Relatively hyperbolic groups with strongly shortcut parabolics are strongly shortcut

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Abstract

We show that a group that is hyperbolic relative to strongly shortcut groups is itself strongly shortcut, thus obtaining new examples of strongly shortcut groups. The proof relies on a result of independent interest: we show that every relatively hyperbolic group acts properly and cocompactly on a graph in which the parabolic subgroups act properly and cocompactly on convex subgraphs.

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1. *Introduction*

Strongly shortcut graphs and groups were introduced by the first named author [**Hod18**] who later generalised the strong shortcut property to rough geodesic metric spaces [**Hod20**]. The strong shortcut property is a very general form of nonpositive curvature condition satisfied by many spaces of interest in geometric group theory, metric graph theory and geometric topology. These include Gromov-hyperbolic spaces [**Hod18**], asymptotically CAT(0) spaces [**Hod20**], hierarchically hyperbolic spaces, coarse Helly metric spaces of uniformly bounded geometry [**HHP20**], 1-skeletons of finite dimensional CAT(0) cube complexes (i.e. median graphs), 1-skeletons of quadric complexes (i.e. hereditary modular graphs), 1-skeletons of systolic complexes (i.e. bridged graphs), standard Cayley graphs of Coxeter groups [**Hod18**] and all of the Thurston geometries except Sol [**HP**, **Kar11**]. Despite this surprisingly unifying nature, there are nonetheless important consequences for groups that act metrically properly and coboundedly on strongly shortcut geodesic metric spaces: finite presentability, polynomial isoperimetric function and thus decidable word problem [**Hod18**, **Hod20**].

The strong shortcut property is essentially about limitations on the scale and precision at which subspaces can approximate circles. Specifically:

Definition 1·1 (Strongly shortcut). A graph Γ is *strongly shortcut* if, for some $K > 1$ there is a bound on the lengths of the K-bilipschitz combinatorial cycles in Γ. A group G is *strongly shortcut* if G acts properly and cocompactly on a strongly shortcut graph.

This turns out to be equivalent to the existence of a metrically proper and cobounded G-action on a strongly shortcut geodesic metric space, which we define in Section 4. Thus the following classes of groups are all strongly shortcut: hyperbolic groups [**Gro87**], asymptotically CAT(0) groups [**Kar11**] (e.g. CAT(0) groups [**BH99**]), hierarchically hyperbolic groups [**BHS17**, **BHS19**] (e.g. mapping class groups of surfaces [**MM99**, **MM00**]), coarse Helly groups [**CCG+20**] (e.g. Artin groups of FC-type, weak Garside groups [**HO19**]), the discrete Heisenberg group [**HP**], systolic groups (e.g. finitely presented $C(6)$ small cancellation groups [**Wis03**]) and quadric groups (e.g. $C(4)$-$T(4)$ small cancellation groups) [**Hod17**].

Our main result is the following.

THEOREM 1·2. *Let G be a finitely generated group that is hyperbolic relative to strongly shortcut groups. Then G is strongly shortcut.*

Theorem 1·2 allows us to obtain examples of strongly shortcut groups that are not known to be strongly shortcut by any other means. For example, let G be the free product of two copies of the discrete Heisenberg group and let $\langle t \rangle$ be a maximal cyclic subgroup generated by a loxodromic element t of the Bass–Serre tree of G. Then the amalgamated free product $G *_{\langle t \rangle} G$ is hyperbolic relative to discrete Heisenberg subgroups by Dahmani [**Dah03**] and thus is strongly shortcut by Theorem 1·2 and [**HP**].

Our approach to proving Theorem 1·2 is to use properties of asymptotic cones of strongly shortcut groups and relatively hyperbolic groups. A result of the first named author characterises strongly shortcut groups as those whose asymptotic cones have no isometrically embedded circles ([**Hod20**, theorem 3·7]), while a result of Osin and Sapir [**DS05**, theorem A·1] guarantees that asymptotic cones of relatively hyperbolic groups are tree-graded. Thus, any isometrically embedded circle in an asymptotic cone of a relatively hyperbolic group has to be contained in a piece, which is impossible if the peripherals are strongly shortcut.

In the course of the proof of Theorem 1·2 we restrict the combinatorial horoball construction of Groves and Manning [**GM08**] to a sufficiently large finite number of levels, thus obtaining the following result which may be of independent interest.

THEOREM 1·3. *Let G be a finitely generated group that is hyperbolic relative to finitely generated subgroups $(H_i)_i$. For each i, let S_i be a finite generating set for H_i. Then there is a connected, free cocompact G-graph Γ with subgraphs $(\Gamma_i)_i$ such that, for each i:*

(1) *Γ_i is a Rips graph of* Cayley(H_i, S_i);

(2) *H_i stabilises Γ_i;*

(3) *the H_i action on Γ_i is free and cocompact; and*

(4) *Γ_i is convex in Γ.*

We use Theorem 1·3 to prove Theorem 4·3, which says that G has a Cayley graph in which the H_i are strongly shortcut subspaces.

Structure of the paper. In Section 2, we recall the Groves and Manning combinatorial horoball construction and their characterisation of relative hyperbolicity. Section 3 is devoted to the proof of Theorem 1·3. In Section 4, we show that a relatively hyperbolic group with strongly shortcut parabolics admits a Cayley graph in which the parabolics are strongly shortcut subspaces. Finally, we recall the notion of asymptotic cones and prove the main result Theorem 1·2 in Section 5.

