A NOTE ON SOME INEQUALITIES

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(Received 17th May, 1958)

1. In the course of some recent work on Fourier series [**5**, **6**] I had occasion to use a number of integral inequalities which were generalizations or limiting cases of known results. These inequalities may perhaps have other applications, and it seems worth while to collect them together in a separate note with one or two further results of a similar nature.

For any number k, used as an index (exponent), and such that $k>1$, we write $k'=k/(k-1)$, so that k and k' are conjugate indices in the sense of Hölder's inequality.

We use B to denote a positive constant depending on the parameters $c, d, \ldots$ concerned in the particular problem in which it appears. If we wish to express the dependence explicitly, we write B in the form $B(c, d, \ldots)$. We use A to denote a positive absolute constant. These constants are not necessarily the same on any two occurrences. We also use suffixes to distinguish particular B's which retain their identity throughout.

Inequalities of the form

$$L \leqslant B(c, d, \ldots)R$$

are to be interpreted as meaning " if the expression R is finite, then the expression L is finite and satisfies the inequality ".

2.1. The first of our inequalities is the following generalization of Hardy's inequality.†

THEOREM 1. *Let $f(t) \geqslant 0$ in $t \geqslant 0$, and let*

$$F(t)=\int_0^t f(u)\,du \quad (\gamma>-1), \quad F(t)=\int_t^\infty f(u)\,du \quad (\gamma<-1).$$

If $q \geqslant p \geqslant 1$ and $\gamma \neq -1$, then

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{F}{t}\right)^q dt\right\}^{1/q} \leqslant B\left\{\int_0^\infty t^{-1-p\gamma} f^p\,dt\right\}^{1/p}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1.1)$$

The case $q=p$ of this inequality is due to Hardy [**8**], while the case $q>p>1$, $\gamma=-1/p$ is due to Hardy and Littlewood [**9**]. Hardy and Littlewood also conjectured the exact value of the constant B in this latter case, and their conjecture was proved by Bliss [**1**]. The complete result above is stated without proof in [**6**].

The case $\gamma<-1$ of Theorem 1 is an easy deduction from the case $\gamma>-1$. For if $\gamma<-1$, and if

$$t=\frac{1}{x} \quad \text{and} \quad \frac{1}{x^2}f\left(\frac{1}{x}\right)=g(x),$$

then

$$F(t)=\int_t^\infty f(u)\,du=\int_0^x \frac{1}{v^2}f\left(\frac{1}{v}\right)dv=\int_0^x g(v)\,dv=G(x),$$

say. Applying (2.1.1) to g (with γ replaced by $-\gamma-2>-1$), we obtain

† Hardy's inequality is the case $q=p>1$, $\gamma=-1/p$ of Theorem 1 (see Hardy, Littlewood, and Pólya [**14**, Theorem 327]).

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{F(t)}{t}\right)^q dt\right\}^{1/q} = \left\{\int_0^\infty x^{-1+q\gamma+2q}\left(\frac{G(x)}{x}\right)^q dx\right\}^{1/q} \leqslant B\left\{\int_0^\infty x^{-1+p(\gamma+2)}g^p(x)\,dx\right\}^{1/p}$$
$$= B\left\{\int_0^\infty t^{-1-p\gamma}f^p(t)\,dt\right\}^{1/p},$$

and this is the result of the theorem for f.†

To prove the inequality (2.1.1) when $\gamma > -1$, we may reduce the general case $q \geqslant p$ to the special case $q = p$ considered by Hardy. Alternatively, we may generalize the inequality by introducing fractional integrals, and then we obtain the following theorem.

THEOREM 2. *Let $f(t) \geqslant 0$ in $t \geqslant 0$, and let $F_\alpha(t)$ be the α-th Riemann-Liouville integral of f with origin* 0, *i.e.*

$$F_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u)\,du.$$

If either $q \geqslant p \geqslant 1$ and $\alpha > \frac{1}{p} - \frac{1}{q}$, or $q > p > 1$ and $\alpha = \frac{1}{p} - \frac{1}{q}$, and if $\gamma > -1$, then

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{F_\alpha}{t^\alpha}\right)^q dt\right\}^{1/q} \leqslant B\left\{\int_0^\infty t^{-1-p\gamma} f^p\,dt\right\}^{1/p}. \quad \text{.........................(2.1.2)}$$

