Can. J. Math., Vol. XXXI, No. 4, 1979, pp. 786–788

A REMARK ON DISINTEGRATIONS WITH ALMOST ALL COMPONENTS NON σ-ADDITIVE

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We extend a theorem of L. E. Dubins on "purely finitely additive disintegrations" of measures (cf. [**4**]) and apply this result to the disintegrations of extremal Gibbs states with respect to the asymptotic algebra enlarging another result of L. E. Dubins on the symmetric coin tossing game.

We recall the following definition of L. E. Dubins (cf. [**3**], [**4**]): Let $(X, \mathscr{A}, \mu)$ be a measure space, $\mathscr{J}$ a sub σ-algebra of $\mathscr{A}$. A real function $\sigma_x(A)$, $(x, A) \in X \times \mathscr{A}$ is called a *measurable $\mathscr{J}$-disintegration* of μ if:

(*i*) $\forall x \in X$, $\sigma_x(.)$ is a finitely additive measure on $\mathscr{A}$.

(*ii*) $\forall A \in \mathscr{A}$, $\sigma.(A)$ is constant on each $\mathscr{J}$-atom.

(*iii*) For each $A \in \mathscr{A}$, $\sigma.(A)$ is measurable with respect to the completion of $\mathscr{A}$ by μ and $\int_X \sigma_x(A)\, d\mu(x) = \mu(A)$

(*iv*) $\sigma_x(B) = 1$ if $x \in B \in \mathscr{J}$.

The disintegration $(\sigma_x)_{x \in X}$ is said to have almost all components non σ-additive if for μ-almost every x of X the measure σ_x is not σ-additive.

Let X be a polish space provided with its Borel σ-algebra $\mathscr{A}$, G be a locally compact separable group or an inductive limit of a sequence of locally compact separable groups. Let $G \times X \ni (g, x) \to \tau_g x \in X$ be a continuous action of G on X, let μ be a σ-finite *G-quasi-invariant* measure on $(X, \mathscr{A})$, i.e.

$$\tau_g \mu \frown \mu, \forall g \in G.$$

Let $Gx = \{\tau_g x | \ g \in G\}$ be the orbit of a point x under G, then Gx is a Borel subset of X (cf. [**5**]). Let

$$\mathscr{J} = \{A \in \mathscr{A} | \ \tau_g A = A, \forall g \in G\}.$$

The system $(X, \mathscr{A}, \mu, G)$ is called a *dynamical system*. The system is said to be *non-transitive* if $\mu(Gx) = 0 \ \forall x \in X$; it is said to be *ergodic* if $\mathscr{J}$ is trivial with respect to μ.

We have

THEOREM 1. *Let $(X, \mathscr{A}, \mu, G)$ be a non-transitive ergodic system, $\mathscr{J}$ the σ-algebra of G-invariant elements of $\mathscr{A}$.*

a) If $\mu(X) = \infty$, no measurable $\mathscr{J}$-disintegration of μ exists.

b) If $\mu(X) < \infty$, every measurable $\mathscr{J}$-disintegration of μ has non σ-additive almost all components non σ-additive.

Received October 28, 1977 and in revised form October 5, 1978.

Proof. Let $(\sigma_x)_{x\in X}$ be a measurable $\mathscr{J}$-disintegration of μ. Let $\mathscr{A}_0 = \{A_n\}_{n\geqq 1}$ be a countable sub-field of $\mathscr{A}$ such that

i) $\mathscr{A}_0$ generates $\mathscr{A}$.
ii) $\mu(A_n) < \infty, \forall n \geqq 1$.

For a fixed A_n, consider the measurable mapping $X \ni x \to f_n(x) = \sigma_x(A_n)$. By the property of $(\sigma_x)_{x\in X}$, we have for every $g \in G$

$$f_n(\tau_g x) = f_n(x),$$

for μ-almost every x of X.

As the system is ergodic, the function f_n is almost everywhere constant, but we have

$$(1) \quad \mu(A_n) = \int_X f_n(x)\, d\mu(x)$$

Case 1. If $\mu(X) = \infty$, the relation (1) implies that $\mu(A_n) = 0\ \forall n \geqq 1$ and therefore $\mu \equiv 0$ contradicting the hypothesis, and part *a*) of the theorem is proved.