2. *Relative hyperbolicity à la Groves and Manning*

Definition 2·1 (Groves and Manning [**GM08**]). Let Λ be a graph. The *combinatorial horoball based on* Λ, denoted by $\mathcal{H}(\Lambda)$, is a graph constructed as follows:

(i) the vertex set is defined as $\mathcal{H}(\Lambda)^{(0)} := \Lambda^{(0)} \times \mathbb{N}_0$, where $\Lambda^{(0)}$ is the vertex set of Λ;

(ii) there are two kinds of edges in $\mathcal{H}(\Lambda)$:

    (i) for each $n \in \mathbb{N}_0$ and each $v \in \Lambda^{(0)}$, there is a *vertical* edge in $\mathcal{H}(\Lambda)$ between (v, n) and $(v, n + 1)$;

    (ii) for each $n \in \mathbb{N}_0$, and each pair of vertices (v, n) and (w, n), there is a *horizontal* edge between (v, n) and (w, n) if and only if $0 < d_\Lambda(v, w) \leq 2^n$.

We denote by $\Lambda \times \{k\}$ the subgraph of $\mathcal{H}(\Lambda)$ spanned by the vertex set $\Lambda^{(0)} \times \{k\}$.

Definition 2·2. A *rough isometry* is a quasi-isometry with multiplicative constant 1.

Definition 2·3. Recall that, for each $k \in \mathbb{N}$, the *Rips graph* $\mathrm{Rips}_k(\Lambda)$ of a graph Λ is the graph with vertex set $\Lambda^{(0)}$ and edges consisting of pairs of vertices at distance at most k in Λ.

Remark 2·4. Observe that the bijection $\Lambda^{(0)} \xrightarrow{\cong} \Lambda^{(0)} \times \{n\} \subset \mathcal{H}(\Lambda)^{(0)}$ given by $v \mapsto (v, n)$ extends to an isomorphism $\mathrm{Rips}_{2^n}(\Lambda) \xrightarrow{\cong} \Lambda \times \{n\} \subset \mathcal{H}(\Lambda)$. In particular, $\Lambda \times \{0\}$ is isomorphic to Λ and, for each n, the subgraph $\Lambda \times \{n\}$ is roughly isometric to Λ with the metric scaled by $1/2^n$.

Definition 2·5 (Groves and Manning [**GM08**]). Let Γ be a graph and $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ be a family of subgraphs of Γ. The *augmented space* $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ is the graph obtained by attaching, for each $\alpha \in \mathscr{A}$, the combinatorial horoball $\mathcal{H}(\Lambda_\alpha)$ to Γ by identifying the subgraph $\Lambda_\alpha \subset \Gamma$ with the subgraph $\Lambda_\alpha \times \{0\} \subset \mathcal{H}(\Lambda_\alpha)$ along the isomorphism $\Lambda_\alpha \xrightarrow{\cong} \Lambda_\alpha \times \{0\}$ given by $v \mapsto (v, 0)$.

Definition 2·6. Let Γ be a graph and $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ be a family of subgraphs of Γ. Then Γ is *hyperbolic relative* to $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ if the *augmented space* $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ is δ-hyperbolic for some δ. In that case, we call each $\Lambda_\alpha = \Lambda_\alpha \times \{0\}$ a *parabolic* subgraph of Γ.

Remark 2·7. The above definition for graphs is motivated by the characterisation of relative hyperbolicity for groups by Groves and Manning (see Definition 2·11 below). Our definition is likely equivalent to metric notions of relative hyperbolicity as investigated in [**Sis12**], but we do not prove nor do we need such an equivalence for the purposes of this paper.

Definition 2·8. Let Γ be a graph and $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ be a family of subgraphs of Γ. The *n-restricted augmentation* $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ is the subgraph of $\mathcal{H}\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ spanned by the vertex set $\Gamma^{(0)} \sqcup \bigsqcup_{\alpha \in \mathscr{A}, k \in \{1,\ldots,n\}} \Lambda_\alpha^{(0)} \times \{k\}$.

Similarly, the *n-restricted horoball* $\mathcal{H}_n(\Lambda)$ is the subgraph of the horoball $\mathcal{H}(\Lambda)$ spanned by the vertex set $\bigsqcup_{k \in \{1,\ldots,n\}} \Lambda \times \{k\}$.

Remark 2·9. If a group G acts properly and cocompactly on Γ and $(\Lambda_\alpha)_\alpha$ is G-invariant then G acts properly and cocompactly on $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$. Moreover, the embedding of Γ in $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ is G-equivariant and, for any α, the stabiliser of $\Lambda_\alpha \times \{n\}$ is equal to the stabiliser of Λ_α.

Remark 2·10. The graph Γ is hyperbolic relative to $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ if and only if for each (any) $n \in \mathbb{N}_0$, $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ is hyperbolic relative to $(\Lambda_\alpha \times \{n\})_{\alpha \in \mathscr{A}}$. Thus, when we speak of the *parabolics* of $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_\alpha\big)$ we will mean the top levels of the *n-restricted horoballs* $(\Lambda_\alpha \times \{n\})_\alpha$.

The following definition is due to Groves and Manning, who prove that it is equivalent to *strong relative hyperbolicity* [**Far98**, **Bow12**]. We refer the reader to [**GM08**, theorem 3·25] for a proof and more details. A detailed study and equivalences of various notions of relative hyperbolicity was done by Hruska in [**Hru10**].

Definition 2·11. Let G be a finitely generated group and let $H_1, \ldots, H_k$ be a family of finitely generated subgroups of G. For $1 \le i \le k$, let S_i be a finite generating set for H_i and let S be a finite generating set for G such that each $S_i \subset S$. Denote by Γ the Cayley graph Cayley(G, S) and, for $1 \le i \le k$, and $g \in G$, denote by $g\Lambda_i$ the subgraph of Γ with vertex set gH_i and edges labelled by gS_i. Then G is *hyperbolic relative to* $\{H_1, \ldots, H_k\}$ if Γ is hyperbolic relative to $\{g\Lambda_i\}_{1 \le i \le k, g \in G}$.

