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Non-integrability of the 1 : 1 : 2-resonance

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Abstract. A Hamiltonian system of n degrees of freedom, defined by the function F, with an equilibrium point at the origin, is called *formally integrable* if there exist formal power series $\hat{f}_1, \ldots, \hat{f}_n$, functionally independent, in involution, and such that the Taylor expansion $\hat{F}$ of F is a formal power series in the $\hat{f}_j$.

Take $n=3$, $\hat{F}=\sum_{k\geq 2} F^{(k)}$, $F^{(k)}$ homogeneous of degree k, $F^{(2)}>0$ and the eigenfrequencies in ratio 1 : 1 : 2. If $F^{(3)}$ avoids a certain hypersurface of 'symmetric' third order terms, then the F-system is *not* formally integrable. If $F^{(3)}$ is symmetric but $F^{(4)}$ is in a non-void open subset, then homoclinic intersection with Devaney spiralling occurs; the angle decays of order 1 when approaching the origin.

1. *Introduction*

Let F be a smooth real-valued function of $2n$ real variables $x=(q,p)$, $q\in\mathbb{R}^n$, $p\in\mathbb{R}^n$, defined in a neighbourhood of the origin. Assume that $dF(0)=0$, that is 0 is an equilibrium for the Hamiltonian system H_F in n degrees of freedom, defined by:

$$\frac{dq_j}{dt}=\frac{\partial F}{\partial p_j}(q,p), \qquad \frac{dp_j}{dt}=-\frac{\partial F}{\partial q_j}(q,p). \tag{1.1}$$

The system (1.1) is called integrable near 0 if there exist n functions $f_1, \ldots, f_n$, defined in an open neighbourhood U of 0, which Poisson commute with each other and are functionally independent, such that F can be written as a function of $f_1, \ldots, f_n$ (one often takes $F=f_n$). Here the Poisson bracket of f and g is defined as

$$\{f,g\}=H_f g=\sum_j \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}-\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}=-\{g,f\}. \tag{1.2}$$

The functional independence means that $df_1, \ldots, df_n$ are linearly independent on a dense open subset of U. This ensures that there $f_j=\text{const.}$ for all j define smooth n-dimensional submanifolds. Because $H_{f_i}f_j=0$, these level manifolds are invariant under the H_{f_i}-flows and because

$$[H_{f_i}, H_{f_k}]=H_{\{f_i,f_k\}}=0$$

these flows commute. On a compact connected component of a level manifold this leads to a transitive action of $\mathbb{R}^n$, making the level manifold diffeomorphic to a torus and the H_F-flow to a quasi-periodic motion on it; (cf. Abraham and Marsden [**2**, 5.2.23]). The compactness condition is in particular satisfied if the quadratic part $F^{(2)}$ is definite because then the level surfaces of F near 0 are small spheres invariant under the H_{f_i}-flows.

The system (1.1) will be called *formally integrable* if there exist formal power series $\hat{f}_1, \ldots, \hat{f}_n$ such that $\{\hat{f}_i, \hat{f}_j\} = 0$ for all i, j, the $\hat{f}_1, \ldots, \hat{f}_n$ are functionally independent and the Taylor expansion $\hat{F}$ of F is a formal power series in $\hat{f}_1, \ldots, \hat{f}_n$. This means that the system will behave asymptotically near the equilibrium like an integrable system up to each order.

Whereas formal integrability leads only to asymptotic information about the flow, the functional independence of $\hat{f}_1, \ldots, \hat{f}_n$ is a stronger statement than the functional independence of the smooth functions $f_1, \ldots, f_n$ of which $\hat{f}_1, \ldots, \hat{f}_n$ are the Taylor series, because formal functional independence already means functional independence of a finite part of the Taylor expansions.

Assume that $F^{(2)}$ is positive definite. Then, by a linear canonical transformation one can arrange that

$$F^{(2)} = \sum_{j=1}^{n} \omega_j (q_j^2 + p_j^2) \tag{1.3}$$

for some $\omega_1, \ldots, \omega_n > 0$, which are the frequencies of the harmonic oscillations in the various degrees of freedom which are exhibited by the linearized system. The Birkhoff normal form theorem (cf. [**2**, 5.6.8]) now states that for every k there is a canonical transformation of coordinates (of which only the Taylor expansion up to order k is needed) after which the Taylor expansion of F up to order k Poisson commutes with $F^{(2)}$. Letting $k \to \infty$ one obtains a formal canonical transformation $\hat{\Phi}$ after which the full Taylor expansion $\hat{F}$ of F Poisson commutes with $F^{(2)}$. $\hat{\Phi}$ can be taken to be the Taylor expansion of a smooth canonical transformation Φ in a neighbourhood of 0, but the condition $\{F^{(2)}, \hat{F}\} = 0$ cannot in general be strengthened to $\{F^{(2)}, F\} = 0$.

If the ω_j are linearly independent over $\mathbb{Q}$ then $\hat{F}$ in normal form Poisson commutes with the functions

$$\rho_j = q_j^2 + p_j^2, \qquad 1 \le j \le n, \tag{1.4}$$

so the system H_F is formally integrable. If the dimension of $\sum_j \mathbb{Q}\omega_j$ over $\mathbb{Q}$ is equal to $n-1$, then the orthogonal complement in $\mathbb{Z}^n$ of ω is generated by a non-zero vector $g \in \mathbb{Z}^n$; let $\nu^{(1)}, \ldots, \nu^{(n-1)}$ be a basis in $\mathbb{R}^n$ of the orthogonal complement of g. Then the functions

$$f_k = \sum_{j=1}^{n} \nu_j^{(k)} \rho_j, \qquad 1 \le k \le n-1, \tag{1.5}$$

Poisson commute with $\hat{F}$. Therefore, also in this case of *simple resonance*, the Hamiltonian system of F is formally integrable. So if one looks for Hamiltonian systems which are not formally integrable at an equilibrium point, then the order of the resonance = the codimension of $\sum_j \mathbb{Q}\omega_j$ over $\mathbb{Q}$, has to be at least 2.

The simplest case with $\omega \neq 0$ for which multiple resonance occurs is in 3 degrees of freedom, and then this condition means that multiplying the ω_j with a common factor (which can be arranged by a change in the time scale) one can take $\omega_j \in \mathbb{Z}$ for $j = 1, 2, 3$. For a survey of the asymptotic analysis of such systems for combinations with small ω_j, see [7].

In this paper we study the resonance 1 : 1 : 2. It will be shown that H_F, in normal form, is not formally integrable unless $F^{(3)}$ is degenerate in the sense that it enjoys an additional symmetry by a linear Hamiltonian circle action leaving $F^{(2)}$ invariant. The key to the proof is the observation that all solutions of the Hamiltonian system of $F^{(3)}$ on the hypersurface $F^{(3)}=0$ are periodic. If P denotes the period function then the complex continuations of the manifolds $P=$ constant turn out to be infinitely branched. This then excludes the existence of a nontrivial analytic integral on any open subset of the complex domain where this infinite branching occurs and whose intersection with the real domain is $H_{F^{(3)}}$-invariant. Although we do use complex analytic extensions, our method is different from the one employed in [**8**].

