WEYL TYPE THEOREMS FOR FUNCTIONS OF OPERATORS

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Abstract. A-Weyl’s theorem and property (ω), as two variations of Weyl’s theorem, were introduced by Rakočević. In this paper, we study a-Weyl’s theorem and property (ω) for functions of bounded linear operators. A necessary and sufficient condition is given for an operator T to satisfy that f(T) obeys a-Weyl’s theorem (property (ω)) for all f ∈ Hol(σ(T)). Also we investigate the small-compact perturbations of operators satisfying a-Weyl’s theorem (property (ω)) in the setting of separable Hilbert spaces.

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1. Introduction. This paper is a continuation of a previous paper of the authors and Feng [17], where the stability of Weyl’s theorem under holomorphic functional calculus is studied. A-Weyl’s theorem and property (ω) as two variations of Weyl’s theorem, which have been recently studied in [3, 4, 5], were introduced by Rakočević [21, 22]. The purpose of this paper is to investigate a-Weyl’s theorem and property (ω) for functions of operators on Banach spaces. A necessary and sufficient condition is given for an operator T to satisfy that f(T) obeys a-Weyl’s theorem (property (ω)) for each function f analytic on some neighbourhood of σ(T).

We first give some notations and terminologies. Throughout this paper, C and N denote the set of complex numbers and the set of natural numbers respectively. \( \mathcal{X} \) will always denote a complex infinite dimensional Banach space. We let \( B(\mathcal{X}) \) denote the algebra of all bounded linear operators on \( \mathcal{X} \), and let \( K(\mathcal{X}) \) denote the ideal of compact operators in \( B(\mathcal{X}) \).

Let \( T \in B(\mathcal{X}) \). We denote by \( \sigma(T) \) and \( \sigma_p(T) \) the spectrum of \( T \) and the point spectrum of \( T \) respectively. Denote by \( n(T) \) and \( \mathcal{R}(T) \) the kernel of \( T \) and the range

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of $T$ respectively. $T$ is called a semi-Fredholm operator, if $\mathcal{R}(T)$ is closed and either $\text{nul } T$ or $\text{nul } T^*$ is finite, where $\text{nul } T := \dim n(T)$ and $\text{nul } T^* := \dim n(T^*)$; in this case, $\text{ind } T := \text{nul } T - \text{nul } T^*$ is called the index of $T$. In particular, if $-\infty < \text{ind } T < \infty$, then $T$ is called a Fredholm operator. It is well known that if $T$ is semi-Fredholm and $K \in \mathcal{K}(\mathcal{X})$, then $T + K$ is also semi-Fredholm and $\text{ind } (T + K) = \text{ind } T$. $T$ is called a Weyl operator if it is Fredholm of index 0.

The Wolf spectrum $\sigma_{\text{fre}}(T)$, the essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_w(T)$ of $T$ are defined as:

$$\sigma_{\text{fre}}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\},$$

$$\sigma_e(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_w(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. $\rho_{\text{semi-F}}(T) := \mathbb{C} \setminus \sigma_{\text{fre}}(T)$ is called the semi-Fredholm domain of $T$. The approximate point spectrum $\sigma_a(T)$ and the essential approximate point spectrum $\sigma_{ea}(T)$ of $T$ are defined as:

$$\sigma_a(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not bounded below} \}$$

and

$$\sigma_{ea}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{X})} \sigma_a(T + K),$$

respectively. The set $\sigma_{ea}(T)$ has been introduced in [20] and studied in [20, 21, 23]. It is easy to see that

$$\sigma_{ea}(T) = \sigma_{\text{fre}}(T) \cup \{ \lambda \in \rho_{\text{semi-F}}(T) : \text{ind } (T - \lambda) > 0 \}.$$  

Given a subset $\sigma$ of $\mathbb{C}$, denote by $\text{iso } \sigma$ and $\text{int } \sigma$ the set of all isolated points of $\sigma$ and the interior of $\sigma$ respectively. We denote

$$\pi_{00}(T) := \{ \lambda \in \text{iso } \sigma(T) : 0 < \text{nul } (\lambda - T) < \infty \}$$

and

$$\pi_{00}'(T) := \{ \lambda \in \text{iso } \sigma_a(T) : 0 < \text{nul } (\lambda - T) < \infty \}.$$  

Following Coburn [9], we say that Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{X})$, denoted by $T \in (\mathcal{W})$, if $\sigma(T) \setminus \sigma_a(T) = \pi_{00}(T)$. Today, Weyl's theorem has been extended to various operators acting on both Hilbert spaces and Banach spaces, and there has been a lot of work (see, for example, [6, 8, 11, 12, 13, 14]). We say that a-Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{X})$, denoted by $T \in (\mathcal{a-W})$, if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}'(T)$. A-Weyl's theorem has been introduced and studied in [21]. $T$ is said to have property $(\omega)$, denoted by $T \in (\omega)$, if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T)$. It is well known that both $T \in (\mathcal{a-W})$ and $T \in (\omega)$ imply that $T \in (\mathcal{W})$ (see [2]).
Let $\text{Hol}(\sigma(T))$ denote the set of all functions $f$ which are analytic on some neighbourhood of $\sigma(T)$ (the neighbourhood depends on $f$) for given $f \in \text{Hol}(\sigma(T))$, $f(T)$ denotes the holomorphic functional calculus of $T$ with respect to $f$.

Let $T \in \mathcal{B}(\mathcal{X})$. If $\sigma$ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain $\Omega$ such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma; T)$ denote the Riesz idempotent of $T$ corresponding to $\sigma$, that is,

$$E(\sigma; T) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} d\lambda,$$

where $\Gamma = \partial \Omega$ is positively oriented with respect to $\Omega$ in the sense of complex variable theory. In this case, we denote $\mathcal{X}(\sigma; T) = \mathcal{R}(E(\sigma; T))$. If $\lambda \in \text{iso } \sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$ and we simply write $\mathcal{X}(\lambda; T)$ instead of $\mathcal{X}(\{\lambda\}; T)$; if, in addition, $\dim \mathcal{X}(\lambda; T) < \infty$, then $\lambda$ is called a normal eigenvalue of $T$. A normal eigenvalue of $T$ is also called a Riesz point of $T$ (see [7]). The set of all normal eigenvalues of $T$ will be denoted by $\sigma_0(T)$.

We denote

$$\rho_{s-F}^0(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is Weyl}\},$$

$$\rho_{s-F}^+(T) := \{\lambda \in \rho_{s-F}(T) : \text{ind } (T - \lambda) > 0\}$$

and

$$\rho_{s-F}^-(T) := \{\lambda \in \rho_{s-F}(T) : \text{ind } (T - \lambda) < 0\}.$$  

Obviously, $\rho_{s-F}(T) = \rho_{s-F}^+(T) \cup \rho_{s-F}^0(T) \cup \rho_{s-F}^-(T)$ and $\rho_{s-F}^0(T) = \mathbb{C} \setminus \sigma_w(T)$.