3. *Horoballs and convexity of parabolics*

It is well known that given a relatively hyperbolic group, its parabolic subgroups are quasiconvex [**DS05**, Lemma 4·15]. The goal of this section is to prove Theorem 3·5, which says that a relatively hyperbolic graph can be modified so that its parabolic subgroups are convex subgraphs. We make use of several previously known results.

LEMMA 3·1 (See Bridson and Haefliger [**BH99**, theorem III·H·1·13]). *Let Γ be a δ-hyperbolic space and let $r > 8\delta + 1$. Then there exists a constant $K = K(\delta, r)$ depending only on δ and r such that the following holds. If γ is a path in Γ and every subpath of length r of γ is a geodesic then γ is a $(2\delta, K)$-quasi-geodesic.*

THEOREM 3·2 (See Bridson and Haefliger [**BH99**, theorem III·H·1·7]). *Let Γ be a δ-hyperbolic graph. Let $L > 0$ and $K \ge 0$. Then there exists a constant $M = M(\delta, L, K)$ such that for any two (L, K)-quasigeodesics β_1 and β_2 with the same endpoints, the images $\mathrm{im}(\beta_1)$ and $\mathrm{im}(\beta_2)$ are at Hausdorff distance at most M.*

LEMMA 3·3. *Let $\mathcal{H}_n(\Lambda)$ be an n-restricted horoball. Let $v_1, v_2 \in \mathcal{H}_n(\Lambda)$ be given. The following hold*:

(i) *there exists a geodesic β between v_1, v_2 whose image consists of at most two vertical segments and one horizontal segment. If the horizontal segment is not contained in*
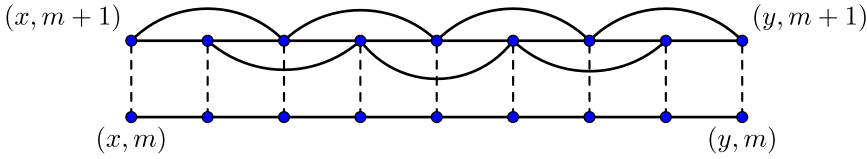
Fig. 1. The $(\Lambda \times \{m\})$-distance between (x, m) and (y, m) is 8 while the $(\Lambda \times \{m+1\})$-distance between $(x, m+1)$ and $(y, m+1)$ is 4.

    $\Lambda \times \{n\}$, *then it is of length at most 3. Further, any geodesic between the two points is at Hausdorff distance at most 4 from* $\mathrm{im}(\beta)$;

(ii) *if the horizontal segment of* $\mathrm{im}(\beta)$ *is contained in* $\Lambda \times \{K\}$, *then the image of any geodesic between* v_1 *and* v_2 *is disjoint from* $\Lambda \times \{K'\}$ *for all* $K' > K$.

(iii) *Moreover, if k is the least number such that either* v_1 *or* v_2 *is contained in* $\Lambda \times \{k\}$, *then the image of any geodesic between the points is contained in* $\mathcal{H}_n(\Lambda) \setminus \mathcal{H}_{k-1}(\Lambda)$.

Lemma 3·3 is essentially a re-statement of Lemma 3·10 of [**GM08**] in the context of restricted horoballs, and our proof below, given for the sake of completeness, is almost identical to theirs.

Let us first make the convention that a *vertical segment* of a path γ is a subpath whose image is the union of vertical edges in a horoball. Similarly a *horizontal segment* is a subpath whose image is disjoint from the set of vertical edges.

Proof. We start the proof with a basic observation. Let $1 \leq m < n$ and let (x, m) and (y, m) be two points in $\Lambda \times \{m\}$. If (x, m) and (y, m) are at $(\Lambda \times \{m\})$-distance D, note that the $(\Lambda \times \{m+1\})$-distance between $(x, m+1)$ and $(y, m+1)$ is $\lceil D/2 \rceil$ (see Figure 1). Similarly, the $(\Lambda \times \{m+k\})$-distance between $(x, m+k)$ and $(y, m+k)$ is $\lceil D/2^k \rceil$. This observation implies the following:

(1) Assume that a geodesic path contains a horizontal segment in $\Lambda \times \{m\}$ of length more than one. Assume that this horizontal segment is not contained in a strictly larger horizontal segment of the geodesic. Then the vertical segment immediately preceding the horizontal segment is an *ascending* segment, in the sense that it is a vertical segment from some $\Lambda \times \{m-k\}$ to $\Lambda \times \{m\}$. Similarly, the immediate successor of the horizontal segment is a *descending* segment. See Figure 2 for an illustration.

(2) Any geodesic path with a descending segment at (x, m) cannot ascend back to $\Lambda \times \{m\}$ in the future (see Figure 3). In other words, no ascending segment follows a descending segment.

(3) Any geodesic path contains at most two maximal descending (respectively ascending) segments. See Figure 4.

Let γ be a geodesic between the points v_1 and v_2 in the statement. By the above observations, if γ contains a horizontal segment of length at least two at some $\Lambda \times \{m\}$, then $\mathrm{im}(\gamma)$ is disjoint from $\Lambda \times \{m'\}$ for all $m < m' \leq n$. Thus, any horizontal segment in γ is either of length one, or is contained in the maximum level $\Lambda \times \{\max\}$ that intersects $\mathrm{im}(\gamma)$ nontrivially.

