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UNIQUENESS OF PREDUALS FOR SPACES OF CONTINUOUS VECTOR FUNCTIONS

BY

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ABSTRACT. A. Grothendieck has shown that if the space $C(X)$ is a Banach dual then X is hyperstonean; moreover, the predual of $C(X)$ is strongly unique. In this article we give a vector analogue of Grothendieck's result. We show that if E^* is a reflexive Banach space and $C(X,(E^*,\sigma^*))$ denotes the space of continuous functions on X to E^* when E^* is provided with its weak* (= weak) topology then the full content of Grothendieck's theorem for $C(X)$ can be established for $C(X,(E^*,\sigma^*))$. This improves a result previously obtained for the case in which E^* is Hilbert space.

1. **Introduction.** Throughout this paper the letter X will denote a compact Hausdorff space, while E, U, and V will stand for Banach spaces. $C(X,E)$ will denote the space of continuous functions on X to E provided with the supremum norm. And, given a dual space E^*, $C(X,(E^*,\sigma^*))$ denotes the Banach space of continuous functions F on X to E^* when the latter space is provided with its weak*topology, again normed by $\|F\|_\infty = \sup_{x\in X}\|F(x)\|$. If E is the one-dimensional field of scalars $\mathbf{K}$, ($\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$), then, unless we wish specifically to call attention to the scalar field involved, we will write $C(X)$ for $C(X,\mathbf{K})$. The notation $E_1 \cong E_2$ indicates that the Banach spaces E_1 and E_2 are isometric.

If X is an extremally disconnected compact Hausdorff space we will call a non-negative, extended real-valued Borel measure μ on X a *category measure* if

(i) every nonempty clopen set has positive measure,

(ii) every nowhere dense Borel set has measure zero, and

(iii) every nonempty clopen set contains a nonempty clopen set with finite measure.

An extremally disconnected compact Hausdorff space on which a category measure is defined will be called *hyperstonean*. This is equivalent to the definition of hyperstonean space obtained via the use of normal measures, [15, p. 95] and [3, p. 26]. Since for hyperstonean X every Borel set B has a unique representation $B = C \triangle D$ with C clopen and D nowhere dense, [3, pp. 1–2] and [8, p. 160], it follows that the null sets for a category measure are precisely the nowhere dense Borel sets. Given a hyperstonean space X with category measure μ, property (iii), together with an application of Zorn's lemma, can be used to show that X is the Stone-Čech compactification of the disjoint union of clopen subsets $X_\gamma, X = \beta(\bigcup_{\gamma\in\Gamma} X_\gamma)$, with $\mu(X_\gamma) < \infty$ for all γ, and for all Borel subsets B of $X, \mu(B) = \sum_{\gamma\in\Gamma}\mu(B\cap X_\gamma)$.

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We will say that a Banach dual U^* has *strongly unique predual* U if, given any isometry T of U^* onto a Banach dual V^* with predual V, then the adjoint mapping T^* carries the canonical image $J(V)$ of V in V^{**} onto the canonical image $J_0(U)$ of U in U^{**}. One easily verifies that, since T is a surjective isometry, it is enough to require that $T^* \circ J(V)$ is contained in $J_0(U)$ – i.e. that T is $\sigma(U^*, U) - \sigma(V^*, V)$ continuous. Equivalently, U^* has strongly unique predual if and only if there exists a unique projection $\pi : U^{***} \longrightarrow U^*$ with $\|\pi\| = 1$ and $\ker(\pi)$ weak*closed.

Two classical results on $C(X)$ as a dual space are due to Dixmier and Grothendieck. In [8], (see also [15, p. 95]), Dixmier proved that when X is hyperstonean then $C(X)$ is a dual space. And Grothendieck provided a strong converse. He showed that if $C(X)$ is a dual then X is hyperstonean; morever, for hyperstonean X the predual of $C(X)$ is strongly unique, [13], or [15, p. 96].

A theorem of Cembranos shows that, if one seeks a vector analogue for these results, the space $C(X, E)$ is an entirely inappropriate object on which to focus. For, Cembranos, [6], has shown that if X is any infinite compact Hausdorff space and E an infinite-dimensional Banach space, then $C(X, E)$ contains a complemented copy of c_0 and thus cannot even be isomorphic to a dual space.