Various cases of this theorem are known. Thus the case $q>p>1$, $\gamma = -1/p$ is due to Hardy and Littlewood [**10**, Theorem 4], the case $q = p > 1$, $\gamma = -1/p$ to Knopp‡ [**15**], and the case $q = p = 1$, $\gamma \geqslant 0$ to Bosanquet [**2**]. Hardy and Littlewood have given a proof of a result [**10**, Theorem 7] which implies (2.1.2) for $-1 < \gamma < 0$, but their proof here appears to be incomplete.§ They have also stated the case $q \geqslant p \geqslant 1$, $\gamma = 0$ as the integral analogue of an inequality for series [**13**, Theorem 5].

The general result (2.1.2) can be proved by appropriate specialization of an argument used by Hardy and Littlewood [**12**] to prove an inequality for series. However, in one of our later applications we require an explicit value for the constant B in one particular case of (2.1.2), and it seems advisable therefore to give the proof of Theorem 2 in full.‖

2.2. Suppose first that $q \geqslant p > 1$. Let

$$J = \left\{\int_0^\infty t^{-1-p\gamma} f^p\,dt\right\}^{1/p},$$

let $\alpha > 1/p - 1/q$, and let λ and η be numbers, to be chosen later, such that $\lambda < 1/p'$, $0 < \eta < 1$, and $p'(\alpha - 1)(1-\eta) > -1$. Applying Hölder's inequality with indices q, p', and $pq/(q-p)$, we have ¶

$$F_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u)\,du$$

† This argument is given by Hardy [**8**].

‡ See also Hardy, Littlewood, and Pólya [**14**], Theorem 329.

§ Their Theorem 7 depends on their Theorem 6, and the discussion of the region A' in part (i) of the proof of the latter theorem does not seem to be valid.

‖ I have used this same argument in [7] to prove the series analogue of Theorem 2, but have not attempted there to determine an explicit value for the constant.

¶ The last factor in (2.2.1) is to be omitted if $p = q$.

$$= \frac{1}{\Gamma(\alpha)}\int_0^t \{(t-u)^{(\alpha-1)\eta}\, u^{\lambda+(1+p\gamma)(q-p)/(pq)} f^{p/q}(u)\}\{(t-u)^{(\alpha-1)(1-\eta)}\, u^{-\lambda}\}\,\{u^{-1-p\gamma} f^p(u)\}^{(q-p)/(pq)}\,du$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\left\{\int_0^t (t-u)^{q(\alpha-1)\eta}\, u^{q\lambda+(1+p\gamma)(q-p)/p} f^p(u)\,du\right\}^{1/q}\left\{\int_0^t (t-u)^{p'(\alpha-1)(1-\eta)}\, u^{-p'\lambda}\,du\right\}^{1/p'}$$

$$\times\left\{\int_0^t u^{-1-p\gamma} f^p(u)\,du\right\}^{(q-p)/(pq)} \qquad \text{...........(2.2.1)}$$

$$\leqslant B_1 J^{1-p/q}\, t^{(\alpha-1)(1-\eta)-\lambda+1/p'}\left\{\int_0^t (t-u)^{q(\alpha-1)\eta}\, u^{q\lambda+(1+p\gamma)(q-p)/p} f^p(u)\,du\right\}^{1/q},$$

where

$$B_1 = \frac{1}{\Gamma(\alpha)}\left\{\int_0^1 (1-v)^{p'(\alpha-1)(1-\eta)}\, v^{-p'\lambda}\,dv\right\}^{1/p'}. \qquad \text{........................(2.2.2)}$$

Write now $\mu = q(\alpha-1)\eta$ and $\omega = q(\lambda+\gamma+1/p)$. Then we have

$$t^{-1-q\gamma-q\alpha} F_\alpha^q(t) \leqslant B_1^q\, J^{q-p}\, t^{-1-\mu-\omega}\int_0^t (t-u)^\mu\, u^{\omega-1-p\gamma} f^p(u)\,du,$$

whence

$$\int_0^\infty t^{-1-q\gamma}\left(\frac{F_\alpha}{t^\alpha}\right)^q dt \leqslant B_1^q\, J^{q-p}\int_0^\infty u^{-1-p\gamma} f^p(u)\, k(u)\,du,$$

where

$$k(u) = u^\omega\int_u^\infty t^{-1-\mu-\omega}(t-u)^\mu\,dt = \int_0^1 v^{-1+\omega}(1-v)^\mu\,dv = B_2, \text{ say.} \qquad \text{...........(2.2.3)}$$