It is paradoxical that the formal non-integrability is proved by looking at a rather dull part of the dynamics, namely the periodic $H_{F^{(3)}}$-motion on $F^{(3)}=0$. From the point of view of dynamical systems it would be more interesting to exhibit some wild behaviour of the solutions as the cause of non-integrability. This will be done by looking on $F^{(3)}=0$ at the hyperbolic periodic solutions of $H_{F^{(2)}+F^{(3)}}$. Their stable (resp. unstable) manifolds Σ^+ (resp. Σ^-) coincide. (The solutions on them are the limits of sequences of periodic solutions on $F^{(3)}=0$ for which the periods tend to ∞.) Adding a generic fourth order term $F^{(4)}$ in normal form, a Melnikov function argument shows that Σ^+ and Σ^- then intersect at an angle which is of linear order in the distance to the equilibrium point. Now Devaney [**3**] has shown that if one has such homoclinic behaviour and if the flows on Σ^+ (resp. Σ^-) are spiralling towards (resp. from) the limit orbit, then wild behaviour, including 'Smale horseshoes', has to occur. This wild behaviour certainly excludes the existence of analytic integrals, but the phenomena themselves are much more interesting from the dynamic point of view.

Unfortunately, the $H_{F^{(3)}}$-flow on $\Sigma^{+,-}$ does not spiral, but is of node type, if $F^{(3)}$ is of the generic non-degenerate type. However, if $F^{(3)}$ is that one of the symmetric cases where the flows on $\Sigma^{+,-}$ are radial, then, for $F^{(4)}$ in a non-void open subset of 4th order terms, there are homoclinic spirals for the H_F-flow. It is another paradox in this paper that wild behaviour is shown by choosing $F^{(3)}$ in the exceptional integrable position, and only taking $F^{(4)}$ in a non-void open set. Both the spiralling coefficient and the angle between the stable and unstable manifold are of linear order in the distance to the equilibrium point. This is in contrast with the famous example of Hénon and Heiles [**4**]. There the numerical appearance of non-integrability only at a finite distance from the equilibrium corresponds with the vanishing of infinite order of non-integrability effects, as a consequence of the formal integrability of this 2 degrees of freedom system in 1 : 1 resonance.

The paper is organized as follows: In § 2 we discuss the normal form of $F^{(3)}$ which turns out to contain only one essential parameter $\mu \in [0, 1]$. The cases $\mu = 0$, resp. $\mu = 1$ are the degenerate symmetric cases. In § 3 it is proved that the $H_{F^{(3)}}$-solutions on $F^{(3)}=0$ are periodic, and that H_F is not formally integrable if $0<\mu<1$. The proof of the necessary properties of the period function is an exercise in complex function theory and is given in the appendix. § 4 contains the proof of the Devaney effect.

I would like to thank Richard Cushman for drawing my attention to Devaney's paper. Also I am grateful to Tonnie Springer for his suggestion to use the mapping (2.12), which immediately clarified the invariants (2.13), (2.14), found before by trial and error. In fact the whole paper grew out of a search for a third integral by brute force calculations, challenged by a statement in [**1**]. Not finding any up to order 6, I gave up and started looking at the $H_{F^{(3)}}$-flow itself. Of course this paper is still far from a complete analysis of the $1:1:2$-resonance. For instance, nothing has been said about the $H_{F^{(3)}}$-flow for $F^{(3)} \neq 0$.

2. *Normal form of the* $1:1:2$*-resonance*

In order to analyse Birkhoff normal forms it is convenient to introduce the complex coordinates

$$z_j = q_j + ip_j, \qquad \zeta_j = q_j - ip_j.$$

Writing $\hat{F}(q,p) = \mathcal{F}(z,\zeta)$, $\hat{G}(q,p) = \mathcal{G}(z,\zeta)$, we get

$$\{\hat{F}, \hat{G}\} = 2i \sum_j \frac{\partial \mathcal{F}}{\partial \zeta_j}\frac{\partial \mathcal{G}}{\partial z_j} - \frac{\partial \mathcal{F}}{\partial z_j}\frac{\partial \mathcal{G}}{\partial \zeta_j}. \tag{2.1}$$

Also $\hat{F}$ has real coefficients if and only if in the expansion

$$\mathcal{F}(z,\zeta) = \sum c_{m,\mu} z^m \zeta^\mu, \tag{2.2}$$

the relation

$$c_{\mu,m} = \overline{c_{m,\mu}} \qquad \text{for all } m, \mu \in \mathbb{N}^n \tag{2.3}$$

holds. Now the $F^{(2)}$ in (1.3) is equal to

$$F^{(2)} = \sum_j \omega_j z_j \zeta_j, \tag{2.4}$$

and the condition that $\{F^{(2)}, \hat{F}\} = 0$ translates into

$$\langle \omega, \mu - m \rangle = 0 \qquad \text{whenever } c_{m,\mu} \neq 0. \tag{2.5}$$

From now on we assume that $n = 3$ and $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 2$. Then the algebra of formal power series which Poisson-commute with

$$F^{(2)} = z_1\zeta_1 + z_2\zeta_2 + 2z_3\zeta_3 = q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2q_3^2 + 2p_3^2 \tag{2.6}$$

is generated by the following 11 functions (with a lot of relations):

$$\begin{gathered} \rho_j = z_j\zeta_j \quad (j = 1,2,3), \quad \sigma = z_1\zeta_2, \quad \bar{\sigma} = z_2\zeta_1, \\ \tau_1 = \tfrac{1}{2} z_3 \zeta_1^2, \quad \tau_2 = \tfrac{1}{2} z_3\zeta_2^2, \quad \tau_3 = z_3\zeta_1\zeta_2, \\ \bar{\tau}_1 = \tfrac{1}{2}\zeta_3 z_1^2, \quad \bar{\tau}_2 = \tfrac{1}{2}\zeta_3 z_2^2, \quad \bar{\tau}_3 = \zeta_3 z_1 z_2. \end{gathered} \tag{2.7}$$

The Birkhoff normal form of the third order term therefore is equal to

$$F^{(3)} = z_3(\tfrac{1}{2}a_1\zeta_1^2 + \tfrac{1}{2}a_2\zeta_2^2 + a_3\zeta_1\zeta_2) + \zeta_3(\tfrac{1}{2}\alpha_1 z_1^2 + \tfrac{1}{2}\alpha_2 z_2^2 + \alpha_3 z_1 z_2). \tag{2.8}$$

$F^{(3)}$ has real coefficients as a power series in q, p if and only if $a_j = \bar{\alpha}_j$, $(j = 1,2,3)$.

On the space of these $F^{(3)}$ one has the action of G = the group of the linear symplectic transformations which leave $F^{(2)}$ invariant. The Lie algebra $\mathfrak{g}$ of G is equal to the space of Hamiltonian vector fields defined by the quadratic functions Q which Poisson-commute with $F^{(2)}$, that is

$$Q = Q_1\rho_1 + Q_2\rho_2 + Q_3\rho_3 + r\sigma + \bar{r}\bar{\sigma}, \qquad Q_j \in \mathbb{R},\ r \in \mathbb{C}. \tag{2.9}$$

Because G commutes with the $H_{F^{(2)}}$-flow, and the action of the $H_{F^{(2)}}$-flow on the space of $F^{(3)}$'s in (2.8) is trivial (by definition), we may reduce to the group G_0 generated by the Q in (2.9) with $Q_3 = 0$. This is the group of those elements of G which leave the coordinates q_3, p_3 fixed. Viewing the standard inner product, resp. the symplectic form in the (q_1, q_2, p_1, p_2)-space as the real, resp. imaginary part of the standard Hermitian inner product in $\mathbb{C}^2 = (z_1, z_2)$-space, $z_1 = q_1 + ip_1$, $z_2 = q_2 + ip_2$, the action of G_0 on these variables is equal to $U(2)$, the unitary group acting on $\mathbb{C}^2$. Now read off the action of G_0 on $F^{(3)}$'s as in (2.8), by looking at its action on

$$\tfrac{1}{2}\alpha_1 z_1^2 + \tfrac{1}{2}\alpha_2 z_2^2 + \alpha_3 z_1 z_2.$$

Identifying this complex quadratic form on $\mathbb{C}^2$ with the complex symmetric (not hermitian!) matrix

$$S = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{pmatrix}, \tag{2.10}$$

the G_0-action on the $F^{(3)}$'s translates into

$$U, S \mapsto {}^tUSU, \qquad U \text{ unitary, } S \text{ symmetric.} \tag{2.11}$$

