ORDER AND NORM CONVERGENCE IN BANACH LATTICES

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Let \( (V, \leq, \| \cdot \|) \) be a Banach lattice, and denote \( V\setminus\{0\} \) by \( V' \). For the definition of a Banach lattice and other undefined terms used below, see Vulikh [4]. Leader [3] shows that, if norm convergence is equivalent to order convergence for sequences in \( V \), then the norm is equivalent to an \( M \)-norm. By assuming the equivalence for nets in \( V \) we can strengthen this result.

**THEOREM.** Let \( (V, \leq, \| \cdot \|) \) be a Banach lattice; then the following statements are equivalent:

(i) Norm convergence is equivalent to order convergence, for nets in \( V \).

(ii) \( V \) is finite-dimensional.

**Proof.** (i) implies (ii). If \( \alpha, \beta \in V' \), write \( \alpha \leq \beta \) to mean \( \| \alpha \| \geq \| \beta \| \). Then \( (V', \leq) \) is a preordered set directed to the right. Let \( x_a = \alpha \) for all \( \alpha \in V' \); then \( \liminf x_a = 0 \), and so \( 0\text{-lim} x_a = 0 \). Hence \( (V, \leq) \) has a strong unit, \( e \) say. Define \( \| \cdot \|_e \) by \( \| x \|_e = \inf \{ \lambda : \| x \| \leq \lambda e \} \), for \( x \in V \). By Birkhoff [1], \( \| \cdot \| \) and \( \| \cdot \|_e \) are equivalent norms. In fact \( (V, \leq, \| \cdot \|_e) \) is a Banach lattice with unity \( e \) and so an \( M \)-space, Birkhoff [1]. So \( (V, \leq, \| \cdot \|_e) \) is isomorphic with \( (C(X), \leq, \text{sup norm}) \), \( X \) compact Hausdorff, by Kelley and Namioka [2].

Let \( x_0 \in X \) and let \( g \) be the characteristic function for the point \( x_0 \). Define

\[
F = \{ f \in C(X) : f \geq 0 \quad \text{and} \quad f(x_0) = 1 \};
\]

then \( (F, \geq) \) is directed to the right. Let \( f_\alpha = \alpha \) for all \( \alpha \in F \). Then, by Urysohn’s Lemma, \( f_\alpha \downarrow g \) pointwise. If \( g \in C(X) \), then \( 0\text{-lim} f_\alpha = g \); otherwise \( 0\text{-lim} f_\alpha = 0 \). Now \( \| \cdot \|_e - \lim f_\alpha = 0 \) is impossible; so \( g \in C(X) \). Hence \( \{ x_0 \} \) is open; so \( X \) is discrete and hence finite.

(ii) implies (i). The proof of this is trivial.

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**REFERENCES**


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