Thus, provided that $\omega > 0$ and $\mu > -1$, we obtain (2.1.2) with $B = B_1 B_2^{1/q}$. Choose η so that $q\eta = p'(1-\eta)$, i.e. so that $\eta = p'/(p'+q)$. Then $0<\eta<1$ and

$$\alpha - 1 > \frac{1}{p} - \frac{1}{q} - 1 = -\left(\frac{1}{p'}+\frac{1}{q}\right) = -\frac{1}{q\eta} = -\frac{1}{p'(1-\eta)},$$

as required. Choose also λ so that $\lambda<1/p'$ and $\omega = q(\lambda+\gamma+1/p)>0$ (this is possible, since $\gamma+1>0$). The various conditions which we have imposed on λ and η are therefore satisfied, and this completes the proof in this case. A similar argument applies if $q\geqslant p=1$, $\alpha>1/p-1/q$. (Take $\lambda=0$, $\eta=1$, and omit the second bracketed factor in (2.2.1).)

2.3. We digress for a moment to determine a value for the constant B in (2.1.2) when $q>p>1$, $\alpha=1-1/q$, $\gamma=-1/p$, and $p'>q$. We are actually interested only in the order of magnitude of B when $p\to 1$, and it is convenient to choose values of λ and η different from those given above. From (2.2.2) and (2.2.3) the constant B is here given by

$$B = B_1 B_2^{1/q} = \frac{1}{\Gamma(1-1/q)}\left\{\frac{\Gamma(1-p'(1-\eta)/q)\,\Gamma(1-p'\lambda)}{\Gamma(2-p'(1-\eta)/q-p'\lambda)}\right\}^{1/p'}\left\{\frac{\Gamma(q\lambda)\,\Gamma(1-\eta)}{\Gamma(q\lambda+1-\eta)}\right\}^{1/q},$$

and λ and η have to satisfy the conditions

$$0<p'\lambda<1,\ 0<\eta<1, \text{ and } p'(1-\eta)/q<1. \qquad \text{....................(2.3.1)}$$

Choose λ and η so that $p'\lambda = p'(1-\eta)/q = \frac{1}{2}$. Then the conditions (2.3.1) are satisfied (since $q<p'$), and, by the duplication formula for the Γ-function,†

† See, for example, Copson [3], § 9.23.

$$B = \frac{\pi^{1/p'}}{\Gamma(1-1/q)}\left\{\frac{\Gamma^2\left(\frac{q}{2p'}\right)}{\Gamma\left(\frac{q}{p'}\right)}\right\}^{1/q} = \frac{\pi^{1/p'}}{\Gamma(1-1/q)}\left\{\frac{\pi^{1/2}\,\Gamma\left(\frac{q}{2p'}\right)}{2^{q/p'-1}\,\Gamma\left(\frac{q}{2p'}+\frac{1}{2}\right)}\right\}^{1/q}$$

$$\leqslant B(q)\,\Gamma^{1/q}\left(\frac{q}{2p'}\right) \leqslant B(q)\left(\frac{2p'}{q}\right)^{1/q} \leqslant \frac{B(q)}{(p-1)^{1/q}}, \qquad \text{(2.3.2)}$$

where the constants $B(q)$ depend only on q.

2.4. We return now to the remaining case of Theorem 2, namely $q>p>1$, $\alpha = 1/p - 1/q$. Write

$$\Gamma(\alpha)F_\alpha(t) = \int_0^t (t-u)^{\alpha-1} f(u)\,du = \int_0^{\frac{1}{2}t} + \int_{\frac{1}{2}t}^t = G(t) + H(t).$$

Then

$$G(t) \leqslant Bt^{\alpha-1}\int_0^{\frac{1}{2}t} f\,du \leqslant Bt^{\alpha-1}\int_0^t f\,du = Bt^{\alpha-1}F_1(t),$$

whence, by the case already proved,

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{G}{t^\alpha}\right)^q dt\right\}^{1/q} \leqslant B\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{F_1}{t}\right)^q dt\right\}^{1/q} \leqslant B\left\{\int_0^\infty t^{-1-p\gamma} f^p\,dt\right\}^{1/p}$$