However, when X is hyperstonean, $C(X, (E^*, \sigma^*))$ is always a dual space [4, Theorem 1], and we thus have an exact vectorial counterpart for the Dixmier result. Specifically, $C(X, (E^*, \sigma^*))$ is the dual of $L^1(\mu, E)$, for μ a category measure on X. And in [5] the full content of Grothendieck's converse was established in the case that E^* is a Hilbert space; i.e. $C(X, (E^*, \sigma^*))$ a dual implies that X is hyperstonean, *and* that the predual of $C(X, (E^*, \sigma^*))$ is strongly unique.

Ehrhard Behrends has supplied a stronger version of the first half of this result, proving that X is hyperstonean when $C(X, (E^*, \sigma^*))$ is a dual, provided that the centralizer $Z(E^*)$ of E^* is $\mathbf{K}$, [2, Theorem 2.7] and [9, p. 490]. The object of this article is to establish that both parts of the Grothendieck result, the hyperstonean nature of X and the strong uniqueness of the predual, follow for duals $C(X, (E^*, \sigma^*))$ when E^* is a reflexive space. We wish to prove the following:

THEOREM 1. *Let X be a compact Hausdorff space and E^* a reflexive Banach space. Then, if $C(X, (E^*, \sigma^*))$ is a dual space, X is hyperstonean and the predual of $C(X, (E^*, \sigma^*))$ is strongly unique.*

Our proof relies heavily upon I. Singer's characterization of $C(X, E)^*$ as the space of all regular Borel vector measures m on X to E^* with finite variation $|m|$, [18], or [7, p. 182], and also upon the notion of the centralizer $Z(E)$ of a Banach space E for which [1] is a reference. The main technical complications in our arguments occur in the proof of Lemma 3, where we employ the concepts of the Cunningham algebra $C_1(E)$ and of the integral module representation of a Banach space E. Here our principal reference is [3]. (See also [12, Section 2] for the case in which the scalars are complex.) We have not been able to find a more elementary proof of Lemma 3.

2. *Proof of the Theorem.* Our theorem will be established by means of a sequence

of lemmas. The first is a somewhat expanded statement of the result labeled Lemma 1 in [5]. The proof here is exactly the same as that given for the corresponding lemma in that article.

LEMMA 1. *Let E^* be any Banach dual with the Radon-Nikodym property, and let μ be a category measure on the hyperstonean space X. If $G \in C(X,(E^*,\sigma^*))$ then there exists an open dense set $O(= O_G)$ of X such that G is continuous from O to E^* when the latter space is given its norm topology. Moreover, the restriction of G to any σ-finite subset of X is μ-measurable (in the sense of [7, p. 41]).*

LEMMA 2. *Let X be a hyperstonean space and E^* a Banach dual with $Z(E^*) = \mathbf{K}$. Then $Z(C(X,(E^*,\sigma^*))) \cong C(X)$.*

PROOF. Let μ be a category measure on X. We first assume that μ is σ-finite so that the results of [12] apply. As previously mentioned $C(X,(E^*,\sigma^*)) \cong L^1(\mu,E)^*$ [4, Theorem 1] and hence $Z(C(X,(E^*,\sigma^*))) \cong C_1(L^1(\mu,E))$, [1, Theorems 5.7(iii) and 5.9]. This latter space is isometric to $L^\infty(\mu)$ because of [12, Corollary 4.3] and our assumption that $Z(E^*)$, (and hence $C_1(E)$), is $\mathbf{K}$. But $L^\infty(\mu) \cong C(X)$, [3, p. 31], and thus $Z(C(X,(E^*,\sigma^*))) \cong C(X)$.