Now, we can list the main results of this paper.

**Main theorem 1.1.** Let $T \in \mathcal{B}(\mathcal{X})$. Then, $f(T) \in \text{(a-W)}$ for all $f \in \text{Hol}(\sigma(T))$ if and only if the following conditions hold.

(i) $T \in \text{(a-W)}$.

(ii) If $\rho_{s-F}^-(T) \neq \emptyset$, then there exists no $\lambda \in \rho_{s-F}(T)$ such that $0 < \text{ind } (T - \lambda) < \infty$.

(iii) If $\sigma_p(T) \cap \rho_{s-F}^-(T) \neq \emptyset$, then $\text{iso } \sigma_p(T) \subseteq \sigma_p(T)$.

It is worth mentioning that Weyl type theorems are closely related to some basic concepts in local spectral theory (see [1]). Oudghiri [18] related Weyl's theorem to the single-valued extension property in local spectral theory. In [2], Aiena gave some sufficient conditions for an operator $T$ to satisfy $f(T) \in \text{(a-W)}$ for all $f \in \text{Hol}(\sigma(T))$ in terms of certain glocal spectral subspaces.

**Main theorem 1.2.** Let $T \in \mathcal{B}(\mathcal{X})$. Then, $f(T) \in \omega$ for all $f \in \text{Hol}(\sigma(T))$ if and only if the following conditions hold.

(i) $T \in \omega$.

(ii) If $\rho_{s-F}^-(T) \neq \emptyset$, then $\sigma_0(T) = \emptyset$ and there exists no $\lambda \in \rho_{s-F}(T)$ such that $0 < \text{ind } (T - \lambda) < \infty$.

(iii) If $\sigma_0(T) \neq \emptyset$, then $\text{iso } \sigma(T) \subseteq \sigma_p(T)$.

If $\mathcal{X}$ is a complex separable Hilbert space and $\dim \mathcal{X} = \infty$, then it is proved in [17] that any operator $T \in \mathcal{B}(\mathcal{X})$ has an arbitrarily small compact perturbation satisfying Weyl's theorem. Since $A \in \omega$ implies that $A$ satisfies Weyl's theorem, the following theorem strengthens the above result.
Main theorem 1.3. Let $\mathcal{X}$ be a complex separable infinite dimensional Hilbert space. Then, given $T \in \mathcal{B}(\mathcal{X})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{X})$ with $\|K\| < \varepsilon$ such that $T + K \in (\omega)$ and $T + K \in (a-W)$.

The rest part of this paper is organized as follows. In Section 2, we shall make some preparations for the proofs of main theorems. Section 3 is devoted to the proof of Main Theorem 1.1. The proofs of main theorem 1.2/1.3 shall be provided respectively in Section 4 and Section 5.

2. Preparations. In this section, we give some useful lemmas.

**Lemma 2.1 ([19], Theorem 2.10).** Let $T \in \mathcal{B}(\mathcal{X})$ and suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_i(i = 1, 2)$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Then, $\mathcal{X}(\sigma_1; T) + \mathcal{X}(\sigma_2; T) = \mathcal{X}$, $\mathcal{X}(\sigma_1; T) \cap \mathcal{X}(\sigma_2; T) = \{0\}$ and $T$ admits the following matrix representation

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \mathcal{X}(\sigma_1; T) \cup \mathcal{X}(\sigma_2; T),$$

where $\sigma(T_i) = \sigma_i(i = 1, 2)$.

Using [15, Corollary 3.22] and the above lemma, we can obtain the following lemma whose proof is left to the reader.

**Corollary 2.2.** Let $\mathcal{X}$ be a complex separable Hilbert space and $T \in \mathcal{B}(\mathcal{X})$. Suppose that $\sigma$ is a clopen subset of $\sigma(T)$. Then

$$T = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \mathcal{X}(\sigma; T) \cup \mathcal{X}(\sigma; T)$$

where $\sigma(A) = \sigma$ and $\sigma(B) = \sigma(T) \setminus \sigma$.

In this paper, if $S, T \in \mathcal{B}(\mathcal{X})$, then we let $S \sim T$ denote that $S$ and $T$ are similar.

**Lemma 2.3 ([10], chapter XI, proposition 6.9).** Let $T \in \mathcal{B}(\mathcal{X})$ and $\lambda_0 \in \text{iso} \sigma(T)$. Then, the following statements are equivalent.

(i) $\lambda_0 \in \sigma_0(T)$.
(ii) $\lambda_0 \in \rho^0_{-F}(T)$.
(iii) $\lambda_0 \in \rho_{-F}(T)$.

Note that an operator $T \in \mathcal{B}(\mathcal{X})$ is bounded below if and only if $\text{nul } T = 0$ and $\mathcal{R}(T)$ is closed, then, by the continuity of the index function $\text{ind } (\cdot)$, the following lemma is clear.

**Lemma 2.4.** Let $T \in \mathcal{B}(\mathcal{X})$ and $\lambda_0 \in \text{iso} \sigma_a(T)$. If $\lambda_0 \in \rho_{-F}(T)$, then $0 < \text{nul } (T - \lambda_0) < \infty$ and $\text{ind } (T - \lambda_0) \leq 0$.

The proof of the following lemma is simple and we omit it.

**Lemma 2.5.** Let $T \in \mathcal{B}(\mathcal{X})$ and $f \in \text{Hol}(\sigma(T))$. Then, $f(\sigma_a(T)) \subseteq \sigma_a(f(T))$ and $f(\sigma_p(T)) \subseteq \sigma_p(f(T))$.

In this paper, we denote by $\text{card } \sigma$ the cardinality of a subset $\sigma$ of $\mathbb{C}$. If $\lambda \in \mathbb{C}$ and $\delta > 0$, then we denote $B_\delta(\lambda) = \{z \in \mathbb{C} : |z - \lambda| < \delta\}$.

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Lemma 2.6. Let $T \in \mathcal{B}(\mathcal{X})$ and $f \in \text{Hol}(\sigma(T))$. If $0 \notin \sigma(f(T))$ and $\text{nul } f(T) < \infty$, then $1 \leq \text{card } \{ \lambda \in \sigma(T): f(\lambda) = 0 \} < \infty$; if, in addition, $0 \in \text{iso } \sigma(f(T))$, then for each $\mu \in \{ \lambda \in \sigma(T): f(\lambda) = 0 \}$ there exists $\delta > 0$ such that $B_{\delta}(\mu) \setminus \{ \mu \} \subseteq \mathbb{C} \setminus \sigma_{\partial}(T)$.