In fact, it can be verified that apart from the horizontal segment at $\Lambda \times \{\max\}$, the image of γ can have at most one more horizontal edge.
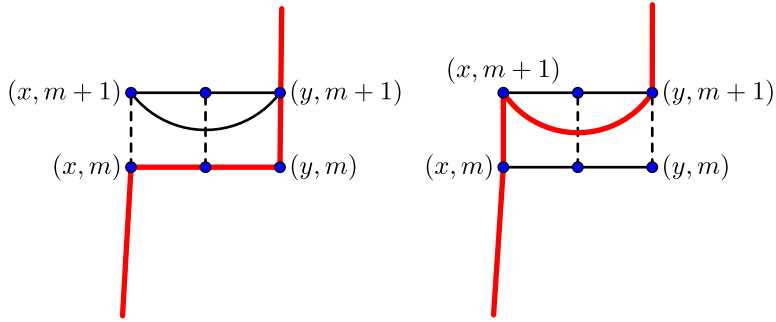
Fig. 2. The thickened path between (x, m) and $(y, m + 1)$ on the left is longer than the thickened path on the right.
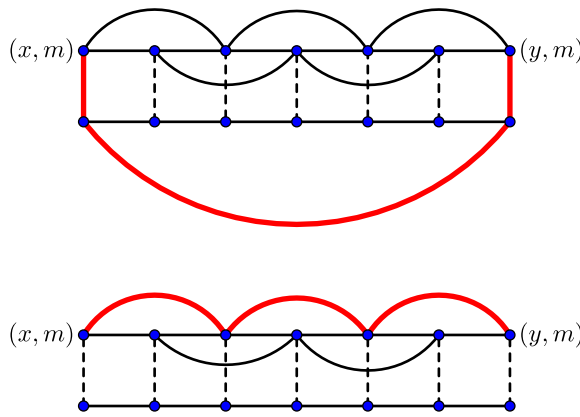

Fig. 3. The thickened path between (x, m) and (y, m) on the bottom panel is shorter than the one on the top panel.
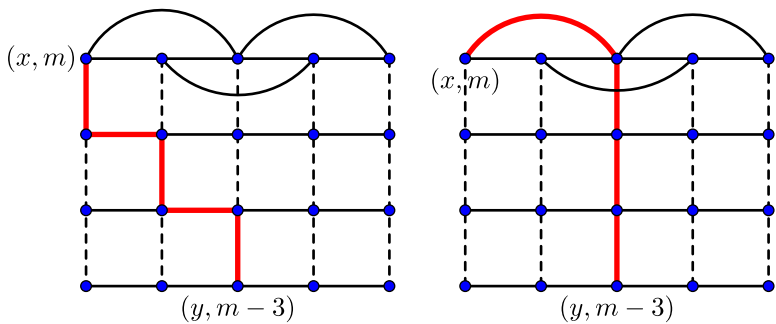

Fig. 4. The thickened path between (x, m) and $(y, m - 3)$ on the left is longer than the one on the right.

Another consequence of the above observations is that if γ contains a horizontal segment of length at least 6, then this segment has to be contained in $\Lambda \times \{n\}$, see Figure 5.

Assume that the horizontal edge not at $\Lambda \times \{\max\}$ is an edge between (x, m) and (y, m) and is followed by an ascending segment from (y, m) to $(y, \max) \in \Lambda \times \{\max\}$. Let γ' be the geodesic obtained from γ by replacing the above by a vertical segment from (x, m)
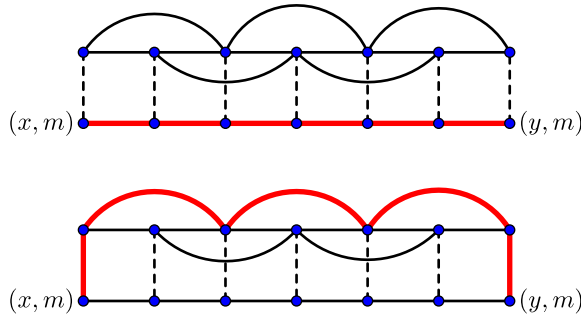
Fig. 5. If $m < n$, then the thickened horizontal path between (x, m) and (y, m) in the top panel is longer than the red path in the bottom panel.

to $(x, \max)$ followed by a horizontal edge to $(y, \max)$. If $\max < n$ and the only horizontal segment of γ' contains 4 or 5 edges, then let β be the geodesic obtained by replacing this horizontal segment by an ascending edge, a horizontal segment in $\Lambda \times \{\max +1\}$ and a descending edge back to $\Lambda \times \{\max\}$, similar to the procedure in Figure 5. We leave it as an exercise to verify that β is as required.

Before stating the main result of this section, we recall a convexity result from [**GM08**] which will be used in the proof.

LEMMA 3·4 ([**GM08**, lemma 3·26]). *Let Γ be a graph that is hyperbolic relative to a family $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ of subgraphs. Let δ be the hyperbolicity constant of $\mathcal{H}\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$. Then for any $k > \delta$ and any $\alpha \in \mathscr{A}$, $\mathcal{H}\big(\Lambda_\alpha\big) \setminus \mathcal{H}_k\big(\Lambda_\alpha\big)$ is convex in $\mathcal{H}\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$.*

THEOREM 3·5. *Let Γ be a graph that is hyperbolic relative to a family $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ of subgraphs. Then, for n large enough, the parabolics (i.e. the top levels) of the restricted horoballs $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ are convex subgraphs.*

Proof. Let δ be the hyperbolicity constant of $\mathcal{H}\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$. Let $r = \lceil 8\delta + 2 \rceil$ and $n \geq 2r + M(\delta, 2\delta, K)$, where K is the constant from Lemma 3·1 and M is the constant from Theorem 3·2 Fix $\alpha_0 \in \mathscr{A}$ and points $x, y \in \Lambda_{\alpha_0} \times \{n\}$. Let $\gamma : P \to \mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ be a geodesic (in $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$) between x and y. Since each n-restricted horoball in $\mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ is a full subgraph, every subpath of γ whose image lies in an n-restricted horoball is a geodesic in that horoball. We will therefore assume that each such geodesic subpath of γ is of the form given by Lemma 3·3.