If now μ is not assumed to be σ-finite, we know that $X = \beta(\bigcup_{\gamma\in\Gamma} X_\gamma)$, where the X_γ are pairwise disjoint clopen sets and $\mu\,|_{X_\gamma}$ is finite for all γ. It is then straightforward to see that $C(X,(E^*,\sigma^*))$ is isometric to $\prod^\infty_{\gamma\in\Gamma} C(X_\gamma,(E^*,\sigma^*))$. The injection of $C(X,(E^*,\sigma^*))$ into $\prod^\infty_{\gamma\in\Gamma} C(X_\gamma,(E^*,\sigma^*))$ is obvious, whereas given an element of the latter space, it can be considered as a bounded, weak*continuous function defined on $\bigcup_{\gamma\in\Gamma} X_\gamma$ to E^*, so that, by the weak*compactness of the unit ball in E^*, it extends to a weak*continuous function on $X = \beta(\bigcup_{\gamma\in\Gamma} X_\gamma)$ to E^*. Thus $Z(C(X,(E^*,\sigma^*))) \cong Z(\prod^\infty_{\gamma\in\Gamma} C(X_\gamma,(E^*,\sigma^*)))$ which (e.g. using the characterization of multipliers as M-bounded operators, [1, p. 54]) is easily seen to be $\prod^\infty_{\gamma\in\Gamma} Z(C(X_\gamma,(E^*,\sigma^*)))$. Finally this space is, by the previous paragraph, $\prod^\infty_{\gamma\in\Gamma} C(X_\gamma) \cong C(\beta(\bigcup_{\gamma\in\Gamma} X_\gamma)) = C(X)$.

Throughout the remainder of this article we assume that E^* is a reflexive Banach space with dual (= predual) E, X is a fixed compact Hausdorff space and that V is a Banach space such that there exists an isometry T mapping $C(X,(E^*,\sigma^*))$ onto V^*. J denotes that canonical injection of V into V^{**}, and, for hyperstonean X with category measure μ, J_0 denotes that of $L^1(\mu,E)$ into $C(X,(E^*,\sigma^*))^*$.

Given $v \in V$, the restriction of $T^* \circ J(v)$ to $C(X,E^*)$ is represented by a regular Borel vector measure on X. When X is hyperstonean with category measure μ we denote by m_v the μ-continuous part of $T^* \circ J(v)|_{C(X,E^*)}$ and by n_v its μ-singular part.

LEMMA 3. *X is hyperstonean and, for each $v \in V, n_v = 0$.*

PROOF. Since E is reflexive, its Cunningham algebra $C_1(E)$ is finite dimensional, [1, 4.25]. And as $Z(E^*) \cong C_1(E)$, [1, 5.7(iii) and 5.9], it follows from [1, 5.3] that $E^* = \prod^\infty E_i^*$ with $Z(E_i^*) = \mathbf{K}, 1 \leqq i \leqq n$. Hence $C(X,(E^*,\sigma^*)) = \prod^\infty C(X,(E_i^*,\sigma^*))$ $(1 \leqq i \leqq n)$. It thus follows from [1, 5.7(ii)] that V is the L^1-direct sum $V = \sum^1 V_i$,

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$(1 \leqq i \leqq n)$ where $V_i^* = T(C(X, (E_i^*, \sigma^*)))$, and hence the fact that X is hyperstonean is a consequence of [2, Theorem 2.7].

Next write, for $v \in V, T^* \circ J(v) |_{C(X,E^*)} = m_v + n_v$. We wish to prove that $n_v = 0$. For this it suffices to show that for all $f \in C(X, \mathbf{R})$ and $e^* \in E^*$ with $\|e^*\| = 1$ we have $\langle f \cdot e^*, n_v \rangle = 0$. But by the previous paragraph we can write $n_v = \sum_{i=1}^n n_{v_i}$, where $v_i \in V_i$, $1 \leqq i \leqq n$ and n_{v_i} vanishes on $C(X, (E^*, \sigma^*))$ for $j \neq i$. Thus, for the remainder of the proof we may assume, without loss of generality, that $n = 1$, i.e. $Z(E^*) = \mathbf{K}$.

Now for $f \in C(X, \mathbf{R})$, $f \longrightarrow \mathrm{Re}\langle f \cdot e^*, T^* \circ J(v) \rangle$ is a continuous linear functional on $C(X, \mathbf{R})$, and is thus given by a finite regular Borel measure $\nu = \nu_{e^*}$, where

$$\nu(f) = \mathrm{Re}\langle f \cdot e^*, T^* \circ J(v) \rangle. \tag{1}$$

And since n_v is supported on a nowhere dense set and m_v is μ-continuous, to show that $\langle f \cdot e^*, n_v \rangle = 0$ for $f \in C(X, \mathbf{R})$ it suffices to show that ν vanishes on nowhere dense sets, (and similarly for the measure defined by replacing Re by Im in (1)).