Proof. It is obvious that $\{ \lambda \in \sigma(T): f(\lambda) = 0 \} \neq \emptyset$. If $\{ \lambda \in \sigma(T): f(\lambda) = 0 \}$ is an infinite subset of $\sigma(T)$, then we can choose a limit point $\lambda_0$ of $\{ \lambda \in \sigma(T): f(\lambda) = 0 \}$. Without loss of generality, we assume that $f$ is analytic on a neighbourhood $\Omega$ of $\sigma(T)$. Then, there is a component $\Omega_1$ of $\Omega$ such that $\lambda_0 \in \Omega_1$ and $f \equiv 0$ on $\Omega_1$. Set $\sigma_1 = \sigma(T) \cap \Omega_1$ and $\sigma_2 = \sigma(T) \setminus \sigma_1$. Then, $\sigma_i(i = 1, 2)$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \neq \emptyset$.

By Lemma 2.1, $T$ can be written as

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \mathcal{X}(\sigma_1;T)$$

where $\sigma(T_1) = \sigma_i(i = 1, 2)$. Hence, $f(T_1) = 0$. Since $\lambda_0$ is a limit point of $\{ \lambda \in \sigma(T): f(\lambda) = 0 \}$, $\lambda_0 \in \sigma_1$ and $\sigma_1$ is a clopen subset of $\sigma(T)$, it is easy to see that $\text{card } \{ \lambda \in \sigma_1: f(\lambda) = 0 \} = \infty$ and $\text{dim } \mathcal{X}(\sigma_1;T) = \infty$. Then, we have

$$f(T) = \begin{bmatrix} f(T_1) & 0 \\ f(T_2) & f(T_2) \end{bmatrix} \mathcal{X}(\sigma_1;T) \mathcal{X}(\sigma_2;T) = \begin{bmatrix} 0 & 0 \\ 0 & f(T_2) \end{bmatrix} \mathcal{X}(\sigma_1;T) \mathcal{X}(\sigma_2;T).$$

It follows immediately that $\text{nul } f(T) \geq \text{dim } \mathcal{X}(\sigma_1;T) = \infty$, a contradiction. Thus, we have proved that $\text{card } \{ \lambda \in \sigma(T): f(\lambda) = 0 \} < \infty$.

Now we assume that $0 \in \text{iso } \sigma_{\partial}(f(T))$ and $\mu \in \sigma(T)$ satisfies that $f(\mu) = 0$. We shall prove that there exists $\delta > 0$ such that $B_{\delta}(\mu) \setminus \{ \mu \} \subseteq \mathbb{C} \setminus \sigma_{\partial}(T)$. If not, then we can choose $\{ \mu_n \}_{n=1}^{\infty} \subseteq \{ \sigma(T) \setminus \{ \mu \} \}$ such that $\mu_n \to \mu$. By Lemma 2.5, $f(\mu_n) \in \sigma_{\partial}(f(T))$ for all $n$ and $f(\mu_n) \to f(\mu) = 0$. Since $\text{card } \{ \lambda \in \sigma(T): f(\lambda) = 0 \} < \infty$, we may assume that $f(\mu_n) \neq 0$ for all $n \geq 1$. Thus, we obtain $0 \notin \text{iso } \sigma_{\partial}(f(T))$, a contradiction. □

Using a similar argument as in the proof of Lemma 2.6, one can obtain the following result.

Corollary 2.7. Let $T \in \mathcal{B}(\mathcal{X})$ and $f \in \text{Hol}(\sigma(T))$. If $0 \in \text{iso } \sigma(f(T))$ and $\text{nul } f(T) < \infty$, then $1 \leq \text{card } \{ \lambda \in \sigma(T): f(\lambda) = 0 \} < \infty$ and $\{ \lambda \in \sigma(T): f(\lambda) = 0 \} \subseteq \text{iso } \sigma(T)$.

Lemma 2.8 ([17], Lemma 2.7). Let $T \in \mathcal{B}(\mathcal{X})$ and $f \in \text{Hol}(\sigma(T))$. If $0 \in \sigma(f(T))$ and $\text{nul } f(T) < \infty$, then, there exists $g \in \text{Hol}(\sigma(T))$ such that $f(T) = g(T)$ and

$$g(\lambda) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_n)^{k_n}g_0(\lambda),$$

where $\lambda_i \in \sigma(T)(1 \leq i \leq n)$, $g_0 \in \text{Hol}(\sigma(T))$ and $g_0(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.

Lemma 2.9. Let $T \in \mathcal{B}(\mathcal{X})$. Suppose that $\lambda_i \in \sigma(T)(1 \leq i \leq n)$ and $f(\lambda) = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_n)^{k_n}g(\lambda)$, where $g \in \text{Hol}(\sigma(T))$ and $g(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$. For each $i$, there exists a $\delta_i > 0$ such that $[B_{\delta_i}(\lambda_i) \setminus \{ \lambda_i \}] \subseteq \{ \mathbb{C} \setminus \sigma_{\partial}(T) \}$. Then, there exists $\delta > 0$ such that $[B_{\delta}(0) \setminus \{ 0 \}] \subseteq \{ \mathbb{C} \setminus \sigma_{\partial}(f(T)) \}$.

Proof. Without loss of generality, we assume that $\{ B_{\delta_i}(\lambda_i) \}_{i=1}^{n}$ are pairwise disjoint and $g$ is well defined on $\bigcup_{i=1}^{n} B_{\delta_i}(\lambda_i)$. Set $\delta_0 = \min\{|g(\lambda)|: \lambda \in \sigma(T)\}$ and $\delta = \frac{\delta_0}{2} \cdot \prod_{i=1}^{n} \delta_i^{k_i}$. Obviously, $\delta > 0$ and $0 \in \sigma(f(T))$. 

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Arbitrarily choose a $\lambda_0 \in B_1(0), \lambda_0 \neq 0$. We shall prove that $\lambda_0 - f(T)$ is bounded below. Without loss of generality, we may assume that $\lambda_0 \in \sigma(f(T))$.

**Claim.** If $\mu \in \sigma(T)$ and $f(\mu) = \lambda_0$, then $\mu \in \bigcup_{i=1}^{n} B_\delta(\lambda_i)$.

In fact, if not, then $|\mu - \lambda_i| \geq \delta_i$ for all $i$. Thus,

$$\delta > |\lambda_0| = |f(\mu)| \geq \delta_0 \cdot |\nu^0_i| \geq \delta_0 \cdot \prod_{i=1}^{n} |\nu^0_i| > \delta,$$

a contradiction.