Denote by $U \subset \mathcal{H}_n\big(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}\big)$ the set $\bigcup_{\alpha \in \mathscr{A}} N_r\big(\Lambda_\alpha \times \{n\}\big)$. The path γ is a concatenation $\gamma_1 \cdot \beta_1 \cdot \gamma_2 \cdots \gamma_k$, where each γ_i is a path with image in U and each β_i is such that its image is disjoint from U, except at the endpoints. See Figure 6 for an illustration.

Note that by Lemma 3·3, each β_i is a path which satisfies the following:

(i) $\operatorname{im}(\beta_i)$ is not contained in any single n-restricted horoball and thus has length at least $2(n - r) > 2r$, and

(ii) for any $\alpha \in \mathscr{A}$, $\operatorname{im}(\beta_i) \cap \mathcal{H}_n\big(\Lambda_\alpha\big)$ is a union of components, where each component is either a vertical segment between $\Lambda_\alpha \times \{0\}$ and $\Lambda_\alpha \times \{n - r\}$ (e.g., $\operatorname{im}(\beta_2) \cap \mathcal{H}_n\big(\Lambda_{\alpha_2}\big)$ in Figure 6), or the image of a geodesic between points of $\Lambda_\alpha \times \{0\}$ (e.g., $\operatorname{im}(\beta_1) \cap \mathcal{H}_n\big(\Lambda_{\alpha_1}\big)$ in Figure 6). In the latter case, we note that this component is disjoint from the image of any γ_j.
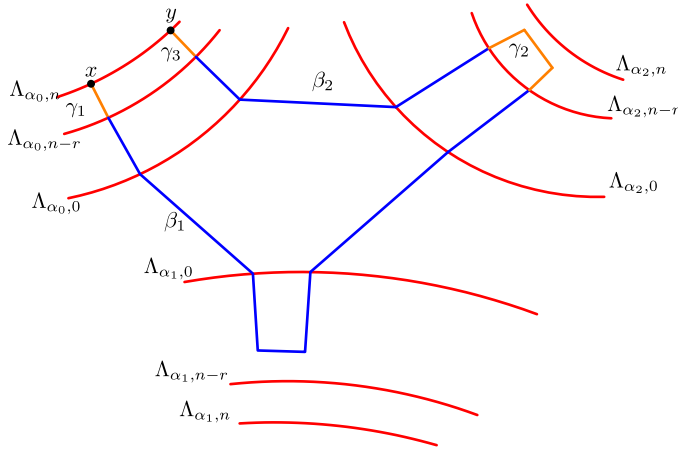
Fig. 6. The path γ is a concatenation of the paths γ_i (geodesics that lie in the r-neighbourhoods of the parabolics) and β_j (geodesics between the γ_i).

Let $\iota : \mathcal{H}_n(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}}) \hookrightarrow \mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ denote the inclusion map. For each i, let γ_i' be a geodesic path in $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ between the endpoints of $\iota \circ \gamma_i$. Let $\gamma' : P' \to \mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ be the path obtained from $\iota \circ \gamma$ by replacing each $\iota \circ \gamma_i$ by γ_i'. We will denote $\iota \circ \beta_i$ by β_i'. Thus $\gamma' = \gamma_1' \cdot \beta_1' \cdot \gamma_2' \cdots \gamma_k'$.

Claim. The path γ' is an r-local geodesic in $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$.

Proof. Each γ_i' is a geodesic, and therefore a local geodesic. Each β_i' is an r-local geodesic since the r-ball around any point in $\mathrm{im}(\beta_i)$ is contained in $\mathcal{H}_n(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$.

As observed above, the image of every subpath of β_i' that lies in a horoball is either a vertical segment or it does not meet any γ_j'. This implies that any subpath of $\beta_{i-1}' \cdot \gamma_i' \cdot \beta_i'$ whose image lies in $\mathcal{H}(\Lambda_\alpha) \setminus \mathcal{H}_r(\Lambda_\alpha)$ is a geodesic in $\mathcal{H}(\Lambda_\alpha) \setminus \mathcal{H}_r(\Lambda_\alpha)$. Since $\mathcal{H}(\Lambda_\alpha) \setminus \mathcal{H}_r(\Lambda_\alpha)$ is convex (by Lemma 3·4), each such subpath is in fact a geodesic in $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$, and therefore an r-local geodesic. This proves the claim.

Thus by Lemma 3·1, γ' is a $(2\delta, K)$-quasi-geodesic and by Theorem 3·2, it lies in an $M = M(\delta, 2\delta, K)$ neighbourhood of any geodesic in $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ between x and y. Since $x, y \in \Lambda_{\alpha_0} \times \{n\}$ with $n - 1 > \delta$, we have that any geodesic between them in $\mathcal{H}(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ lies in $\mathcal{H}(\Lambda_{\alpha_0}) \setminus \mathcal{H}_{n-1}(\Lambda_{\alpha_0})$ (again, by Lemma 3·4). This implies that γ' lies in $N_M(\mathcal{H}(\Lambda_{\alpha_0}) \setminus \mathcal{H}_{n-1}(\Lambda_{\alpha_0})) \subset \mathcal{H}(\Lambda_{\alpha_0}) \setminus \mathcal{H}_{2r}(\Lambda_{\alpha_0})$.