It is therefore now enough to prove that

$$\left\{\int_0^\infty t^{-1-q\gamma}\left(\frac{H}{t^\alpha}\right)^q dt\right\}^{1/q} \leqslant B\left\{\int_0^\infty t^{-1-p\gamma} f^p\,dt\right\}^{1/p}. \qquad \text{(2.4.1)}$$

To prove this, it is sufficient, by the converse of Hölder's inequality, to prove that

$$\int_0^\infty t^{-1/q-\gamma-\alpha}\,hH\,dt \leqslant B\left\{\int_0^\infty t^{-1-p\gamma} f^p\,dt\right\}^{1/p}\left\{\int_0^\infty h^{q'}\,dt\right\}^{1/q'} \qquad \text{(2.4.2)}$$

for every h belonging to $L^{q'}(0, \infty)$. The integral on the left of (2.4.2) is equal to

$$\int_0^\infty t^{-1/q-\gamma-\alpha}\,h(t)\,dt\int_{\frac{1}{2}t}^t (t-u)^{\alpha-1} f(u)\,du$$

$$= \int_0^\infty\int_{\frac{1}{2}t}^t u^{-1/p-\gamma} f(u)\,h(t) u^{1/p+\gamma}\, t^{-1/q-\gamma-\alpha}\,(t-u)^{\alpha-1}\,du\,dt$$

$$\leqslant \int_0^\infty\int_{\frac{1}{2}t}^t u^{-1/p-\gamma} f(u)\,h(t)\,(t-u)^{\alpha-1}\,du\,dt \qquad \text{(2.4.3)}$$

(since in the region of integration $u^{1/p+\gamma} \leqslant Bt^{1/p+\gamma} = Bt^{1/q+\gamma+\alpha}$). Since $\alpha - 1 = 1/p + 1/q' - 2$, it follows by a well-known theorem of Hardy and Littlewood [**10**, Theorem 3]† that the last integral in (2.4.3) does not exceed the right-hand side of (2.4.2) (for some B). This completes the proof of (2.4.1), and of Theorem 2.

2.5. We note in passing that the case $q = p > 1$ of Theorem 2 is a particular case of the following general inequality (Hardy, Littlewood, and Pólya [**14**, Theorem 319]).

THEOREM A. *Suppose that $p>1$, that $K(t, u)$ is non-negative and homogeneous of degree -1, that*‡

† See also Hardy, Littlewood, and Pólya [**14**], Theorem 382.

‡ The equality of the two integrals in (2.5.1) is a consequence of the homogeneity of K.

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$$\int_0^\infty K(1, u)\, u^{-1/p}\, du = \int_0^\infty K(t, 1)\, t^{-1/p'}\, dt = k, \qquad \text{.....................(2.5.1)}$$

and that $f(t) \geqslant 0$ *in* $t \geqslant 0$. *Then*

$$\int_0^\infty dt \left(\int_0^\infty K(t, u) f(u)\, du \right)^p \leqslant k^p \int_0^\infty f^p\, dt.$$

Taking $K(t, u) = t^{-1/p-\gamma-\alpha}\, u^{1/p+\gamma}\, (t-u)^{\alpha-1}$ in $0 < u \leqslant t$, and 0 elsewhere, and observing that

$$\int_0^\infty K(1, u)\, u^{-1/p}\, du = \int_0^1 u^\gamma (1-u)^{\alpha-1}\, du = \frac{\Gamma(\alpha)\, \Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}$$

when $\alpha > 0$ and $\gamma > -1$, we obtain (2.1.2) with

$$B = \frac{\Gamma(\alpha)\, \Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}.$$

2.6. The result of Theorem 2† enables us to fill a gap in a theorem of the author on fractional integrals. Let f be periodic with period 2π and integrable in $(-\pi, \pi)$, let

$$\phi(t) = \Phi_0(t) = f(\theta + t) + f(\theta - t) \text{ and } \Lambda_1(t) = t\phi(t) - \int_0^t \phi\, du,$$

and for $t > 0$ let

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du \quad (\alpha > 0)$$

and

$$\Lambda_\alpha(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-u)^{\alpha-2} \Lambda_1(u)\, du \quad (\alpha > 1)$$

