densities. To date, we have been unable to devise a non-trivial packing procedure for which these lemmas would both hold.

Empirical evidence from computer simulations (Blaisdell and Solomon (1970), Akeda and Hori (1976), Jodrey and Tory (1979)) indicates that the conjecture itself is false and hence that x, y-coordinates are dependent, but it is not easy to see the effect of this dependence. If X is a random variable representing the x-coordinate of the centre of a rectangle and Y represents the corresponding y-coordinate (in the time-sequence mode of labelling), then symmetry implies that $\mathrm{Cov}(X, Y) = 0$. Though the simulated densities differ from the conjectured by eleven standard deviations, the actual difference in densities (0.0032) is small (Jodrey and Tory (1979)). Figure 1 suggests that filling B before A and C creates a slight tendency for the new rectangles to line up with those already in place. We speculate that this causes the small increase in packing density over that conjectured.

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Yours sincerely,

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Dear Editor,

On Weiner's proof of the Palásti conjecture

In a recent paper Weiner (1978) claims to have proved the Palásti conjecture (see Palásti (1960)) respecting the asymptotic mean density of random sequential packing in the plane or the higher-dimensional space. This conjecture has previously been tested by the use of Monte Carlo simulation techniques. Earlier

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