We are thus forced to conclude that $\gamma' = \gamma_1'$ (and therefore $\gamma = \gamma_1$). Indeed, if not, then β_1 is a geodesic in $\mathcal{H}_n(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ with endpoints on $\Lambda_{\alpha_0} \times \{n - r\}$ and such that $\mathrm{im}(\beta_1) \subset \mathcal{H}_n(\Lambda_{\alpha_0})$. But as observed above, $\mathrm{im}(\beta_1)$ is not contained in any single n-restricted horoball, which is a contradiction.

Using Lemma 3·3 once again, we conclude that $\gamma \subset \Lambda_{\alpha_0} \times \{n\}$.

COROLLARY 3·6. *Let Γ be a graph that is hyperbolic relative to a family $(\Lambda_\alpha)_{\alpha \in \mathscr{A}}$ of subgraphs. Let n be such that the parabolics $(\Lambda_\alpha \times \{n\})_\alpha$ of $\mathcal{H}_n(\Gamma, (\Lambda_\alpha)_{\alpha \in \mathscr{A}})$ are convex subgraphs, as in Theorem 3·5. Then for each $\alpha \in \mathscr{A}$, the subspace $\Lambda_\alpha^{(0)} \times \{0\}$ is roughly isometric to the subgraph $\Lambda_\alpha \times \{n\}$.*

Proof. Let $(x, 0), (y, 0) \in \Lambda_\alpha \times \{0\}$ be vertices at the bottom level of the combinatorial horoball based on Λ_α in Γ and let $(x, n), (y, n) \in \Lambda_\alpha \times \{n\}$ be the corresponding vertices at the nth level. We have

$$\left| d_{\mathcal{H}_n}\big((x, 0), (y, 0)\big) - d_{\mathcal{H}_n}\big((x, n), (y, n)\big) \right| \leq 2n$$

by the triangle inequality. It follows that the map $(\Lambda_\alpha^{(0)} \times 0, d_{\mathcal{H}_n}) \to (\Lambda_\alpha^{(0)} \times \{n\}, d_{\mathcal{H}_n})$ given by $(x, 0) \mapsto (x, n)$ is a rough isometry and $\Lambda_\alpha^{(0)} \times \{n\}$ is a convex subgraph of $\mathcal{H}_n$.

We now recall and prove Theorem 1·3.

THEOREM 1·3. *Let G be a finitely generated group that is hyperbolic relative to finitely generated subgroups $(H_i)_i$. For each i, let S_i be a finite generating set for H_i. Then there is a connected, free cocompact G-graph Γ with subgraphs $(\Gamma_i)_i$ such that, for each i:*

(i) *Γ_i is a Rips graph of* Cayley(H_i, S_i);

(ii) *H_i stabilises Γ_i;*

(iii) *the H_i action on Γ_i is free and cocompact; and*

(iv) *Γ_i is convex in Γ.*

Proof. Let S be a finite generating set of G containing each of the S_i. Let Γ be the Cayley graph of G with respect to S. Then the Cayley graphs $\Gamma_i' = $ Cayley(H_i, S_i) are subgraphs of Γ and G is hyperbolic relative to the family $(g\Gamma_i')_{g,i}$ of G-translates of these subgraphs. By Theorem 3·5, there is an n for which the parabolics of $\mathcal{H}_n(\Gamma, (g\Gamma_i')_{g,i})$ are convex. For each i, let Γ_i be the parabolic in the restricted horoball with base Γ_i'. Then $\mathcal{H}_n$ and the Γ_i satisfy all the required conditions.

4. *A Cayley graph with strongly shortcut parabolics*

Let G be a finitely generated group that is strongly shortcut relative to strongly shortcut subgroups $(H_i)_i$. In this section we will show that there exists a generating set S for G such that the H_i are strongly shortcut metric subspaces of the Cayley graph Cayley(G, S). In order to do this, we will first need to define what it means for a metric space to be strongly shortcut. The following definition appears in earlier work of the first named author under the name *nonapproximability of n-gons* [**Hod20**, definition 3·2].

Definition 4·1. Let C_n denote the cycle graph of length n (i.e., a circle subdivided into n edges and n vertices) and let $C_n^{(0)}$ denote the vertex set of C_n. A metric space X is *strongly shortcut* if there exists a $K > 1$, an $n \in \mathbb{N}$ and an $M > 0$ such that there is no K-bilipschitz embedding of $(C_n^{(0)}, \lambda d_{C_n})$ in X with $\lambda \geq M$.

THEOREM 4·2 ([**Hod20**, corollary 3·6]). *A graph Γ is strongly shortcut as a graph if and only if it is strongly shortcut as a metric space.*

Our aim in this section is to prove the following.

THEOREM 4·3. *Let G be a finitely generated group that is hyperbolic relative to a family of strongly shortcut groups $(H_i)_i$. Then G has a finite generating set S for which the H_i are strongly shortcut metric subspaces of* Cayley(G, S).

In order to prove Theorem 4·3 we will rely on Theorem 3·5 and the following refined version of the Milnor–Švarc Lemma. This version of the Milnor–Švarc Lemma gives us arbitrary control on the multiplicative constant of the quasi-isometry, up to scaling the metric on the Cayley graph. This arbitrary control on the multiplicative constant of the quasi-isometry comes at the cost of having to choose larger and larger finite generating sets and accepting larger and larger additive quasi-isometry constants.