Since we are assuming that $Z(E^*) = \mathbf{K}$, by Lemma 2 we have $Z(C(X, (E^*, \sigma^*))) \cong C(X)$. This isometry is, in fact, effected by the map $g \longrightarrow \hat{g}$ for $g \in C(X)$, where $\hat{g}$ denotes the operator which is multiplication by the function g. Now $\hat{g} \longrightarrow \check{g} := T \circ \hat{g} \circ T^{-1}$ carries $Z(C(X, (E^*, \sigma^*)))$ isometrically onto $Z(V^*)$, and, again by [1, 5.7(iii) and 5.9], $C_1(V)$ is isometric to $Z(V^*)$ under the map which sends an element of $C_1(V)$ to its adjoint. We thus have that $g \longrightarrow \check{g} \longrightarrow \tilde{g}; = (\check{g})_*$ is an isometry of $C(X)$ onto $C_1(V)$, where, of course, $(\check{g})_*$ denotes that element whose adjoint is $\check{g}$. Therefore, [3, Chap. 3], V has an integral module representation Ψ on X such that $\Psi(\tilde{g}v) = g \cdot \Psi(v)$ for $g \in C(X, \mathbf{R})$.

As $|\nu|$ is regular and closed sets are compact, to show that ν vanishes on nowhere dense sets it is sufficient to show that, for a closed nowhere dense set B, inf $\{|\nu|(D)$: D is clopen and contains $B\} = 0$. Thus given B, let D be a clopen set containing B. Since $|\nu|(D)$ is the norm of the restriction of ν to $C(D, \mathbf{R})$ we have

$$\begin{aligned} |\nu|(D) &= \sup\{\nu(g) : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\} \\ &= \sup\{\mathrm{Re}\langle g \cdot e^*, T^* \circ J(v) \rangle : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\} \\ &= \sup\{\mathrm{Re}\langle v, T(g \cdot e^*) \rangle : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\}. \end{aligned}$$

We define $v^* \in V^*$ by $v^* = T(\mathbf{e}^*)$, (where $\mathbf{e}^*$ denotes that element of $C(X, (E^*, \sigma^*))$ that is constantly equal to e^*). Then the above quantity is

$$\begin{aligned} &\sup\{\mathrm{Re}\langle v, T \circ \hat{g} \circ T^{-1} v^* \rangle : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\} \\ &= \sup\{\mathrm{Re}\langle v, \check{g} v^* \rangle : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\} \\ &= \sup\{\mathrm{Re}\langle \tilde{g} v, v^* \rangle : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\}. \end{aligned}$$

And recalling that e^*, hence v^*, has norm 1, the latter quantity is less than or equal to

$$\begin{aligned} &\sup\{\|\tilde{g} v\| : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\} \\ &= \sup\{\|g \cdot \Psi(v)\| : g \in C(X, \mathbf{R}), |g| \leqq \chi_D\} \leqq \|\chi_D \cdot \Psi(v)\|, \end{aligned}$$

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according to the definition of the norm in an integral module, [3, p. 35]. Then, [3, p. 24],

$$\begin{aligned}&\inf\{\|\chi_D \cdot \Psi(v)\| : D \text{ is clopen and contains } B\}\\ &= \|\chi_{Int(B)} \cdot \Psi(v)\|\\ &= \|\chi_\phi \cdot \Psi(v)\| = 0.\end{aligned}$$

This concludes the proof of the lemma.

We thus have shown that for $v \in V$ we have $T^* \circ J(v)|_{C(X,E^*)} = m_v$ with $m_v \ll \mu$. The elements of $C(X, (E^*, \sigma^*))$ are integrable with respect to m_v, for by Lemma 1 they are μ-measurable on μ-σ-finite sets, and as $|m_v|$ is finite, the μ-continuous measure m_v has μ-σ-finite support. Therefore $F \longrightarrow \int F dm_v$ defines a continuous linear functional on $C(X, (E^*, \sigma^*))$, which we continue to denote by m_v. Then $\Phi_v = T^* \circ J(v) - m_v \in C(X, (E^*, \sigma^*))^*$ with $\Phi_v \in C(X, E^*)^\perp$ and we have $T^* \circ J(v) = m_v + \Phi_v$. Whenever we write, for $v \in V, T^* \circ J(v) = m_v + \Phi_v$ it will be understood that m_v is the vector measure determined by the restriction of $T^* \circ J(v)$ to $C(X, E^*)$ and that $\Phi_v \in C(X, E^*)^\perp$.