(so that Φ_α and Λ_α are the αth and $(\alpha-1)$th integrals of ϕ and Λ_1, respectively). Then we have

Theorem 3. *Let f be periodic with period 2π and integrable in $(-\pi, \pi)$, and let the mean value of f over this interval be* 0. *Let also*

$$\mathfrak{F}_{k,\alpha}(\theta) = \left\{ \int_0^\pi |\chi(t)|^k\, t^{-1-k\alpha}\, dt \right\}^{1/k},$$

where either $\chi = \Phi_\alpha$ and $\alpha \geqslant 0$, or $\chi = \Lambda_\alpha$ and $\alpha \geqslant 1$. Then for $1 < k \leqslant 2$ and $p > 1$

$$\left\{ \int_{-\pi}^\pi |f|^p\, d\theta \right\}^{1/p} \leqslant B \left\{ \int_{-\pi}^\pi \mathfrak{F}_{k,\alpha}^p\, d\theta \right\}^{1/p} + B \int_{-\pi}^\pi |f|\, d\theta.$$

This is proved in [**6**] for $\chi = \Phi_\alpha$ and $\alpha > 1/k$ and for $\chi = \Lambda_\alpha$ and $\alpha > 1 + 1/k$. Since $\mathfrak{F}_{k,\beta} \leqslant B(k, \alpha, \beta)\, \mathfrak{F}_{k,\alpha}$ for $\beta > \alpha$ (by Theorem 2), the complete result follows.

3.1. We pass next to some results which are substitutes for Theorems 1 and 2 when one (or more) of the parameters takes a limiting value. We have first

Theorem 4. *Let $f(t) \geqslant 0$ in $0 \leqslant t \leqslant a$, where a is finite, and let $F_\alpha(t)$ be the αth Riemann-Liouville integral of f with origin* 0. *If $q > 1$ and $\alpha = 1 - 1/q$, then*

† We require only the case $q = p > 1$ considered in § 2.5.

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$$\left\{\int_0^a F_\alpha^q\,dt\right\}^{1/q} \leqslant B\int_0^a f(\log^+ f)^{1/q}\,dt + B,$$

where the B's depend only on q and a.

This result is the substitute for Theorem 2 when all three of the parameters p, α, γ have their limiting values (i.e. $p=1$, $\alpha=1-1/q$, $\gamma=-1$). Zygmund [**18**] has proved the corresponding result for the Weyl fractional integral, and has remarked that his proof can be modified to apply to the Riemann-Liouville integral. We give here an alternative proof of Theorem 4 which is somewhat simpler than that of Zygmund.†

Setting $f=0$ in $t>a$, we see from § 2.3 that when $1<p<\text{Min}\,(q, q')$ and $\alpha=1-1/q$,

$$\left\{\int_0^a t^{-q/p'} F_\alpha^q\,dt\right\}^{1/q} \leqslant B(q)\,(p-1)^{-1/q}\left\{\int_0^a f^p\,dt\right\}^{1/p}.$$

Since $t^{-1/p'} > \text{Min}\,(a^{-1}, 1)$ in $0 \leqslant t \leqslant a$, we have therefore

$$\left\{\int_0^a F_\alpha^q\,dt\right\}^{1/q} \leqslant B(a, q)\,(p-1)^{-1/q}\left\{\int_0^a f^p\,dt\right\}^{1/p}$$

for $1<p<\text{Min}(q, q')$ and $\alpha=1-1/q$, and the result of Theorem 4 now follows immediately from the following extrapolation theorem.

THEOREM 5. *Let T be a transformation which transforms a real function integrable in $(0, a)$, where a is finite, into a function measurable in $(0, a)$, and has the properties*

(i) $$|T(f)| = |T(-f)|,$$

(ii) *for any real f_1 and f_2,*

$$|T(f_1+f_2)| \leqslant |T(f_1)| + |T(f_2)|,$$

(iii) *for any infinite sequence of non-negative functions f_n,*

$$\left|T\left(\sum_1^\infty f_n\right)\right| \leqslant \sum_1^\infty \left|T(f_n)\right|,$$

(iv) *there exist constants $k>0$, $p_0>1$, $q\geqslant 1$, and $C>0$, such that for every real f of $L^p(0, a)$, where $1<p<p_0$,*

$$\left\{\int_0^a |T(f)|^q\,dt\right\}^{1/q} \leqslant C(p-1)^{-k}\left\{\int_0^a |f|^p\,dt\right\}^{1/p}.$$

Then, for any integrable f,

$$\left\{\int_0^a |T(f)|^q\,dt\right\}^{1/q} \leqslant B\int_0^a |f|\,(\log^+|f|)^k\,dt + B,$$

where the constants B depend only on k, p_0, and C.