THEOREM 4·4 (Fine Milnor–Švarc Lemma [**Hod20**, theorem H]). *Let (X, d) be a geodesic metric space. Let G be a group acting metrically properly and coboundedly on X by isometries. Fix $x_0 \in X$. For $t > 0$ let S_t be the finite set defined by*

$$S_t = \left\{ g \in G : d(x_0, gx_0) \leq t \right\}$$

and consider the word metric d_{S_t} defined by S_t. (For those t where S_t does not generate G, we allow d_{S_t} to take the value ∞). Let K_t be the infimum of all $K > 1$ for which

$$(G, t d_{S_t}) \longrightarrow X$$
$$g \longmapsto g \cdot x_0$$

is a (K, C_K)-quasi-isometry for some $C_K \geq 0$. Then $K_t \to 1$ as $t \to \infty$.

LEMMA 4·5. *Let G be a finitely generated group that is hyperbolic relative to finitely generated subgroups $(H_i)_i$. For each i, let S_i be a finite generating set for H_i. Then, for any $L > 1$, there is a $t > 0$ and a finite generating set S for G such that each inclusion*

$$(H_i, d_{S_i}) \hookrightarrow (G, t d_S)$$

is a quasi-isometric embedding with multiplicative constant L, where d_S and the d_{S_i} are the word metrics.

Proof. Let $S' \supseteq \bigcup_i S_i$ be a finite generating set for G. Let $\Gamma' = \mathrm{Cayley}(G, S')$ and let $\Lambda_{g,i} = g\mathrm{Cayley}(H_i, S_i)$. By Theorem 3·5, for some n, the top level subgraphs $\Lambda_{g,i} \times \{n\}$ of the restricted horoballs of $\mathcal{H}_n = \mathcal{H}_n(\Gamma', \{\Lambda_{g,i}\}_{g,i})$ are convex. Moreover, by Remark 2·9, the group G acts properly and cocompactly on $\mathcal{H}_n$.

By Corollary 3·6 and Remark 2·4, there is a rough isometry $(\Lambda_{e,i}^{(0)} \times \{0\}, d_{\mathcal{H}_n}) \to (H_i, 1/2^n d_{S_i})$. By Theorem 4·4, there is a generating set S for G and a scaling factor $t' > 0$ such that the inclusion $(G, t' d_S) \hookrightarrow \mathcal{H}_n$ is a quasi-isometry with multiplicative constant L, where d_S is the word metric coming from S. But the image of H_i under this inclusion is $\Lambda_{e,i}^{(0)} \times \{0\}$ and so the composition of the restriction $(H_i, t' d_S) \hookrightarrow (\Lambda_{e,i}^{(0)} \times \{0\}, d_{\mathcal{H}_n})$ and the rough isometry $(\Lambda_{e,i}^{(0)} \times \{0\}, d_{\mathcal{H}_n}) \to (H_i, 1/2^n d_{S_i})$ gives us a quasi-isometry $(H_i, t' d_S) \to (H_i, 1/2^n d_{S_i})$ with multiplicative constant L. Scaling the domain and the codomain by 2^n, taking the quasi-inverse and composing it with the isometric embedding $(H_i, 2^n t' d_S) \hookrightarrow (G, 2^n t' d_S)$ we obtain a quasi-isometry $(H_i, d_{S_i}) \hookrightarrow (G, 2^n t' d_S)$ with multiplicative factor L.

Finally, we will need the next two theorems about strongly shortcut spaces and groups.

THEOREM 4·6 ([**Hod20**, proposition 3·4]). *Let X be a strongly shortcut metric space. Then there exists an $L_X > 1$ such that whenever Y is a metric space and $C > 0$ and $f : Y \to X$ is an (L_X, C)-quasi-isometry up to scaling, then Y is also strongly shortcut.*

THEOREM 4·7 ([**Hod20**], theorem C]). *A group G is strongly shortcut if and only if G has a finite generating set S for which* Cayley(G, S) *is strongly shortcut.*

Proof of Theorem 4·3. Let G be a finitely generated group that is hyperbolic relative to strongly shortcut groups $(H_i)_i$. By Theorem 4·7, we can choose finite generating sets S_i of H_i so that the Cayley graphs Cayley(H_i, S_i) are strongly shortcut. Then, by Theorem 4·6, for each i, there exists an $L_i > 1$ such that any metric space that, up to scaling, is quasi-isometric to (H_i, d_{S_i}) with multiplicative constant L_i is also strongly shortcut. By Lemma 4·5, there is a finite generating set S of G and a $t > 0$ such that, for each i, if d_S is the word metric coming from S then (H_i, td_S) is quasi-isometric to (H_i, d_{S_i}) with multiplicative constant $L = \min_i L_i$. Thus each (H_i, d_S) is strongly shortcut.

5. *Asymptotic cones and the proof of the main result*

In this section we will recall the definition of asymptotic cones of metric spaces. Then we will state the theorem of Osin and Sapir on tree-gradedness of asymptotic cones of relatively hyperbolic groups and a theorem of the second named author giving an asymptotic cone characterization of the strong shortcut property. We will use these theorems and the results of the previous sections to prove Theorem 1·2.

For an exposition of asymptotic cones, see Drutu and Kapovich [**DK18**].

Definition 5·1. A *non-principal ultrafilter* ω over $\mathbb{N}$ is a set of subsets of $\mathbb{N}$ satisfying the following properties:

(i) for each $A \subseteq \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$, but not both;

(ii) no finite subset of $\mathbb{N}$ is in ω;

(iii) if $A, B \in \omega$, then $A \cap B \in \omega$;

(iv) if $A \in \omega$ and $A \subseteq B$, then $B \in \omega$.

The existence of non-principal ultrafilters is a consequence of Zorn's Lemma (see [**DK18**, lemma 10·18] for instance).

Definition 5·2. Let ω be a non-principal ultrafilter. Let $(x_n)_n$ be a sequence of points in a topological space X. An element $x \in X$ is an ω-*limit* of $(x_n)_n$, denoted $\lim_\omega x_n$, if for every open set $U \ni x$, the set $A_U = \{n \in \mathbb{N} | x_n \in U\}$ is contained in ω.