Here ${}^tU$ denotes the transposed matrix of U, the unitary condition means that $\overline{{}^tU} = {}^t\bar{U} = U^{-1}$. Now the mapping

$$S \mapsto \bar{S}S = S^*S, \tag{2.12}$$

from the space $\mathcal{S}$ of symmetric complex 2×2 matrices to the space $\mathcal{P}$ of non-negative self-adjoint matrices, intertwines the action of U on $\mathcal{S}$ with conjugation by U^{-1} on $\mathcal{P}$. Indeed,

$$\overline{{}^tUSU} \circ {}^tUSU = U^{-1}\bar{S}\bar{U}\,{}^tUSU = U^{-1}(\bar{S}S)U.$$

This leads immediately to the invariants

$$T = \text{Trace}\,(\bar{S}S) = |\alpha_1|^2 + 2|\alpha_3|^2 + |\alpha_2|^2, \tag{2.13}$$

$$D = \text{Det}\,(\bar{S}S) = (|\alpha_1|^2 + |\alpha_3|^2)(|\alpha_3|^2 + |\alpha_2|^2) - |\bar{\alpha}_1\alpha_3 + \bar{\alpha}_3\alpha_2|^2, \tag{2.14}$$

for the action of G on the space of $F^{(3)}$'s. Also we may conclude that, using the G-action, we can bring $F^{(3)}$ into the position where $\bar{S}S$ is a diagonal matrix, that is we can arrange that

$$\bar{\alpha}_1\alpha_3 + \bar{\alpha}_3\alpha_2 = 0. \tag{2.15}$$

In order not to destroy (2.15) our only further action is to take

$$U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\psi} \end{pmatrix}, \qquad \phi, \psi \in \mathbb{R} \bmod 2\pi. \tag{2.16}$$

This leads to $\alpha_1 \mapsto e^{i\phi}\alpha_1$, $\alpha_2 \mapsto e^{i\psi}\alpha_2$, $\alpha_3 \mapsto e^{i(\phi+\psi)}\alpha_3$. Using this action we can arrange that α_3 is real and then (2.15) implies that either $\alpha_3 = 0$ or $\alpha_3 \neq 0$ and $\bar{\alpha}_1 + \alpha_2 = 0$. In the first case we can, again using this action, arrange that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. In the second case we can arrange that α_1 is also real. It is an easy exercise to show that the matrix

$$S = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & -\alpha_1 \end{pmatrix}, \qquad \alpha_1, \alpha_3 \in \mathbb{R},$$

is equivalent to

$$\begin{pmatrix} \sqrt{(\alpha_1^2+\alpha_3^2)} & 0 \\ 0 & -\sqrt{\alpha_1^2+\alpha_3^2} \end{pmatrix}$$

by the action of a real orthogonal transformation U (for which ${}^tUSU = {}^tU^{-1}SU$), which can then be turned into $\sqrt{\alpha_1^2+\alpha_3^2}\cdot I$. So we have proved:

THEOREM 2.1. *By a linear symplectic transformation leaving $F^{(2)}$ invariant, the third order term of F in Birkhoff normal form can be brought into the form*

$$F^{(3)} = q_3\{\beta_1(q_1^2-p_1^2)+\beta_2(q_2^2-p_2^2)\}+2p_3\{\beta_1 q_1 p_1+\beta_2 q_2 p_2\}, \tag{2.17}$$

with $\beta_1 \ge \beta_2 \ge 0$. The relation with the coefficients in (2.8) *is given via the formulae* (2.13), (2.14) *by*

$$T = \beta_1^2+\beta_2^2, \qquad D = \beta_1^2\beta_2^2. \tag{2.18}$$

Each level manifold T = constant, D = constant is equal to exactly one G-orbit, which contains exactly one normal form of the type (2.17) *with $\beta_1 \ge \beta_2 \ge 0$. If $\beta_1 > \beta_2 > 0$ then the group of linear symplectic transformations leaving $F^{(2)}$ and $F^{(3)}$ invariant, modulo the $H_{F^{(2)}}$-action, is discrete. If $\beta_1 = \beta_2 = \beta > 0$ then we have the independent third integral*

$$G = \operatorname{Im} z_2\zeta_1 = q_1p_2 - p_1q_2, \tag{2.19}$$

and if $\beta_1 > 0$, $\beta_2 = 0$ then

$$G = q_2^2 + p_2^2 \tag{2.20}$$

is an independent third integral.

It can be observed that if $\beta_1 > 0$ then a rescaling $x = (1/\beta_1)\tilde{x}$ of the variables leaves $F^{(2)}$ invariant, but makes $\beta_1 = 1$. So

$$\mu = \beta_2/\beta_1 \in [0, 1] \tag{2.21}$$

is the only essential parameter in $F^{(3)}$, which originally had 56 coefficients.

3. *The $H_{F^{(3)}}$-flow on $F^{(3)} = 0$*

From now on we assume that $F^{(2)}$ (resp. $F^{(3)}$) is as in (2.6) (resp. (2.17)), $\beta_1 \ge \beta_2 > 0$. When analyzing the $H_{F^{(3)}}$-flow we keep in mind that it commutes with the $H_{F^{(2)}}$-flow, which is a circle action, and leaves the level surfaces of $F^{(2)}$ invariant. So we may view the $H_{F^{(3)}}$-flow as acting on the space of $H_{F^{(2)}}$-orbits on a hypersurface $F^{(2)}$ = constant, leading to a Hamiltonian action on the so-called reduced phase space which is 4-dimensional, cf. [**2**]. However, since the reduced phase space has singularities, in practice it is more convenient to work in the original coordinates and to remember the invariance of $F^{(2)}$ and the $H_{F^{(2)}}$-symmetry at the appropriate moment.

Because $F^{(3)}$ is linear in (q_3, p_3), it follows that for the $H_{F^{(3)}}$-flow:

$$\frac{dq_3}{dt}\cdot p_3 - \frac{dp_3}{dt}\cdot q_3 = F^{(3)} = \text{constant}. \tag{3.1}$$

That is, the curve $t \to (q_3(t), p_3(t))$ in $\mathbb{R}^2$ satisfies Kepler's law of sweeping out equal area in equal time. In particular, if $F^{(3)}=0$ then, in $\mathbb{R}^2\backslash\{0\}$, this curve moves on a straight line through the origin. On the other hand, if $q_3=p_3=0$, $dq_3/dt=dp_3/dt=0$, then we are dealing with the (q_3, p_3)-coordinates of a solution of the $H_{F^{(3)}}$-flow at a critical point of $F^{(3)}$, so in that case the curve remains at the origin in the (q_3, p_3)-plane. So for each $H_{F^{(3)}}$-solution on $F^{(3)}=0$ there is a straight line through the origin on which the (q_3, p_3)-coordinates remain for all time. Using the $H_{F^{(2)}}$-action we can turn this line into the position $q_3=0$, which we will assume we have done from now on. The $H_{F^{(3)}}$-system now reads:

$$\begin{aligned} \frac{dq_1}{dt} &= 2\beta_1 p_3 q_1 & \frac{dp_1}{dt} &= -2\beta_1 p_3 p_1 \\ \frac{dq_2}{dt} &= 2\beta_2 p_3 q_2 & \frac{dp_2}{dt} &= -2\beta_2 p_3 p_2, \end{aligned} \tag{3.2}$$