Since $|\lambda_0| < |f(\lambda)|$ on $\bigcup_{i=1}^{n} \partial B_\delta(\lambda_i)$, by Rouché’s theorem, we deduce that $f(\lambda)$ and $f(\lambda) - \lambda_0$ have the same number of zeros in $\bigcup_{i=1}^{n} B_\delta(\lambda_i)$, where each zero is counted as many times as its multiplicity. Hence, we may assume that

$$f(\lambda) - \lambda_0 = (\lambda - \lambda_1) \cdots (\lambda - \lambda_m) f_0(\lambda),$$

where $f_0 \in \text{Hol}(\sigma(T))$ and $f_0(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$. Here, $\lambda_i (1 \leq i \leq m)$ may repeat according to multiplicity. Then,

$$f(T) - \lambda_0 = (T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_m) f_0(T),$$

where $f_0(T)$ is invertible.

Note that $\lambda_i \neq \lambda_j$ for all $i$ and if (otherwise $\lambda_0 = f(\mu_i) = f(\lambda_j) = 0$, a contradiction).

Then, by our claim, we have

$$\{\mu_i : 1 \leq i \leq m\} \subseteq \bigcup_{i=1}^{n} [B_\delta(\lambda_j) \setminus \{\lambda_j\}].$$

Therefore, $T - \lambda_i$ is bounded below for all $i$ and hence $f(T) - \lambda_0$ is bounded below. □

Using a similar argument as in the proof of Lemma 2.9, one can obtain the following result.

**Corollary 2.10.** Let $T \in B(\mathcal{X})$ and suppose that $\lambda_i \in \text{iso} \sigma(T) (1 \leq i \leq n)$. If $f(\lambda) = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_n)^{k_n} g(\lambda)$, where $g \in \text{Hol}(\sigma(T))$ and $g(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$, then $0 \notin \text{iso} \sigma(f(T))$.

3. Proof of Main Theorem 1.1.

**Lemma 3.1.** Let $T \in B(\mathcal{X})$. Then,

$$\sigma_a(T) \setminus \sigma_{\text{eu}}(T) = [\rho^0_{a-F}(T) \cup \rho^0_{a-F}(T)] \cap \sigma_p(T).$$

Hence,

$$T \in (a-W) \iff [\rho^0_{a-F}(T) \cup \rho^0_{a-F}(T)] \cap \sigma_p(T) = \pi^a_{\text{eu}}(T)$$

and

$$T \in (\omega) \iff [\rho^0_{a-F}(T) \cup \rho^0_{a-F}(T)] \cap \sigma_p(T) = \pi_0(T).$$

**Proof.** Obviously, we need only prove that

$$\sigma_a(T) \setminus \sigma_{\text{eu}}(T) = [\rho^0_{a-F}(T) \cup \rho^0_{a-F}(T)] \cap \sigma_p(T).$$
The inclusion relation “⊇” is obvious. We only prove that the inclusion relation “⊆” holds. \( \lambda_0 \in [\sigma_d(T) \setminus \sigma_{ev}(T)] \) implies that \( \lambda_0 \notin \sigma_{ev}(T) \), that is, there exists \( K \in \mathcal{K}(X) \) such that \( T + K - \lambda_0 \) is bounded below. Hence, \( \text{ind}((T - \lambda_0) = \text{ind}(T + K - \lambda_0) \leq 0 \). Note that \( \lambda_0 \in \sigma_d(T) \), then \( \text{null}(T - \lambda_0) > 0 \). Thus, we obtain \( \lambda_0 \in [\rho_{s-f}(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \).

Recall that a set, which is made up only of isolated points, is called a discrete set. The following result provides a necessary and sufficient condition for an operator to satisfy a-Weyl’s theorem.

**Lemma 3.2.** Let \( T \in \mathcal{B}(X) \). Then, \( T \in (a-W) \) if and only if the following conditions hold:

(i) \( [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \) is a discrete set;

(ii) \( \pi_{000}(T) \subseteq \rho_{s-f}(T) \).

**Proof.** “⇒”. By Lemma 3.1, \( T \in (a-W) \) implies that \( \pi_{000}(T) \subseteq \rho_{s-f}(T) \) and \( [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \subseteq \sigma_d(T) \). Hence, \( [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \) is a discrete set.

“⇐”. By Lemma 2.4, it follows easily from \( \pi_{000}(T) \subseteq \rho_{s-f}(T) \) that \( \pi_{000}(T) \subseteq [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \). By Lemma 3.1, it suffices to prove that \( [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \) is an open set.

Arbitrarily choose a \( \lambda_0 \in [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \). Then, by (i), there exists a \( \delta_1 > 0 \) such that \( \text{null}(\lambda - T) = 0 \) for all \( \lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\} \). Note that \( \lambda_0 \in [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \) is open, then there exists \( \delta_2 > 0 \) such that \( B_\delta(\lambda_0) \subseteq \rho_{s-f}(T) \cup \rho_{s-f}(T) \). Set \( \delta = \min(\delta_1, \delta_2) \). Then, it is easy to see that \( \lambda - T \) is bounded below for all \( \lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\} \). Then, it follows from \( \lambda_0 \in [\rho_{s-f}^0(T) \cup \rho_{s-f}^0(T)] \cap \sigma_p(T) \) that \( \lambda_0 \in \pi_{000}(T) \).

**Corollary 3.3.** Let \( T \in \mathcal{B}(X) \) and suppose that \( T \in (a-W) \). If \( \lambda \in \rho_{s-f}(T) \) and \( \text{ind}(\lambda - T) \leq 0 \), then either \( \lambda \notin \sigma_d(T) \) or \( \lambda \in \pi_{000}(T) \).

Now we are going to give the proof of Main Theorem 1.1.

**Proof of Main Theorem 1.1** “⇒”. Assume that \( f(T) \in (a-W) \) for all \( f \in \text{Hol}(\sigma(T)) \).

(i) Set \( f_1(\lambda) = \lambda \). Then, evidently, \( T = f_1(T) \in (a-W) \).

(ii) If \( f \) does not hold, then we can choose \( \lambda_1, \lambda_2 \in \mathbb{C} \) such that \( 0 < \text{ind}(T - \lambda_1) < \infty \) and \(-\infty \leq \text{ind}(T - \lambda_2) < 0 \). Obviously, we can choose \( k \in \mathbb{N} \) such that \( \text{ind}(T - \lambda_1) + k \cdot \text{ind}(T - \lambda_2) < 0 \). Define \( f_2(z) = (z - \lambda_1)(z - \lambda_2)^k \). Then, \( f_2(T) \) is semi-Fredholm and

\[ \text{ind}(f_2(T)) = \text{ind}(T - \lambda_1) + k \cdot \text{ind}(T - \lambda_2) < 0, \]

that is, \( 0 \in \rho_{s-f}(f_2(T)) \).

\[ \text{ind}(T - \lambda_1) > 0 \] implies that there exists \( \delta > 0 \) such that \( \text{ind}(T - \mu) > 0 \) for all \( \mu \in B_\delta(\lambda_1) \). Then, \( \text{null}(T - \mu) \geq \text{ind}(T - \mu) > 0 \) for all \( \mu \in B_\delta(\lambda_1) \). Hence, we have \( B_\delta(\lambda_1) \subseteq \sigma_p(T) \). By Lemma 2.5, it follows that \( f_2(B_\delta(\lambda_1)) \subseteq \sigma_p(f_2(T)) \).