The case $q=1$ of this result is due to Yano [**17**].‡ The proof of the general case is similar to that given by Yano, but for the sake of completeness we give the proof here.

We may suppose $f\geqslant 2$, for, if this is not so, write

$$f_1 = \begin{cases} f+2 & \text{if } f \geqslant 0, \\ 2 & \text{if } f < 0, \end{cases} \qquad f_2 = \begin{cases} 2 & \text{if } f \geqslant 0, \\ -f+2 & \text{if } f < 0. \end{cases}$$

† In our proof we use only the relatively simple case $\alpha=1-1/q$, $\gamma=-1/p$ of Theorem 2 discussed in §§ 2.2–3. Zygmund, on the other hand, makes use of the case $q>p>1$, $\alpha=1/p-1/q$, $\gamma=-1/p$ of Theorem 2, which depends on a difficult theorem of Hardy and Littlewood (see § 2.4).

‡ It is also implicit in work of Titchmarsh [**16**].

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Then $f_1 \geqslant 2$, $f_2 \geqslant 2$, $f = f_1 - f_2$, and in virtue of (i) and (ii) it is enough to prove the theorem for f_1 and f_2.

Suppose then that $f \geqslant 2$. Let E_n be the subset of $(0, a)$ in which $n - 1 \leqslant f < n$, and let f_n be equal to f in E_n and to 0 elsewhere. Then, by (iii) and Minkowski's inequality,

$$\left\{\int_0^a |T(f)|^q \, dt\right\}^{1/q} \leqslant \left\{\int_0^a \left(\sum_3^\infty |T(f_n)|\right)^q dt\right\}^{1/q} \leqslant \sum_3^\infty \left\{\int_0^a |T(f_n)|^q \, dt\right\}^{1/q},$$

both sides possibly being $+\infty$. If now we take $p = 1 + \delta/\log n$ in (iv), where δ is positive and so small that $1 + \delta/\log n < p_0$ for $n \geqslant 3$, we obtain

$$\left\{\int_0^a |T(f_n)|^q \, dt\right\}^{1/q} \leqslant \frac{C}{(p-1)^k}\left\{\int_0^a |f_n|^p \, dt\right\}^{1/p} \leqslant \frac{C}{(p-1)^k} n \, |E_n|^{1/p}$$

$$= C\delta^{-k} n (\log n)^k \, |E_n|^{1-\delta/(\delta + \log n)}$$

$$\leqslant B n (\log n)^k \, |E_n| + B n^{-2} (\log n)^k,$$

where both B's are of the form $B(C, k, \delta)$ [to prove that

$$|E_n|^{1-\delta/(\delta+\log n)} \leqslant B\,|E_n| + Bn^{-3},$$

consider the cases $|E_n| \geqslant 1/n^3$ and $|E_n| \leqslant 1/n^3$ separately]. Hence

$$\left\{\int_0^a |T(f)|^q \, dt\right\}^{1/q} \leqslant B \sum_3^\infty n (\log n)^k \, |E_n| + B$$

$$\leqslant B \sum_3^\infty (n-1)\{\log(n-1)\}^k \, |E_n| + B$$

$$\leqslant B \int_0^a f (\log^+ f)^k \, dt + B,$$

where again both B's are of the form $B(C, k, \delta)$, and this proves the theorem.

3.2. If we combine Theorem 4 with the case $p = q > 1$, $\gamma = -1/p$ of Theorem 2, and observe that $(F_\alpha)_\beta = F_{\alpha+\beta}$ for $\alpha > 0$ and $\beta > 0$, we obtain the case $q > 1$ of the following result, which was stated without proof in [**6**].