and

$$\frac{dp_3}{dt} = \beta_1(p_1^2 - q_1^2) + \beta_2(p_2^2 - q_2^2). \tag{3.3}$$

We also have to keep in mind that

$$q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2p_3^2 = F^{(2)} = \text{constant}, \tag{3.4}$$

and

$$\beta_1 q_1 p_1 + \beta_2 q_2 p_2 \equiv 0. \tag{3.5}$$

Writing $u(t) = 4\int_0^t p_3(s)\, ds$, (3.2) is equivalent to

$$q_j(t) = e^{\frac{1}{2}\beta_j u(t)} q_j(0), \quad p_j(t) = e^{-\frac{1}{2}\beta_j u(t)} p_j(0), \qquad (j = 1, 2),$$

so (3.3) now reads

$$\begin{aligned} \frac{1}{4}\frac{d^2u}{dt^2} &= \sum_{j=1}^{2} \beta_j (e^{-\beta_j u} p_j(0)^2 - e^{\beta_j u} q_j(0)^2) \\ &= -\frac{d}{du}\sum_{j=1}^{2} (e^{-\beta_j u} p_j(0)^2 + e^{\beta_j u} q_j(0)^2). \end{aligned} \tag{3.6}$$

This is a Newton equation for u with mass $=\frac{1}{4}$ and potential function

$$V(u) = \sum_{j=1}^{2} (e^{-\beta_j u} p_j(0)^2 + e^{\beta_j u} q_j(0)^2). \tag{3.7}$$

The total energy of this Newton system is equal to

$$E = \frac{1}{8}\left(\frac{du}{dt}\right)^2 + V(u) = 2p_3(0)^2 + \sum_{j=1}^{2} (p_j(0)^2 + q_j(0)^2) = F^{(2)}, \tag{3.8}$$

because E is a constant of motion and $u(0)=0$. The potential energy function is convex, and it is a well with infinitely high walls if

$$(p_1, p_2) \neq 0 \quad \text{and} \quad (q_1, q_2) \neq 0. \tag{3.9}$$

In this case $t \mapsto u(t)$ is periodic, which implies that the solution of (3.2), (3.3) is periodic as well. The period is given by

$$P = \sqrt{2} \int_{u_-}^{u_+} [E - V(u)]^{-\frac{1}{2}} \, du. \tag{3.10}$$

Here u_- and u_+ are the two zeros of $u \to E - V(u)$. The formula only holds on the part of the hypersurface $F^{(3)} = 0$ where (3.9) is valid. It is also assumed in (3.10) that $q_3 = 0$, but the period can be found at the other points of $F^{(3)} = 0$ by using the invariance of P under the $H_{F^{(2)}}$-action.

If we let the initial values (on the manifold determined by (3.4), (3.5)) converge to a point where $q_1(0) = q_2(0) = 0$, $(p_1(0), p_2(0)) \neq 0$ then the period runs to $+\infty$. The limiting solution satisfies $q_1(t) \equiv 0$, $q_2(t) \equiv 0$,

$$\lim_{t \to \pm\infty} (p_1(t), p_2(t)) = 0,$$

whereas $p_3(t)$ is strictly increasing,

$$\lim_{t \to -\infty} p_3(t) = -\sqrt{\tfrac{1}{2}E}, \qquad \lim_{t \to \infty} p_3(t) = \sqrt{\tfrac{1}{2}E}.$$

So the limiting solution is a saddle connection between the equilibrium points $q_1 = q_2 = p_1 = p_2 = 0$, $q_3 = 0$, $p_3 = \pm\sqrt{\frac{1}{2}E}$ of $H_{F^{(3)}}$. For $H_{F^{(2)}+F^{(3)}} = H_{F^{(2)}} + H_{F^{(3)}}$ these are the initial points on the transversal section $q_3 = 0$ of the stable = unstable manifold of the hyperbolic periodic solution in the (q_3, p_3)-plane.

The next argument will show that if G is a smooth function on an open set U such that $V = U \cap (F^{(3)} = 0)$ is $H_{F(3)}$-invariant, and $\{F^{(2)}, G\}$, $\{F^{(3)}, G\}$ and the total derivative $d\{F^{(3)}, G\}$ all vanish on V, then G is a function of $F^{(2)}$ and P on V.

The function P is real analytic and, in view of the above convergence to ∞, it is not constant on $F^{(2)} = E$, $F^{(3)} = 0$, for any $E > 0$. It follows that for any $E > 0$, $dP \neq 0$ on an open dense subset Ω of the manifold determined by (3.4), (3.5), (3.9). The assumptions for G ensure that, on V, H_G commutes with $H_{F^{(3)}}$, so $H_G P = 0$. It follows that G is invariant under the flows of $H_{F^{(2)}}$, $H_{F^{(3)}}$ and H_P, the latter regarded as the Hamiltonian flow on $F^{(3)} = 0$ of any smooth extension $\tilde{P}$ of P to an open neighbourhood of $F^{(3)} = 0$, the flow on $F^{(3)} = 0$ modulo $H_{F^{(3)}}$ is independent of the choice of the extension. Now on $\Omega \cap U$, dP is linearly independent of $dF^{(2)}$ and $dF^{(3)}$, so the $H_{F^{(2)}}$, $H_{F^{(3)}}$, H_P-flows together sweep out a 3-dimensional submanifold of the $6-3=3$-dimensional manifold $F^{(2)} = E$, $F^{(3)} = 0$, $P =$ constant. The conclusion is that, on $\Omega \cap U$, G is locally constant on $F^{(2)} = E$, $F^{(3)} = 0$, $P =$ constant. That is, G is a function of $F^{(2)}$, $F^{(3)}$ and P on each connected component of $\Omega \cap U \cap (F^{(3)} = 0)$.

Now suppose that G is analytic on U, so it has a complex analytic extension to some open neighbourhood $\tilde{U}$ of U in $\mathbb{C}^{2n}$. If G is not functionally dependent on $F^{(2)}$ on $F^{(3)} = 0$ then the manifolds $F^{(2)} = E$, $F^{(3)} = 0$, $G = c$ extend to closed complex analytic manifolds in $\tilde{U}$ for the generic values of E, c, which coincide with the complex analytic continuations of manifolds $F^{(2)} = E$, $F^{(3)} = 0$, $P =$ constant. If the latter analytic continuations exhibit infinite branching near a point $x \in \tilde{U}$, then we arrive at a contradiction, so G has to be a function of $F^{(2)}$ on $F^{(3)}=0$. Now treating $F^{(3)}$ as a coordinate we can locally near x write $G=G_0+G_1F^{(3)}$ where G_0 is a function of $F^{(2)}$ and G_1 is analytic. Because $G_1=(G-G_0)/F^{(3)}$ Poisson-commutes with $F^{(2)}$ and $F^{(3)}$, the same argument gives that G_1 is a function of $F^{(2)}$ on $F^{(3)}=0$. Continuing this procedure we obtain that G is a power series in $F^{(3)}$ with coefficients which are functions of $F^{(2)}$, that is G is a function of $F^{(2)}$ and $F^{(3)}$ near x. By analytic continuation this conclusion is then true in the connected component of x in U.