Evidently \( f_2 \) is an open mapping, then \( f_2(B_\delta(\lambda_1)) \) is an open neighbourhood of \( 0 \). Note that \( 0 \in \rho_{s-f}(f_2(T)) \), then, we can choose \( \varepsilon > 0 \) such that \( B_\varepsilon(0) \subseteq f_2(B_\delta(\lambda_1)) \cap \rho_{s-f}(f_2(T)) \subseteq \sigma_p(f_2(T)) \cap \rho_{s-f}(f_2(T)) \). By Lemma 3.2, we have \( f_2(T) \notin (a-W) \), a contradiction.
(iii) If (iii) does not hold, then we can choose \( \lambda_1 \in \sigma_p(T) \cap [\rho_{s-F}(T) \cup \rho_{s-F}^0(T)] \) and \( \lambda_2 \in \text{iso } \sigma_a(T) \) such that \( \lambda_2 \notin \sigma_p(T) \). By Lemma 3.2, it follows from \( T \in (a-W) \) that \( \lambda_1 \in \pi_{00}^a(T) \). It follows from Lemma 2.4 that \( \lambda_2 \in \text{iso}_{(a)}(T) \).

Define \( f_3(z) = (z - \lambda_1)(z - \lambda_2) \). Then, \( 0 < \text{null } f_3(T) < \infty \). By Lemma 2.5 and Lemma 2.9, it follows from \( \lambda_1 \in \text{iso } \sigma_a(T) \) and \( \lambda_2 \in \text{iso } \sigma_a(T) \) that \( 0 \in \text{iso } \sigma_a(f_3(T)) \). Hence, we have \( 0 \in \pi_{00}(f_3(T)) \). Since \( f_3(T) \in (a-W) \), by Lemma 3.2, \( f_3(T) \) is semi-Fredholm and \( \text{ind } f_3(T) \leq 0 \). Note that \( f_3(T) = (T - \lambda_1)(T - \lambda_2) \), then we deduce that \( T - \lambda_2 \) is semi-Fredholm, a contradiction.

”. Arbitrarily choose an \( f \in \text{Hol}(\sigma(T)) \). It suffices to prove that \( f(T) \in (a-W) \). 

**Step 1.** \( \sigma_a(f(T)) \setminus \sigma_{ea}(f(T)) \subseteq \pi_{00}(f(T)) \).

Let \( \lambda_0 \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T)) \) be fixed. Then, by Lemma 3.1, \( 0 < \text{null } f(T) - \lambda_0 < \infty \) and \( \text{ind } (f(T) - \lambda_0) \leq 0 \). It suffices to prove that \( \lambda_0 \in \pi_{00}(f(T)) \). By Lemma 2.6, we have \( \text{card } \{ z \in \sigma(T) : f(z) - \lambda_0 = 0 \} < \infty \). Assume \( \{ \lambda_i \}_{i=1}^n \) is an enumeration of \( \{ z \in \sigma(T) : f(z) - \lambda_0 = 0 \} \). Then, by Lemma 2.8, we may assume that \( f(z) - \lambda_0 = (z - \lambda_1)^{k_1} \cdots (z - \lambda_n)^{k_n} g(z) \), where \( g(z) \neq 0 \) for all \( z \in \sigma(T) \). Hence, \( f(T) - \lambda_0 = (T - \lambda_1)^{k_1} \cdots (T - \lambda_n)^{k_n} g(T) \), where \( g(T) \) is invertible.

It follows from \( \text{ind } (f(T) - \lambda_0) \leq 0 \) that \( \lambda_i \in \rho_{s-F}(T) \) and \( \text{ind } (\lambda_i - T) < \infty \) for all \( i \). Then, \( \sum_{i=1}^n k_i \cdot \text{ind } (T - \lambda_i) = \text{ind } (\lambda_0 - f(T)) \leq 0 \). It follows from condition (ii) that \( \text{ind } (T - \lambda_i) \leq 0 \) for \( 1 \leq i \leq n \). By Corollary 3.3, for each \( 1 \leq i \leq n \), we have either \( \lambda_i \notin \sigma_a(T) \) or \( \lambda_i \in \text{iso } \sigma_a(T) \). Then, by Lemma 2.9, either \( \lambda_0 \notin \sigma_a(f(T)) \) or \( \lambda_0 \in \text{iso } \sigma_a(f(T)) \). Since \( 0 < \text{null } f(T) - \lambda_0 < \infty \), we can conclude that \( \lambda_0 \in \pi_{00}(f(T)) \).

**Step 2.** \( \pi_{00}(f(T)) \subseteq \sigma_a(f(T)) \setminus \sigma_{ea}(f(T)) \).

Arbitrarily choose a \( \lambda_0 \in \pi_{00}(f(T)) \). Then, \( 0 < \text{null } f(T) - \lambda_0 < \infty \) and \( \lambda_0 \in \text{iso } \sigma_a(f(T)) \). By Lemma 3.1 and Lemma 2.4, it suffices to prove that \( \lambda_0 \in \rho_{s-F}(f(T)) \). Note that \( \text{null } f(T) - \lambda_0 < \infty \), then, by Lemma 2.6 and Lemma 2.8, we may assume that \( \{ \lambda_i \}_{i=1}^n \) is an enumeration of \( \{ \lambda \in \sigma(T) : f(\lambda) - \lambda_0 = 0 \} \) and \( f(z) - \lambda_0 = (z - \lambda_1)^{k_1} \cdots (z - \lambda_n)^{k_n} g(z) \), where \( g(z) \neq 0 \) for all \( z \in \sigma(T) \). Then, \( f(T) - \lambda_0 = (T - \lambda_1)^{k_1} \cdots (T - \lambda_n)^{k_n} g(T) \), where \( g(T) \) is invertible.

Since \( \lambda_0 \in \text{iso } \sigma_a(f(T)) \), it follows from Lemma 2.6 that there exist \( \delta_i > 0 \) such that \( B_{\delta_i}(\lambda_i) \setminus \{ \lambda_i \} \subseteq \mathbb{C} \setminus \sigma_a(T) \) (\( 1 \leq i \leq n \)). Then, for each \( i \), either \( \lambda_i \notin \sigma_a(T) \) or \( \lambda_i \in \text{iso } \sigma_a(T) \). So, it remains to prove that \( \lambda_i \in \rho_{s-F}(T) \) for all \( i \). Now let \( i \) be fixed and, without loss of generality, we assume that \( \lambda_i \in \text{iso } \sigma_a(T) \).