Remark 5·3. If X is a Hausdorff space, then an ω-limit is unique whenever it exists. If X is compact, then for every sequence, an ω-limit exists.

Let (X, d) be a metric space and let ω be a non-principal ultrafilter over $\mathbb{N}$. Let $(r_n)_n$ be a sequence of real numbers such that $\lim_\omega r_n = \infty$. Fix a sequence of basepoints $(p_n)_n \in X^{\mathbb{N}}$.

Let $d_\infty : X^{\mathbb{N}} \times X^{\mathbb{N}} \to [0, \infty]$ be defined as $d_\infty((x_n)_n, (y_n)_n) := \lim_\omega (d(x_n, y_n)/r_n)$. Let $X_B^{\mathbb{N}}((r_n)_n, (p_n)_n) := \{(x_n)_n \in X^{\mathbb{N}} | d_\infty((x_n)_n, (p_n)_n) < \infty\}$.

Remark 5·4. Note that $(X_B^{\mathbb{N}}((r_n)_n, (p_n)_n), d_\infty)$ is a pseudo-metric space.

Definition 5·5. The *asymptotic cone* $\text{Cone}_\omega(X, (r_n)_n, (p_n)_n)$ of X is the quotient of $X_B^{\mathbb{N}}((r_n)_n, (p_n)_n)$ identifying $(x_n)_n$ and $(y_n)_n$ whenever $d_\infty((x_n)_n, (y_n)_n) = 0$. We let $[x_n]$ denote the point of $\text{Cone}_\omega(X, (r_n)_n, (p_n)_n)$ represented by $(x_n)_n$.

Remark 5·6. For a group G equipped with a left invariant metric, any asymptotic cone $\mathrm{Cone}_\omega(G, (r_n)_n, (p_n)_n)$ is isometric to $\mathrm{Cone}_\omega(G, (r_n)_n, (1)_n)$, where $(1)_n$ is the constant basepoint sequence at the identity. Thus in this case we will simply write $\mathrm{Cone}_\omega(G, (r_n)_n)$.

Definition 5·7. (Drutu and Sapir [**DS05**]) A complete geodesic metric space X is a *tree graded space* with respect to a collection of closed geodesic subspaces, called pieces, if the following two properties are satisfied:

(i) any two distinct pieces intersect in at most a single point, and

(ii) every non-trivial simple geodesic triangle (i.e., the concatenation of the three geodesics is a simple loop) in X is contained in a piece.

Theorem 5·8 (Osin and Sapir [**DS05**, theorem A·1]). *Let G be a finitely generated group and let d_S be a word metric coming from a finite generating set S of G. If G is hyperbolic relative to a family of subgroups $(H_i)_i$ then every asymptotic cone $\mathcal{A} = \mathrm{Cone}_\omega\big((G, d_S), (r_n)\big)$ of (G, d_S) is tree graded with respect to the ω-limits*

$$\lim_\omega (g_n H_i)_n = \big\{[x_n]_n \in \mathcal{A} : x_n \in g_n H_i\big\}$$

of the $(g_n H_i)_n$ with $[g_n]_n \in \mathcal{A}$.

Remark 5·9. The $\lim_\omega (g_n H_i)_n$ are isometric to asymptotic cones of the (H_i, d_S). Indeed, the asymptotic cone $\mathcal{A}$ is a group with multiplication given by

$$[x_n]_n \cdot [y_n]_n = [x_n y_n]_n$$

and d_∞ is a left-invariant metric with respect to this group structure. Thus $\lim_\omega (g_n H_i)_n$ is isometric to

$$[g_n^{-1}]_n \lim_\omega (g_n H_i)_n = \lim_\omega (H_i)_n$$

which is $\mathrm{Cone}_\omega\big((H_i, d_S), (r_n)\big)$.

A *Riemannian circle* C is S^1 equipped with a geodesic metric of some length $|C|$. In other words C is the quotient of $\mathbb{R}$ by the action of $|C|\mathbb{Z}$.

Theorem 5·10. ([**Hod20**, theorem 3·7]). *A metric space X is strongly shortcut if and only if no asymptotic cone of X contains an isometric copy of the Riemannian circle of unit length.*

We are now ready to prove our main result, which we first recall:

Theorem 1·2. *Let G be a finitely generated group that is hyperbolic relative to strongly shortcut groups. Then G is strongly shortcut.*

Proof. Let G be a finitely generated group that is hyperbolic relative to strongly shortcut groups $(H_i)_i$. By Theorem 4·3, there is a finite generating set S of G such that (H_i, d_S) is strongly shortcut for each i, where d_S is the word metric coming from S. We will show that the Cayley graph $\mathrm{Cayley}(G, S)$ is strongly shortcut. By Theorem 4·2 and Theorem 5·10

it will suffice to prove that no asymptotic cone $\mathcal{A}$ of Cayley(G, S) contains a Riemannian circle of unit length.

By Theorem 5·8, any embedded copy of C in $\mathcal{A}$ is contained in some $\lim_\omega (g_n H_i)_n$ with $[g_n]_n \in \mathcal{A}$. Thus it suffices to show that $\lim_\omega (g_n H_i)_n$ does not contain an isometric copy of the Riemannian circle of unit length. But by Remark 5·9, the ω-limit $\lim_\omega (g_n H_i)_n$ is isometric to an asymptotic cone $\mathcal{A}'$ of $(H_i, d_{S'})$, which is strongly shortcut. Hence $\mathcal{A}'$ cannot contain an isometric copy of the Riemannian circle of unit length, by Theorem 5·10.

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