We shall prove that if $\beta_1>\beta_2>0$, then the analytic continuation of the manifolds $F^{(2)}=E>0$, $F^{(3)}=0$, $P=\text{constant}$ have infinite branching near the points $q_1=q_2=p_1=p_2=0$. This then leads to the following:

THEOREM 3.1. *Let $\beta_1>\beta_2>0$. Let G be an analytic function on a connected neighbourhood of the manifold $q_1=q_2=q_3=0$, $p_1^2+p_2^2+2p_3^2=E>0$ or of the manifold $p_1=p_2=q_3=0$, $q_1^2+q_2^2+2p_3^2=E>0$. If G Poisson-commutes with $F^{(2)}$ and $F^{(3)}$, then G is functionally dependent on $F^{(2)}$ and $F^{(3)}$. In particular, H_F is not formally integrable if $F=F^{(2)}+F^{(3)}+F^{(4)}+\cdots$.*

For the last conclusion, let $\hat{G}$ be a formal power series which Poisson commutes with $\hat{F}$. Assume by induction on k that $\hat{G}=\sum_{l\geq k}G^{(l)}+$a function of $F^{(2)}$ and $\hat{F}$. Regarding the homogeneous terms of order k (resp. $k+1$) in $0=\{\hat{F},\hat{G}\}$, one obtains the equations $\{F^{(2)},G^{(k)}\}=0$ (resp. $\{F^{(2)},G^{(k+1)}\}+\{F^{(3)},G^{(k)}\}=0$). Using the fact that $F^{(2)}$ Poisson commutes with $F^{(3)}$ it follows that

$$(\operatorname{ad}F^{(2)})^2(G^{(k+1)})=0.$$

Because $\operatorname{ad}F^{(2)}:G\mapsto\{F^{(2)},G\}$ is a semi-simple linear transformation of the space of polynomials of degree $k+1$, the conclusion is that

$$(\operatorname{ad}F^{(2)})(G^{k+1})=0,$$

and therefore $\{F^{(3)},G^{(k)}\}=0$ as well. Now the first part of the theorem yields that $G^{(k)}$ is a function of $F^{(2)}$ and $F^{(3)}$ on a non-void open subset. By the implicit function theorem, $G^{(k)}=\Gamma(F^{(2)},F^{(3)})$ for an analytic function Γ on a non-void open subset of regular values for the mapping $\Delta=(F^{(2)},F^{(3)}):\mathbb{C}^6\to\mathbb{C}^2$. Γ has a unique analytic continuation to the set of all regular values of Δ because the fibres of Δ are connected. Now Δ is surjective and for every converging sequence w_j in $\mathbb{C}^2$ one can find a converging sequence z_j in $\mathbb{C}^6$ such that $\Delta(z_j)=w_j$. Therefore Γ extends to an entire analytic function: $\mathbb{C}^2\to\mathbb{C}$. Comparing homogeneous terms in the Taylor expansion of $G^{(k)}=\Gamma(F^{(2)},F^{(3)})$ at the origin in $\mathbb{C}^6$, one obtains

$$\Gamma(F^{(2)},F^{(3)})=\sum_{2i+3j=k}c_{ij}(F^{(2)})^i(F^{(3)})^j,$$

so Γ is actually a polynomial. Replacing $F^{(3)}$ by $\hat{F}-F^{(2)}$ modulo terms of order ≥ 4, one obtains that $G^{(k)}$ is a function of $F^{(2)}$ and $\hat{F}$, modulo terms of order $\geq k+1$.

In order to investigate the manifolds $P=\text{constant}$ we rewrite

$$P=\sqrt{\frac{2}{E}}\frac{1}{\beta_1}\cdot\mathcal{P}(\varepsilon,\eta),\tag{3.11}$$

with

$$\mathcal{P}(\varepsilon, \eta) = \int_{x_-}^{x_+} [1-(e^{-x}+e^{-\mu x}\varepsilon+e^x\mu^2\varepsilon\eta+e^{\mu x}\eta)]^{-\frac{1}{2}}\,dx, \tag{3.12}$$

$$\mu = \frac{\beta_2}{\beta_1}, \quad \varepsilon = \frac{p_2^2}{E}\left(\frac{p_1^2}{E}\right)^{-\mu}, \quad \eta = \frac{q_2^2}{E}\left(\frac{p_1^2}{E}\right)^{\mu}. \tag{3.13}$$

With this notation, (3.5) is equivalent to

$$\mu^2\varepsilon\eta = q_1^2 p_1^2/E^2. \tag{3.14}$$

First consider the restriction of P to the submanifold of $F^{(2)} = E$, $F^{(3)} = 0$, where $q_1 = 0$, $p_2 = 0$. This corresponds in (3.12) with $\varepsilon = 0$ and the conclusion is that P is a function of $q_2^2 \cdot p_1^{2\mu}$. So the manifolds P = constant exhibit arbitrarily often branching, arbitrarily close to $q_1 = q_2 = p_1 = p_2 = 0$, if μ is irrational.

We may therefore assume from now on that μ is rational, say:

(3.15) $\mu = k/l$ with k, l positive integers, relatively prime, $k > l$.

In the appendix we shall prove the following properties of the function $\mathcal{P}(\varepsilon, \eta)$ in (3.12), with μ as in (3.15):

LEMMA 3.2. (i) *$\mathcal{P}$ has an analytic continuation to $\{(\varepsilon, \eta) \in \mathbb{C}^2;\ \varepsilon\eta \cdot \Delta(\varepsilon, \eta) \neq 0\}$, where Δ is a polynomial in 2 variables. Near $(\varepsilon, \eta) = (0, 0)$ the zero set of Δ has k branches, 'ε is a continuous function of η', and*

$$\lim_{(\varepsilon,\eta)\to(0,0),\Delta(\varepsilon,\eta)=0} \left(\left(1-\frac{k}{l}\right)\eta\right)^{l-k} \Big/ \left(-\frac{k}{l}\varepsilon\right)^{k} = 1. \tag{3.16}$$

(ii) *There is a neighbourhood U of $]0, \infty[\times\{0\}$ in $\mathbb{C}^2$ such that on $\{(\varepsilon, \eta) \in U;\ \eta \neq 0\}$ we can write*

$$\mathcal{P}(\varepsilon, \eta) = \phi(\varepsilon, \eta) \cdot \log \eta + \psi(\varepsilon, \eta^{1/l}). \tag{3.17}$$

Here ϕ can be extended to a complex analytic function on a full neighbourhood of $(0, 0)$ in $\mathbb{C}^2$ and ψ is a complex analytic function on $\{(\varepsilon, \zeta) \in \mathbb{C}^2;\ (\varepsilon, \zeta^l) \in U\}$. Furthermore,

$$\phi(\varepsilon, 0) \equiv -1, \qquad \frac{\partial^2\phi}{\partial\eta\,\partial\varepsilon}(0, 0) = -\tfrac{3}{4} + \tfrac{3}{4}(k^2/l^2),$$

and if $k = 1$: (3.18)

$$\phi(0, \eta) = -1 - \prod_{j=0}^{l} (\tfrac{1}{2}+j)\frac{1}{l!}\eta^l + O(\eta^{2l}) \qquad \text{if } \eta \to 0.$$

(iii) *Along every branch of $\Delta(\varepsilon, \eta) = 0$ the analytic continuation of $\mathcal{P}$ develops logarithmic singularities with non-zero coefficients.*

We shall now prove that lemma 3.2 implies that the analytic continuation of the tangent bundle of the curves $\mathcal{P}$ = constant has infinite branching over η, for η near 0. This then completes the arguments preceding theorem 3.1.