Since \( 0 < \text{null } f(T) - \lambda_0 < \infty \), there exists some \( i_0 \) (\( 1 \leq i_0 \leq n \)) such that \( 0 < \text{null } (T - \lambda_{i_0}) < \infty \). Hence, \( \lambda_{i_0} \in \pi_{00}^a(T) \). Since \( T \in (a-W) \), by Lemma 3.2, we have \( \lambda_{i_0} \in \rho_{s-F}(T) \) and \( \text{ind } \lambda_{i_0} = \text{null } \lambda_{i_0} - f(T) \). By condition (iii), we have \( \lambda_i \in \sigma_p(T) \). Note that \( 0 < \text{null } (\lambda_i - T) \leq \text{null } (\lambda_0 - f(T)) < \infty \), then \( \lambda_i \in \pi_{00}^a(T) \) and, using Lemma 3.2 again, we have \( \lambda_i \in \rho_{s-F}(T) \). Thus, we conclude the proof.
If one checks the proof of Main Theorem 1.1, then one can easily obtain the following result.

**Corollary 3.4.** Let \( T \in B(X) \). Then, \( f(T) \in (a-W) \) for all \( f \in \text{Hol}(\sigma(T)) \) if and only if \( p(T) \in (a-W) \) for each polynomial \( p(\lambda) \).

### 4. Proof of Main Theorem 1.2

**Lemma 4.1.** Let \( T \in B(X) \). Then, \( T \in (\omega) \) if and only if 
\[(i) \quad \sigma(T) = \sigma_w(T) \cup \sigma_0(T),
(ii) \quad \pi_{\sigma_0}(T) \subseteq \sigma_0(T),\] and
\[(iii) \quad \rho_{\sigma_0}(T) \cap \sigma_0(T) = \emptyset.\]

**Proof.** By Lemma 3.1, \( T \in (\omega) \) if and only if \([\rho_{\sigma_0}(T) \cup \rho_0^0(T)] \cap \sigma_0(T) = \pi_{\sigma_0}(T)\).

“\( \Rightarrow \)”. It follows from \( T \in (\omega) \) that 
\[\pi_{\sigma_0}(T) = [\rho_{\sigma_0}(T) \cup \rho_0^0(T)] \cap \sigma_0(T) = [\rho_{\sigma_0}(T) \cap \sigma_0(T)] \cup [\rho_0^0(T) \cap \sigma_0(T)].\]

Since \( \pi_{\sigma_0}(T) \subseteq \sigma_0(T) \) and \( \rho_{\sigma_0}(T) \subseteq \sigma_0(T) \), it follows that \( \rho_{\sigma_0}(T) \cap \sigma_0(T) = \emptyset \) and \([\rho_0^0(T) \cap \sigma_0(T)] \subseteq \sigma_0(T) \). By Lemma 2.3, \( [\rho_0^0(T) \cap \sigma_0(T)] \subseteq \sigma_0(T) \). Hence, 
\[\sigma(T) = \sigma_w(T) \cup \sigma_0(T).\]

On the other hand, \( \pi_{\sigma_0}(T) \subseteq [\sigma_0(T) \setminus \sigma_{\text{ev}}(T)] \) implies that \( \pi_{\sigma_0}(T) \subseteq \rho_{\sigma_0}(T) \). Then, by Lemma 2.3, we have \( \pi_{\sigma_0}(T) \subseteq \sigma_0(T) \).

“\( \Leftarrow \)”. By (i) and (iii), it is obvious that 
\[\sigma_0(T) \setminus \sigma_{\text{ev}}(T) = \sigma_0(T) \subseteq \pi_{\sigma_0}(T).\] This combining (ii) implies that \( \sigma_0(T) \setminus \sigma_{\text{ev}}(T) = \pi_{\sigma_0}(T) \).

**Proof of Main Theorem 1.2 \( \Rightarrow \)”. Assume that \( f(T) \in (\omega) \) for all \( f \in \text{Hol}(\sigma(T)) \).

(i) Since \( f(T) \in (\omega) \) for all \( f \in \text{Hol}(\sigma(T)) \), we have \( T = f_1(T) \in (\omega) \), where \( f_1(\lambda) = \lambda \).

(ii) If (ii) does not hold, then we can choose \( \lambda_1 \in \rho_{\sigma_0}(T) \) and \( \lambda_2 \in [\rho_{\sigma_0}(T) \cap \sigma_0(T)] \) such that \( 0 \leq \text{ind}(T - \lambda_2) < \infty \). Obviously we can choose \( k \in \mathbb{N} \) such that \( k \cdot \text{ind}(T - \lambda_1) + \text{ind}(T - \lambda_2) < 0 \). Set \( f_2(\lambda) = (\lambda - \lambda_1)^k(\lambda - \lambda_2) \). Then, \( f_2(T) = (T - \lambda_1)^k(T - \lambda_2) \) is a semi-Fredholm operator and 
\[\text{ind} f_2(T) = \text{ind}(T - \lambda_1)^k(T - \lambda_2) = k \cdot \text{ind}(T - \lambda_1) + \text{ind}(T - \lambda_2) < 0.\]

Evidently, \( \text{nul} f_2(T) \geq \text{nul}(T - \lambda_2) > 0 \). Thus \( 0 \in [\rho_{\sigma_0}(f_2(T)) \cap \sigma_0(f_2(T))] \neq \emptyset \). By Lemma 4.1, we obtain \( f_2(T) \notin (\omega) \), a contradiction.

(iii) If (iii) does not hold, then we can choose \( \lambda_1 \in \sigma_0(T) \) and \( \lambda_2 \in \sigma_0(T) \) such that \( \lambda_2 \notin \sigma_0(T) \). Thus, \( T - \lambda_2 = 0 \) and, by Lemma 2.3, we have \( \lambda_2 \in \sigma_{\text{ev}}(T) \).

Set \( f_3(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \). It is easy to verify that \( 0 \in \sigma_{\text{ev}}(f_3(T)) \) and \( 0 < \text{nul} f_3(T) = \text{nul}(T - \lambda_1) < \infty \). On the other hand, since \( \lambda_1, \lambda_2 \in \sigma(T) \), by Corollary
By Lemma 4.1, we have $0 \in \sigma(f_3(T))$. Thus, we obtain that $0 \in \pi_{00}(f_3(T))$. Since $0 \in \sigma_{\rho_{t-F}}(f_3(T))$, by Lemma 4.1, it follows that $f_3(T) \notin (\omega)$, a contradiction.