THEOREM 6. *Let $f(t) \geqslant 0$ in $0 \leqslant t \leqslant a$, where a is finite, and let $F_1(t) = \int_0^t f \, du$. If $q \geqslant 1$, then*

$$\left\{\int_0^a t^{-1} F_1^q \, dt\right\}^{1/q} \leqslant B_3 \int_0^a f (\log^+ f)^{1/q} \, dt + B_4,$$

where B_3 and B_4 are of the form $B(a, q)$.

This result is, of course, a substitute for Theorem 1 when $p = 1$ and $\gamma = -1$. The case $q = 1$ has been proved by Hardy and Littlewood [**11**] by an application of Young's inequality, and their method extends easily to give an alternative proof of the case $q > 1$. Another proof of the case $q = 1$ is given in [**4**], and this shows that in this case we may take the constants B_3 and B_4 to be of the form A and Aa, respectively.

3.3. We conclude this note by using the case $q = 1$ of Theorem 6 to prove a substitute for Theorem A in the case $p = 1$. This was stated without proof in [**5**].

THEOREM 7. *Suppose that $K(t, u)$ is non-negative and homogeneous of degree -1, that*

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$$\sup_{0<u\leqslant 1} K(1, u) = k_1 < \infty \quad \text{and} \quad \int_0^1 K(t, 1)\, dt = k_2 < \infty,$$

and that $f(t) \geqslant 0$ *in* $(0, a)$, *where* a *is finite. Then*

$$\int_0^a dt \left\{ \int_0^a K(t, u) f(u)\, du \right\} \leqslant A(k_1 + k_2) \left\{ \int_0^a f \log^+ f\, dt + a \right\}$$

and

$$\left\{ \int_0^a dt \left(\int_0^a K(t, u) f(u)\, du \right)^\mu \right\}^{1/\mu} \leqslant B(k_1, k_2, a, \mu) \int_0^a f\, dt \quad (0 < \mu < 1).$$

In addition to the result of Theorem 6 we require also the following inequality (Zygmund [19, § 10.22]) : *if* $f \geqslant 0$ *in* $(0, a)$, *where* a *is finite, and if* $0 < \mu < 1$, *then*

$$\left\{ \int_0^a \left(\frac{F_1}{t} \right)^\mu dt \right\}^{1/\mu} \leqslant B(a, \mu) \int_0^a f\, dt. \qquad \text{(3.3.1)}$$

For $0 < t \leqslant a$ we have

$$\int_0^a K(t, u) f(u)\, du = \frac{1}{t} \int_0^t tK(t, u) f(u)\, du + \int_t^a K(t, u) f(u)\, du = P(t) + Q(t),$$

say. Writing $u = tv$, where $0 < v \leqslant 1$, we have

$$tK(t, u) = tK(t, tv) = K(1, v) \leqslant k_1,$$

so that

$$P(t) \leqslant \frac{k_1}{t} \int_0^t f\, du = k_1 \left(\frac{F_1(t)}{t} \right).$$

Hence, by the case $q = 1$ of Theorem 6 (with constants as determined in [4]) and (3.3.1),

$$\int_0^a P\, dt \leqslant Ak_1 \int_0^a f \log^+ f\, dt + Aak_1$$

and

$$\left\{ \int_0^a P^\mu\, dt \right\}^{1/\mu} \leqslant B(a, \mu)\, k_1 \int_0^a f\, dt.$$

Next, we have

$$\left\{ \int_0^a Q^\mu\, dt \right\}^{1/\mu} \leqslant B(a, \mu) \int_0^a Q\, dt$$

for $0 < \mu < 1$, and

$$\int_0^a Q\, dt = \int_0^a f(u)\, du \int_0^u K(t, u)\, dt = \int_0^a f(u)\, du \int_0^1 uK(uv, u)\, dv$$

$$= \int_0^a f(u)\, du \int_0^1 K(v, 1)\, dv = k_2 \int_0^a f(u)\, du$$

$$\leqslant Ak_2 \int_0^a f \log^+ f\, du + k_2 a,$$

and the result now follows from these relations and the fact that

$$\int_0^a dt \left\{ \int_0^a K(t, u) f(u)\, du \right\}^\mu \leqslant \int_0^a P^\mu\, dt + \int_0^a Q^\mu\, dt$$

for $0 < \mu \leqslant 1$.

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