The tangent at (ε, η) of the complex curve $\mathscr{P} = $ constant is given by

$$\frac{\frac{\partial \mathscr{P}}{\partial \varepsilon}(\varepsilon, \eta)}{\frac{\partial \mathscr{P}}{\partial \eta}(\varepsilon, \eta)} = \frac{\frac{\partial \phi}{\partial \varepsilon}(\varepsilon, \eta) \cdot \log \eta + \frac{\partial \psi}{\partial \varepsilon}(\varepsilon, \eta^{1/l})}{\frac{\partial \phi}{\partial \eta}(\varepsilon, \eta) \cdot \log \eta + \phi(\varepsilon, \eta) \cdot \eta^{-1} + \frac{\partial}{\partial \eta}\psi(\varepsilon, \eta^{1/l})}. \tag{3.19}$$

Winding around 0 with η arbitrarily many times, one sees that (3.19) can only be finitely valued if

$$\frac{\partial \mathscr{P}}{\partial \varepsilon}(\varepsilon, \eta) \Big/ \frac{\partial \mathscr{P}}{\partial \eta}(\varepsilon, \eta) = \frac{\partial \phi}{\partial \varepsilon}(\varepsilon, \eta) \Big/ \frac{\partial \phi}{\partial \eta}(\varepsilon, \eta), \tag{3.20}$$

that is, the level curves of $\mathscr{P}$ coincide with the level curves of ϕ. This property continues if we replace $\mathscr{P}$ by any analytic continuation $\tilde{\mathscr{P}}$. If $\tilde{\mathscr{P}}$ has a logarithmic singularity with non-zero coefficient along a smooth complex curve B, then the curves $\tilde{\mathscr{P}} = $ const. for large values of the constant approach B. It follows that ϕ is constant along B. Therefore ϕ is constant along $\eta = 0$, where we already knew it to be equal to -1, and on every branch through $(0, 0)$ of $\Delta(\varepsilon, \eta) = 0$.

But (3.18) implies that $(0, 0)$ is a non-degenerate critical point of ϕ, so near $(\varepsilon, \eta) = (0, 0)$ the set $\phi(\varepsilon, \eta) = -1$ is the union of the set $\eta = 0$ and the set $\varepsilon = \chi(\eta)$ for some analytic function χ with $\chi(0) = 0$. ϕ being constant on the zero set of Δ, we read off from lemma 3.2 (i) that k has to be equal to 1. Furthermore, if $k = 1$ then (3.18) and (3.16) imply that

$$\lim_{\eta \to 0} \chi(\eta)/\eta^{l-1} = -(2l+1)!/2^{2l+1} l! l! \tfrac{3}{4}(1 - (1/l^2)). \tag{3.21}$$

Writing $c_l = l \cdot (1 - (1/l))^{l-1}$ for the limit of $-\varepsilon/\eta^{l-1}$ in (3.16) for $k = 1$, and $-d_l$ for the right hand side in (3.21), it follows that $c_l = d_l$.

However, it turns out that $c_{l+1}/d_{l+1} > c_l/d_l$ and because $c_4/d_4 = \frac{27}{28} < 1$, $c_5/d_5 = 262\,144/240\,625 > 1$, $c_l \neq d_l$ for all l. So for each k, l we arrive at a contradiction with the assumption that (3.19) is finitely branched over η.

Remark. The comparison of c_l with d_l would not be needed if $\mathscr{P}$ also developed logarithmic singularities around $\varepsilon = 0$. However, it turns out that $\mathscr{P}$ is finitely branched around $\varepsilon = 0$.

4. *Homoclinic spirals*

The equations (3.2) show that the limit behaviour as $t \to \pm\infty$ on the stable and unstable manifolds $q_1 = q_2 = q_3 = 0$ (resp. $p_1 = p_2 = q_3 = 0$) is of node type if $\beta_1 > \beta_2 > 0$. Therefore small perturbations can only lead to spiralling homoclinic orbits if $\beta_1 = \beta_2$, which we assume from now on. Rescaling we may even take $\beta_1 = \beta_2 = 1$. A straightforward calculation shows that on $q_1 = q_2 = q_3 = 0$ the solutions with $p_3(0) = 0$ are given by:

$$p_1(t) = \frac{p_1(0)}{\cosh \sqrt{2} rt}, \quad p_2(t) = \frac{p_2(0)}{\cosh \sqrt{2} rt}, \quad p_3(t) = \frac{r}{\sqrt{3}} \tanh \sqrt{2} rt; \tag{4.1}$$

here $r = \sqrt{p_1(0)^2 + p_2(0)^2} = \sqrt{E}$, if E is the constant value of $F^{(2)}$. The system $H_{F^{(3)}}$

with $\beta_1=\beta_2$ has the third integral

$$G=-q_1p_2+p_1q_2, \qquad H_G=\left(q_2\frac{\partial}{\partial q_1}-q_1\frac{\partial}{\partial q_2}\right)+\left(p_2\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial p_2}\right). \tag{4.2}$$

So the Hamiltonian flow defined by the 4th order term $F^{(2)}G$ also commutes with the $H_{F^{(3)}}$-flow. Furthermore, $H_{F^{(2)}G}=F^{(2)}H_G$ on $q_1=q_2=q_3=0$. So the solutions of $H_{F^{(3)}+cF^{(2)}G}$ on $q_1=q_2=q_3=0$, with $p_3(0)=0$, are given by:

$$\begin{pmatrix}p_1(t)\\p_2(t)\end{pmatrix}=\frac{1}{\cosh\sqrt{2E}\,t}\begin{pmatrix}\cos cEt & \sin cEt\\-\sin cEt & \cos cEt\end{pmatrix}\begin{pmatrix}p_1(0)\\p_2(0)\end{pmatrix},$$
$$p_3(t)=(E/\sqrt{3})\tanh\sqrt{2E}\,t. \tag{4.3}$$

This motion is obviously spiralling from and towards $(p_1,p_2)=(0,0)$ if $c\neq 0$. Now consider the fourth order term

$$K=(q_1q_2+p_1p_2)^2. \tag{4.4}$$

If $x(t)$ is a homoclinic solution for $H_{F^{(3)}+cF^{(2)}G+dK}$, then the integral of

$$(H_{F^{(3)}+cF^{(2)}G+dK}G)(x(t))=d\cdot\{K,G\}(x(t)) \tag{4.5}$$

over the whole t-axis is equal to

$$G(x(\infty))-G(x(-\infty))=0. \tag{4.6}$$

Seeking, for small d, the homoclinic solution near the solution $x_0(t)$ of $H_{F^{(3)}+cF^{(2)}G}$ described in (4.3), one is therefore led to look at the zeros of the function introduced by Melnikov [**5**]:

$$\int_{-\infty}^{\infty}\{K,G\}(x_0(t))\,dt=\int_{-\infty}^{\infty}2p_1(t)p_2(t)(p_2(t)^2-p_1(t)^2)\,dt$$
$$=-\frac{E^{3/2}\cdot\pi}{24\sqrt{2}}\cdot\frac{\omega}{\sinh(\pi/2)\omega}\cdot(1+\tfrac{1}{4}\omega^2)\cdot\sin 4\theta. \tag{4.7}$$

Here $E=p_1(0)^2+p_2(0)^2$, $p_1(0)=\sqrt{E}\cos\theta$, $p_2(0)=\sqrt{E}\sin\theta$, $\omega=4c\sqrt{E/2}$.