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\[ \lambda \]

“\( \Leftarrow \)”. Arbitrarily choose an $f \in \text{Hol}(\sigma(T))$. We shall prove that $f(T) \in (\omega)$.

\textbf{Step 1.} $[\sigma_{\alpha}(f(T)) \setminus \sigma_{\alpha}(f(T))] \subseteq \pi_{00}(f(T))$.

Arbitrarily choose a $\lambda_0 \in [\sigma_{\alpha}(f(T)) \setminus \sigma_{\alpha}(f(T))]$. Then, by Lemma 3.1, $0 < \nul(f(T) - \lambda_0) < \infty$ and $\text{ind}(\lambda_0 - f(T)) \leq 0$. It suffices to prove that $\lambda_0 \notin \sigma(f(T))$.

By Lemma 2.6 and Lemma 2.8, we may assume that $\{\lambda_i\}_{i=1}^{n}$ is an enumeration of $\{z \in \sigma(T) : f(z) - \lambda_0 = 0\}$ and

\[ f(z) - \lambda_0 = (z - \lambda_1)^{k_1} \cdots (z - \lambda_n)^{k_n}g(z), \]

where $g(z) \neq 0$ for all $z \in \sigma(T)$. Hence,

\[ f(T) - \lambda_0 = (T - \lambda_1)^{k_1} \cdots (T - \lambda_n)^{k_n}g(T), \]

where $g(T)$ is invertible.

It follows from $\text{ind}(f(T) - \lambda_0) \leq 0$ that $\lambda_i \in \rho_{t-F}(T)$ and $\text{ind}(\lambda_i - T) < \infty$ for all $i$. We claim that $\text{ind}(\lambda_i - T) \geq 0$ for all $1 \leq i \leq n$. In fact, if not, then there exists some $i_0$ such that $\text{ind}(T - \lambda_i) < 0$. By condition (ii), $\sigma_0(T) = \emptyset$ and $\text{ind}(T - \lambda_i) \leq 0$ for all $i$. By Lemma 4.1, it follows from $T \in (\omega)$ that $\sigma(T) = \sigma_\omega(T)$, $\text{ind}(T - \lambda_i) < 0$ and $T - \lambda_i$ is bounded below for all $1 \leq i \leq n$. Furthermore, $f(T) - \lambda_0$ is bounded below. Then, $\lambda_0 \notin \sigma_\alpha(f(T))$, a contradiction. Thus, we have proved that $\text{ind}(T - \lambda_i) \geq 0$ for all $1 \leq i \leq n$.

Since $\sum_{i=1}^{n} k_i \cdot \text{ind}(T - \lambda_i) = \text{ind}(\lambda_0 - f(T)) \leq 0$, we deduce that $\text{ind}(T - \lambda_i) = 0$ for all $i$. Note that $T \in (\omega)$ and $\lambda_i \in \sigma(T)$, it follows from Lemma 4.1 that $\lambda_i \in \sigma_\alpha(T)$ for all $i$. In view of the form of $f(\lambda)$, it follows from Corollary 2.10 that $\lambda_0 \notin \sigma(f(T))$.

\textbf{Step 2.} $\pi_{00}(f(T)) \subseteq [\sigma_{\alpha}(f(T)) \setminus \sigma_{\alpha}(f(T))]$.

Arbitrarily choose a $\lambda_0 \in \pi_{00}(f(T))$. Then, $0 < \nul(f(T) - \lambda_0) < \infty$ and $\lambda_0 \in \sigma(f(T))$. By Lemma 2.4 and Lemma 3.1, it suffices to prove that $\lambda_0 \in \rho_{t-F}(f(T))$.

Note that $\nul(f(T) - \lambda_0) < \infty$, then, by Corollary 2.7 and Lemma 2.8, we may assume that $\{\lambda_i\}_{i=1}^{n}$ is an enumeration of $\{\lambda \in \sigma(T) : f(\lambda) - \lambda_0 = 0\}$ and

\[ f(z) - \lambda_0 = (z - \lambda_1)^{k_1} \cdots (z - \lambda_n)^{k_n}g(z), \]

where $g(z) \neq 0$ for all $z \in \sigma(T)$ and $\lambda_i \in \sigma(T)$ for all $i$. Then,

\[ f(T) - \lambda_0 = (T - \lambda_1)^{k_1} \cdots (T - \lambda_n)^{k_n}g(T), \]

where $g(T)$ is invertible.

Since $0 < \nul(f(T) - \lambda_0) < \infty$, there exists some $i_0(1 \leq i_0 \leq n)$ such that $0 < \nul(T - \lambda_{i_0}) < \infty$. Hence, $\lambda_{i_0} \in \pi_{00}(T)$. Since $T \in (\omega)$, by Lemma 4.1, we have $\lambda_{i_0} \in \sigma_0(T)$. So, $\sigma_0(T) \neq \emptyset$ and, by condition (iii), $\sigma(T) \subseteq \sigma_\rho(T)$. It follows that $\lambda_{i_0} \in \sigma_\rho(T)$ for all $i$. Note that $\nul(T - \lambda_i) \leq \nul(f(T) - \lambda_0) < \infty$ for all $i$, we deduce that $\lambda_{i_0} \in \pi_{00}(T)$. Using Lemma 4.1 again, we obtain $\lambda_i \in \sigma_\alpha(T)$ for all $i$. Therefore, we conclude that $\lambda_0 \in \rho_{t-F}(f(T))$.

It can be seen from the proof of Main Theorem 1.2 that the following corollary is clear.

\textbf{Corollary 4.2.} Let $T \in B(X)$. Then, $f(T) \in (\omega)$ for all $f \in \text{Hol}(\sigma(T))$ if and only if $p(T) \in (\omega)$ for each polynomial $p(\lambda)$. 

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5. Proof of Main Theorem 1.3. In this section, it is always assumed that $\mathcal{X}$ is a complex separable infinite dimensional Hilbert space. We first give several useful lemmas.

**Lemma 5.1** ([15], Theorem 3.48). Let $T \in B(\mathcal{X})$. Then, given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{X})$ such that $\|K\| < \varepsilon + \max\{\text{dist}[(\lambda, \partial \rho_{s-F}(T)) : \lambda \in \sigma_0(T)\}$ and $\min \text{ind} (T + K - \lambda) = 0$ for all $\lambda \in \rho_{s-F}(T)$.

For $T \in B(\mathcal{X})$ and $\lambda \in \rho_{s-F}(T)$, the minimal index (see [15]) of $\lambda - T$ is defined by $\min \text{ind} (\lambda - T) := \min\{\text{nul} (\lambda - T), \text{nul} (\lambda - T)\}^*.$

**Lemma 5.2** ([16], Proposition 3.4). Let $T \in B(\mathcal{X})$, if $\sigma_0(T) = \emptyset$, then, given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{X})$ with $\|K\| < \varepsilon$ such that $\sigma_p(T + K) = \rho_{s-F}^+(T)$.