Except for the need to distinguish between E^* and its predual E in our notation, the proof of the following lemma, which completes the proof of our theorem, is exactly the same as that given for the case when $E = E^*$ is Hilbert space in [5, Lemma 4].

LEMMA 4. *For $v \in V$ we have $T^* \circ J(v) = G_v d\mu$ for some $G_v = L^1(\mu, E)$ with $\|G_v\|_1 = \|v\|$. Consequently $V \cong L^1(\mu, E)$ under the mapping $J_0^{-1} \circ T^* \circ J$.*

3. **Remarks and Problems.** Our result, of course, cannot possibly hold for arbitrary Banach duals E^*. For if X is a one-point space then $C(X, (E^*, \sigma^*)) \cong E^*$. Thus if E^* fails to have strongly unique predual, e.g. if $E^* = l^1$, the same may be true of $C(X, (E^*, \sigma^*))$. In fact it can be established that, if λ is Lebesgue measure on [0, 1], then the L^1-direct sum $V = L^1(\lambda, c_0) \oplus_1 L^1(\lambda, c)$ is a predual of $L^1(\lambda, c_0)^*$ although V is not isometric to $L^1(\lambda, c_0)$. (V is not even isometric to *any* space $L^1(\lambda, E)$ with E a predual of l^1, as an investigation of its L-summands reveals.) However, one may ask whether our theorem holds for all duals E^* with strongly unique predual. We remark that it still seems to be unknown whether *strong* uniqueness of the predual is implied by uniqueness of the predual (any two preduals are isometrically isomorphic.)

The authors are indebted to the referee for providing the following simplified proof of a special case of our Theorem 1, (which contains the result for E^* = Hilbert space found in [5].)

THEOREM 2. *Let X be a compact Hausdorff space and E^* a reflexive space with a 1-unconditional basis. Then, if $C(X, (E^*, \sigma^*))$ is a dual space, X is hyperstonean and the predual of $C(X, (E^*, \sigma^*))$ is strongly unique.*

PROOF. Let e_1 be a norm-one element of the unconditional basis of E^*. Then the subspace $\{f \cdot e_1 : f \in C(X)\} \cong C(X)$ is constrained (i.e. the range of a projection of norm one) in $C(X, (E^*, \sigma^*))$. Now a Banach space U is constrained in a dual space V^* if and only if it is constrained in its own second dual. (Let j be the injection of U into

V^*, π the norm-one projection of V^* onto $j(U)$ and q the canonical projection of V^{***} onto V^* given by the restriction map. Then $j^{-1} \circ \pi \circ q \circ j^{**}$ is a norm-one projection of U^{**} onto U. Alternately, see [17].) Thus $C(X)$ is constrained in $C(X)^{**} \cong C(X_0)$ with X_0 hyperstonean, by Grothendieck's result and the characterization of the second dual of $C(X)$ originally due to Kakutani [14].

Hence let P be a norm-one projection of $C(X_0)$ onto (the canonical image in $C(X)^{**}$ of) $C(X)$. Let U, V be Banach spaces with $U \subseteq V$ and let L be a bounded linear operator from U to $C(X) \subseteq C(X_0)$. Then [15, p. 88] L has a norm-preserving linear extension $\hat{L} : V \longrightarrow C(X_0)$ and hence $P \circ \hat{L}$ is a norm-preserving linear extension of L mapping V to $C(X)$. Thus, [15, p. 92], X is extremally disconnected.

Since [16, p. 2] E^* is a Banach lattice $C(X, (E^*, \sigma^*))$ is a Banach lattice for its natural ordering. And it is proven in [11, Théorème 8], (see also [10]), that if a Banach lattice U is isometric to V^* where V is a *Banach space*, then V is contained in the space of order continuous linear forms on U. Consideration of the subspace $\{f \cdot e_1 : f \in C(X)\} \subseteq C(X, (E^*, \sigma^*))$ then shows that the order continuous linear forms on $C(X)$ separate $C(X)$, and thus [15, p. 95] X is hyperstonean. Finally E is a weakly sequentially complete Banach space and thus so is $L^1(\mu, E)$, [19, Theorem 11]. It thus follows from [11] that $L^1(\mu, E)$ is a strongly unique predual for $C(X, (E^*, \sigma^*))$.

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