Since (4.7) has simple zeros as a function of θ at $\omega=2\pi k/8$, $k=0,1,\ldots,7$, one can apply an implicit function argument to prove that nearby (at distance $O(d)$) the point $(0,0,0,\sqrt{E}\cos\theta,\sqrt{E}\sin\theta,0)$ in (q,p)-space, there exists, for d sufficiently small, a homoclinic solution of $H_{F^{(3)}+cF^{(2)}G+dK}$, and that along this homoclinic solution the stable and unstable manifold of the hyperbolic solution in the (q_3,p_3)-plane intersect at an angle of order d. By continuity this homoclinic solution will still spiral if $c\neq 0$ and d is sufficiently small. Also note that for $c=0$, H_K vanishes on the star $\sin 4\theta=0$, so in this case (4.3) describes the homoclinic solution exactly. Also note that if we let $\sqrt{E}=$ the distance to the equilibrium point go to zero then both the spiralling ratio and the angle between the stable and unstable manifold will asymptotically be linear in $\sqrt{E}$ with a non-zero coefficient, because the order of $H_{F^{(4)}}$ is $\sqrt{E}$ times the order of $H_{F^{(3)}}$. This also ensures that higher order terms in the Taylor expansion of F cannot destroy the homoclinic spirals. Summarizing, we have proved:

THEOREM 4.1. *Assume that* $\{F^{(2)},F\}=0$. *There is a non-void open subset* Ω *of homogeneous polynomials of degree* 4 *(in normal form) such that if the Taylor expansion* *of F at the equilibrium-point up to degree 4 is of the form* $F^{(2)}+F^{(3)}+F^{(4)}$, *with* $F^{(2)}$ *as in* (2.6), $F^{(3)}$ *as in* (2.17) *with* $\beta_1=\beta_2>0$ *and* $F^{(4)}\in\Omega$, *then the stable and unstable manifolds of the hyperbolic solution in the* (q_3, p_3)*-plane, for the* H_F*-flow close to the equilibrium, intersect along a homoclinic spiral, if intersected with the transversal hypersurface* $q_3=0$. *Both the spiralling coefficient and the angle of intersection are of linear order in the distance to the equilibrium point, with a non-zero coefficient.*

The assumption that F has $F^{(2)}$ as an exact integral allows the reduction to the flow on the space of $H_{F^{(2)}}$-orbits on the manifold $F^{(2)}=\text{constant}$. The reduced flow is a Hamiltonian system with 2 degrees of freedom with a hyperbolic equilibrium at the (q_3, p_3)-plane. It is to this system that one can apply the results of Devaney [3] and conclude the existence of horseshoes and corresponding wild behaviour. For the original system itself the set where the wild behaviour occurs is $H_{F^{(2)}}$-invariant and therefore of one dimension higher.

If $F^{(2)}$ is only a formal integral for the H_F-system, then it is expected that the 3-dimensional stable and unstable manifolds in the 5-dimensional space $F=\text{constant}$ intersect only along 1-dimensional orbits. The intersection of the stable and unstable manifold then has 2 angles, one decreasing linearly but the other vanishing of infinite order as one approaches the origin. Also, since one cannot reduce to a 2 degrees of freedom system, the results of Devaney can no longer be applied by mere citation, and it is an interesting question what aspects of the wild behaviour will survive.

As a final remark, a complete description should tell what happens if we take the coefficients of $F^{(2)}+F^{(3)}+F^{(4)}$ in a full transversal slice to the action of the group of local canonical transformations of coordinates. However this may be asking for too much to be feasible.

Appendix: *the period function if* $\mu=k/l$

In (3.12) the singularities of the integrand at the endpoints x_-, x_+ are of order $|x-x_\pm|^{-\frac{1}{2}}$ and therefore absolutely integrable. We begin with the substituting of variables $e^{x/l}=z$, so that

$$\mathcal{P}(\varepsilon,\eta)=l\int_{z_-}^{z_+}\left[1-\left(z^{-l}+z^{-k}\varepsilon+z^{l}\frac{k^2}{l^2}\varepsilon\eta+z^{k}\eta\right)\right]^{-\frac{1}{2}}\frac{dz}{z} \tag{A.1}$$

where z_-, z_+ are the two consecutive positive real zeros of the rational function under the square root sign. Consider the closed curve γ in the complex plane which travels along the real axis from z_-+0 to z_+-0, then around z_+ in the positive direction, where the integrand picks up a minus sign, then back along the real axis, and finally around z_- in the positive direction, so that the integrand is single-valued along γ. So

$$\mathcal{P}(\varepsilon,\eta)=\tfrac{1}{2}\int_{\gamma}\left[1-\left(z^{-l}+z^{-k}\varepsilon+z^{l}\frac{k^2}{l^2}\varepsilon\eta+z^{k}\eta\right)\right]^{-\frac{1}{2}}\frac{dz}{z} \tag{A.2}$$

and the right hand side does not change under a homotopy of γ in the complement in $\mathbb{C}$ of 0 and the zeros of the polynomial

$$\pi(\varepsilon,\eta,z)=z^{l}-(1+z^{l-k}\varepsilon+z^{2l}(k^2/l^2)\varepsilon\eta+z^{l+k}\eta); \tag{A.3}$$

(the restriction about the origin can be removed if l is even). If we now let ε, η

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move in $\mathbb{C}^2$ then $z_- = z_-(\varepsilon, \eta)$, $z_+ = z_+(\varepsilon, \eta)$ move along and we can homotope γ with it as long as neither $z_-(\varepsilon, \eta)$ nor $z_+(\varepsilon, \eta)$ coalesce with a zero of $\pi(\varepsilon, \eta, \cdot)$ other than z_-, z_+ (coalescence with 0 cannot occur). If we write $\Delta(\varepsilon, \eta)$ for the discriminant of $\pi(\varepsilon, \eta, \cdot)$ from which we delete common factors ε (resp. η) of all terms, then this argument already shows that $\mathcal{P}$ has a multi-valued analytic extension to the set of $(\varepsilon, \eta) \in \mathbb{C}^2$ such that $\varepsilon\eta \cdot \Delta(\varepsilon, \eta) \neq 0$. More precisely, $\mathcal{P}$ is a function of the class introduced by Nilsson [**6**]: the local branches span a finite-dimensional space and the singularities of $\mathcal{P}$ when approaching the zero set of $\varepsilon\eta\Delta(\varepsilon, \eta)$ are 'of tempered growth'.

Let us now investigate what happens to the roots of $\pi(\varepsilon, \eta, \cdot)$ if $(\varepsilon, \eta) \in \mathbb{C}^2$ is sufficiently close to $(0, 0)$. In this case we have l roots of the form

$$z_{-,m} = e^{2\pi i m/l} + O(|\varepsilon| + |\eta|), \qquad m = 0, \ldots, l-1. \tag{A.4}$$

If $|z| = \theta \cdot \frac{1}{2} \min \{|(k^2/l^2)\varepsilon\eta|^{-1/l}, |\eta|^{-1/k}\} := R$ with $0 < \theta < 1$, then

$$|z^l - (z^{2l}(k^2/l^2)\varepsilon\eta + z^{l+k}\eta)| \geq |z|^l \cdot (1-\theta),$$

and

$$|1 + z^{l-k}\varepsilon| \leq 1 + |z|^{l-k}\varepsilon.$$

So applying Rouché's theorem, the number of roots of $\pi(\varepsilon, \eta, \cdot)$ within the circle of radius R is equal to the number of roots of

$$z^l(1 - (z^l(k^2/l^2)\varepsilon\eta + z^k\eta))$$

in that region, which is qual to l. This already happens if we keep ε bounded and then take η sufficiently small. So the roots of π not given by (A.4) have absolute value at least of order $\min \{|\varepsilon\eta|^{-1/l}, |\eta|^{-1/k}\}$. This means that the roots $z_{-,m}$ are all simple and that coalescence of roots can only occur between the other, large roots. For $\varepsilon \neq 0$, $\eta \neq 0$, $\Delta(\varepsilon, \eta) = 0$ if and only if there exist $z \in \mathbb{C}$ such that

$$\pi(\varepsilon, \eta, z) = 0, \qquad \frac{\partial \pi}{\partial z}(\varepsilon, \eta, z) = 0. \tag{A.5}$$

For given small (ε, η), the equations (A.5) are equivalent to the equations

$$z^k \cdot (1 - (k/l))\eta = 1 - r_1, \qquad -z^l \cdot (1-(k/l))(k/l)\varepsilon\eta = 1 - r_2, \tag{A.6}$$

where $r_1 = 2(z^{-l} + z^{-k}\varepsilon) + (1-(k/l))z^{1-k} \cdot \varepsilon$ and $r_2 = ((l/k)+1)(z^{-l} + z^{-k}\varepsilon) + ((l/k)-1)z^{1-k} \cdot \varepsilon$ are asymptotically small.