**Proof of Main Theorem 1.3.** For given $\varepsilon > 0$, set $\sigma_1 = \{\lambda \in \sigma_0(T) : \text{dist}(\lambda, \partial \rho_{s-F}(T)) \geq \varepsilon\}$. Then, $\sigma_1$ is a finite, clopen subset of $\sigma(T)$. Set $\sigma_2 = \sigma(T) \setminus \sigma_1$. By Corollary 2.2, $T$ admits the following representation

$$T = \begin{bmatrix} T_1 & E \\ 0 & T_2 \end{bmatrix} \mathcal{X}(\sigma_1; T) \mathcal{X}(\sigma_1; T)^\perp,$$

where $\sigma(T_i) = \sigma_i(i = 1, 2)$. Then, one can verify that $\max\{\text{dist}[(\lambda, \partial \rho_{s-F}(T_2)) : \lambda \in \sigma_0(T_2)]\} < \frac{\varepsilon}{2}$. Then, by Lemma 5.1, there exists a compact operator $K_1$ on $\mathcal{X}(\sigma_1; T)^\perp$ such that $\|K_1\| < \frac{\varepsilon}{2}$ and $\min \text{ind} (T_2 + K_1 - \lambda) = 0$ for all $\lambda \in \rho_{s-F}(T_2)$. Then,

$$\sigma(T_2 + K_1) = \sigma_{\text{ire}}(T_2 + K_1) \cup [\rho_{s-F}(T_2 + K_1) \cap \sigma(T_2 + K_1)]$$

$$= \sigma_{\text{ire}}(T_2) \cup \rho_{s-F}^+(T_2 + K_1) \cup \rho_{s-F}^-(T_2 + K_1) \cup [\rho_{s-F}(T_2 + K_1) \cap \sigma(T_2 + K_1)]$$

$$= \sigma_{\text{ire}}(T_2) \cup \rho_{s-F}^+(T_2 + K_1) \cup \rho_{s-F}^-(T_2 + K_1) \subset \sigma(T_2). \quad (1)$$

In particular, $\sigma_0(T_2 + K_1) = \emptyset$ and $\sigma(T_2 + K_1) \cap \sigma(T_1) = \emptyset$.

Using Lemma 5.2, one can find a compact $K_2$ with $\|K_2\| < \varepsilon/2$ such that $\sigma_p(T_2 + K_1 + K_2) = \rho_{s-F}^+(T_2 + K_1 + K_2)$. Set

$$T_2 = T_2 + K_1 + K_2 \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 \\ 0 & K_1 + K_2 \end{bmatrix} \mathcal{X}(\sigma_1; T) \mathcal{X}(\sigma_1; T)^\perp.$$

Then, $K \in \mathcal{K}(\mathcal{X})$, $\|K\| < \varepsilon$ and

$$T + K = \begin{bmatrix} T_1 & E \\ 0 & T_2 \end{bmatrix} \mathcal{X}(\sigma_1; T) \mathcal{X}(\sigma_1; T)^\perp.$$

Also, we claim that

(i) $\sigma(T + K) = \sigma(T_1) \cup \sigma(T_2)$ and $\sigma(T_1) \cap \sigma(T_2) = \emptyset$,

(ii) $\sigma_{\text{ire}}(T + K) = \sigma_{\text{ire}}(T_2)$ and $\text{ind} (T + K - \lambda) = \text{ind} (T_2 - \lambda)$ for all $\lambda \in \rho_{s-F}(T + K)$,

(iii) $\sigma_p(T + K) = \sigma_p(T_1) \cup \sigma_p(T_2) = \sigma(T_1) \cup \rho_{s-F}^+(T_2) = \sigma(T_1) \cup \rho_{s-F}^+(T + K)$,

(iv) $\sigma_0(T + K) = \sigma(T_1) = \pi_{00}(T) = \pi_{00}(T)$ (since $\sigma_p(T_2) = \rho_{s-F}^+(T_2)$).

Now let us explain in detail the above facts (i)--(iv).

(i) Using a similar argument as in the equality (1), one can see that $\sigma(T_1) \cap \sigma(T_2) = \emptyset$. By [15, Corollary 3.22], $T + K$ and $T_1 \oplus T_2$ are similar. Then, $\sigma(T + K) = \sigma(T_1) \cup \sigma(T_2).$
(ii) Since \( \dim \mathcal{X}(\sigma_1; T) < \infty \), one can see that \( T_1 \) and \( E \) are both compact. Thus, \( T + K \) is a compact perturbation of the following operator

\[
\begin{bmatrix}
0 & 0 \\
0 & T_2
\end{bmatrix}
\mathcal{X}(\sigma_1; T) \mathcal{X}(\sigma_1; T)'.
\]

Thus, the facts in (ii) are clear.

(iii) We have proved in (i) that \( T + K \) and \( T_1 \oplus T_2 \) are similar. Then, \( \sigma_p(T + K) = \sigma_p(T_1) \cup \sigma_p(T_2) \). Note that \( T_1 \) is acting on a finite-dimensional space, we have \( \sigma(T_1) = \sigma_p(T_1) \). On the other hand, we have proved that \( \sigma_p(T_2) = \rho_{a-W}(T_2) = \rho_{a-W}(T + K) \). This proves (iii).

(iv) Since \( T + K \) and \( T_1 \oplus T_2 \) are similar, using the facts (i)-(iii), one can easily verify the conditions in (iv).

Based on the facts (i)–(iv), we obtain

\[
[\rho_{a-W}(T + K) \cup \rho_0^{a-W}(T + K)] \cap \sigma_p(T + K) = \rho_0^{a-W}(T + K) \cap \sigma_p(T + K) = \sigma_0(T + K) = \sigma(T_1) = \pi \sigma_0(T) = \pi \sigma_0(T).
\]

By Lemma 3.1, we deduce that \( T + K \in (a-W) \) and \( T + K \in (\omega) \).

Remark 5.3. As we have seen in the above proof, the result of Theorem 1.3 greatly depends on the work by D. Herrero \[16\] on perturbations of Hilbert space operators, and therefore, the result is established only in the setting of separable Hilbert spaces.

We conclude this paper with the following question.

**Question 5.4.** Let \( \mathcal{X} \) be a complex infinite dimensional Banach space. Then, given \( T \in B(\mathcal{X}) \) and \( \varepsilon > 0 \), can one find \( A \in B(\mathcal{X}) \) with \( \| A - T \| < \varepsilon \) such that \( A \in (W) \) (or \( A \in (\omega) \), or \( A \in (a-W) \))?  

**REFERENCES**