Case 2. If $\mu(X) < \infty$, to simplify the notations, we assume that $\mu(X) = 1$. The relation (1) implies

$f_n(x) = \mu(A_n)$ for almost every x of X, or

$\sigma_x(A_n) = \mu(A_n)$ for almost every x of X.

We deduce the existence of a μ-null subset N of X such that

$$(2) \quad \sigma_x(A_n) = \mu(A_n), \forall n \geqq 1, \forall x \in X\backslash N.$$

The relation (2) implies that for every $x \in X\backslash N$ the measure σ_x cannot be σ-additive: assume that for a particular $x \in X\backslash N$ the measure σ_x is σ-additive. By the uniqueness of the extension of finite σ-additive measure the relation (2) implies

$$(3) \quad \sigma_x(A) = \mu(A), \forall A \in \mathscr{A}.$$

But this contradicts the relations $\sigma_x(Gx) = 1$ and $\mu(Gx) = 0$. Therefore almost every σ_x is not σ-additive.

The theorem is proved.

Application to disintegration of extremal Gibbs states. Let $E = \mathbf{R}^n$, or $\mathbf{Z}^n$ or a finite set, and let E represent the state of a particle. Considering the d-dimensional lattice $\mathbf{Z}^d$, we take can $X = E^{\mathbf{Z}^d}$ as the space of configurations of particles on the lattice.

In each case, the space E is a locally compact separable group. Let $G = E^{(\mathbf{Z}^d)}$. G is an inductive limit of a sequence of locally compact separable groups and the continuous action of G on X is defined by

$$\tau_{(e_i)}(x_i) = (e_i + x_i), \forall (e_i) \in E^{(\mathbf{Z}^d)}, (x_i) \in E^{\mathbf{Z}^d}.$$

The G-invariant σ-algebra $\mathscr{J}$ is precisely the *asymptotic* (or *tail*) σ-algebra.

In statistical mechanics (see [**8**], [**2**], [**6**], [**7**]), one is interested in Gibbs states on X. These states are G-quasi-invariant probability measures on X, and the extremal Gibbs states are those for which the asymptotic σ-algebra is trivial.

We deduce from Theorem 1 the following

THEOREM 2. *Every measurable disintegration of an extremal Gibbs state μ with respect to the asymptotic σ-algebra has almost all components non σ-additive.*

This theorem can be applied to the following cases:

a) *the coin-tossing case* (symmetric or not): $E = \{0, 1\}$, $X = \{0, 1\}^{\mathbf{Z}}$ or $\{0, 1\}^{\mathbf{N}}$, μ a product probability measure (the symmetric case was proved by L. E. Dubins).

b) $E = \mathbf{R}^n$, $X = E^{\mathbf{Z}^d}$, $\mu = \nu^{\otimes \mathbf{Z}^d}$ where ν is a probability measure on $\mathbf{R}^n$ equivalent to the Lebesgue measure.

REFERENCES

1. D. Blackwell and L. E. Dubins, *On existence and non-existence of proper, regular conditional distributions*, Ann. of Prob. *3* (1975), 741–753.

2. N. Dang-Ngoc and G. Royer, *Markov property of extremal local fields*, Proc. Amer. Math. Soc. (1978), 185–188.

3. L. E. Dubins, *Finitely additive conditional probabilities, conglomerability and disintegrations*, Ann. of Prob. *3* (1975), 89–99.

4. ——— *Measurable tail disintegrations of the Haar integral are purely finitely additive*, Proc. Amer. Math. Soc. *62* (1977), 34–36.

5. E. G. Effros, *Transformation groups and C*-algebras*, Ann. of Math. *81* (1965), 38–55.

6. H. Follmer, *Phase transition and Martin boundary*, Séminaire de Probabilités IX. Lecture Notes in Math. (Springer-Verlag, 1975).

7. H. O. Georgii, *Two Remarks on extremal equilibrium states*, Comm. Math. Phys. *32* (1973), 107–118.

8. D. Ruelle, *Statistical mechanics* (Benjamin, New York, 1969).

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