These equations have at most one solution, and they have one if and only if

$$((1-(k/l))\eta)^{l-k}/(-(k/l)\varepsilon)^k = (1-r_1)^l/(1-r_2)^k. \tag{A.7}$$

This proves lemma 3.2(i).

We now investigate what happens if ε remains close to a positive real value and $\eta \to 0$. In this case the large roots of $\pi(\varepsilon, \eta, \cdot)$ are asymptotically given by the 1st roots of $1/((k^2/l^2)\varepsilon\eta)$. Letting η (starting at a positive real value) run through a small circle around 0 in the positive direction, the large roots turn over an angle asymptotically equal to $-2\pi/l$. Homotoping γ along, $\mathcal{P}(\varepsilon, \eta)$ has changed to (A.2) with γ replaced by a closed curve $\tilde{\gamma}$ which winds once around z_-, once around a large root z_+ which is approximately equal to $e^{-2\pi i/l}z_+$ and not around any of the other roots or 0.

Repeating the procedure we end up with a branch of $\mathcal{P}(\varepsilon, \eta)$ which is equal to (A.2) as above but with z_+ replaced by any of the large roots of $\pi(\varepsilon, \eta, \cdot)$.

In particular, if we substitute $\eta = \zeta^l$ and let turn ζ around the origin once, $\mathcal{P}(\varepsilon, \eta)$ has changed to $\tilde{\mathcal{P}}(\varepsilon, \eta) = $ (A.2) with γ replaced by the curve $\tilde{\gamma}$ which is equal to:

(i) From $z_- - 0$ to $z_- + 0$ along a small semicircle around z_- in the lower half plane, in the positive direction.

(ii) The circle δ around the origin, from $z_- + 0$ back to $z_- + 0$ in the negative direction, with radius just big enough that all bounded roots of $\pi(\varepsilon, \eta, \cdot)$ are enclosed. The integrand is single valued along δ.

(iii) From $z_- + 0$ to $z_+ - 0$ along the real axis.

(iv) Once around z_+ in the positive direction, so that the integrand has picked up a minus sign.

(v) From $z_+ - 0$ to $z_- + 0$ along the real axis.

(vi) δ in the positive direction.

(vii) From $z_- + 0$ to $z_- - 0$ along a small semi-circle around z_- in the upper half plane, in the positive direction.

The integral over (i) + (iii) + (iv) + (v) + (vii) is equal to the integral over γ, the one over (ii) + (vi) is equal to -2 times the integral over δ. Therefore

$$\tilde{\mathcal{P}}(\varepsilon, \eta) = \mathcal{P}(\varepsilon, \eta) + 2\pi i l \phi(\varepsilon, \eta),$$

with

$$\phi(\varepsilon, \eta) = \frac{-1}{2\pi i}\int_\delta \left[1 - \left(z^{-l} + z^{-k}\varepsilon + z^l \frac{k^2}{l^2}\varepsilon\eta + z^k\eta\right)\right]^{-\frac{1}{2}} \frac{dz}{z}. \tag{A.8}$$

It follows that

$$(\varepsilon, \zeta) \to \mathcal{P}(\varepsilon, \zeta^l) - \phi(\varepsilon, \zeta^l) \cdot \log \zeta \underset{\text{def}}{=} \psi(\theta, \zeta) \tag{A.9}$$

is single valued around $\zeta = 0$. Since a straight forward estimate yields that $\mathcal{P}$ is bounded by a constant times $\log(1/|\zeta|)$, the conclusion is that ψ is analytic at $\zeta = 0$.

Furthermore, putting $\eta = 0$ in (A.8) and using the substitution of variables $z = 1/y$ one obtains that

$$\phi(\varepsilon, 0) = -[1 - (y^l + y^k\varepsilon)]^{-\frac{1}{2}}|_{y=0} = -1. \tag{A.10}$$

A similar calculation yields

$$\frac{\partial^2 \phi}{\partial\varepsilon\, \partial\eta}(0, 0) = -\tfrac{3}{4} + \tfrac{3}{4}(k^2/l^2), \tag{A.11}$$

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial\eta}\right)^j (0,0) = 0 \qquad \text{if } 1 \le j \le l, \\ &\left(\frac{\partial}{\partial\eta}\right)^l (0,0) = -\tfrac{1}{2}\cdots(\tfrac{1}{2} + l) \qquad \text{if } k = 1, \end{aligned}\right. \tag{A.12}$$

which completes the proof of lemma 3.2(ii).

If we approach a branch B of $\Delta = 0$ then two large roots of $\pi(\varepsilon, \eta, \cdot)$ coalesce. The previous description of the behaviour of the large roots as η runs around 0 shows that $\mathcal{P}$ has an analytic continuation to an integral $\tilde{\mathcal{P}}$ as in (A.2) with γ replaced by a closed curve $\tilde{\gamma}$ which winds around one of the two coalescing roots. Going with (ε, η) around B means that these roots make a full turn around their common midpoint. Homotoping $\tilde{\gamma}$ along we get a new curve $\tilde{\gamma}$ which is equal to $\tilde{\gamma}$ plus ± 2 times a loop λ around both roots. If the roots coalesce at $z = p$ then the value of the integral is equal to $\pm 2\pi i \frac{l}{2} \cdot p^{\frac{l}{2}-1}((\partial^2/\partial z^2)\pi(\varepsilon, \eta, p))^{-\frac{1}{2}}$. This proves lemma 3.2(iii).

Added in proof. The theorem that H_F is not formally integrable if $\beta_1 > \beta_2 > 0$ has been proved in a much more algebraic fashion by A. Stimemann in his Diplomarbeit at the ETH Zurich, 1984. I also found the following mistakes in the present paper: In (3.15), $k > 1$ should be $k < 1$.
A more serious mistake is that the value for $\partial^2\varphi/\partial\varepsilon\, \partial\eta(0, 0)$ in (3.18) and (A.11) is

$$-\tfrac{3}{4} - \tfrac{3}{4}\frac{k^2}{l^2} \qquad \textit{instead of} \qquad -\tfrac{3}{4} + \tfrac{3}{4}\frac{k^2}{l^2}.$$

As a result the comparison of the coefficients c_l, d_l at the end of §3 becomes $c_2 = d_2$, $c_l > d_l$ for $l > 2$. So the conclusions of theorem 3.1 remain valid for $\beta_1 > \beta_2 > 0$, $\beta_1 \neq 2\beta_2$.

On the other hand, for $\beta_1 = 2\beta_2$, the function

$$G = (p_1 q_2 - q_1 p_2)^2 \cdot (p_2^2 + q_2^2) + 2[\tfrac{1}{2} q_3 (q_2^2 - p_2^2) + p_3 q_2 p_2]^2$$

satisfies $\{F^{(2)}, G\} = 0$ and $\{F^{(3)}, G\} = 0$, making the $H_{F^{(2)}+F^{(3)}}$-flow completely integrable. Apparently I missed this case in my search for third integrals of order 6. In this case the complete integrability is not explained by an obvious symmetry.

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