THE GENERALISED $f$-PROJECTION OPERATOR
WITH AN APPLICATION

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In this paper, we introduce a new concept of generalised $f$-projection operator which extends the generalised projection operator $\pi_K : B^* \to K$, where $B$ is a reflexive Banach space with dual space $B^*$ and $K$ is a nonempty, closed and convex subset of $B$. Some properties of the generalised $f$-projection operator are given. As an application, we study the existence of solution for a class of variational inequalities in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Let $B$ be a Banach space with dual space $B^*$. As usual, $\langle \varphi, x \rangle$, denotes the duality pairing of $B^*$ and $B$, where $\varphi \in B^*$ and $x \in B$. If $B$ is a Hilbert space, $\langle \varphi, x \rangle$ denotes an inner product on $B$. Let $K$ be a nonempty, closed and convex subset of $B$. The metric projection operator $P_K : B^* \to K$ has been used in many areas such as optimisation theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problems, et cetera (see, for example, [10, 11, 12, 13, 14, 16, 17, 20] and the references therein).

In 1994, Alber [1] introduced the generalised projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [2], Alber presented some applications of the generalised projections to approximate solving variational inequalities and Von-Neumann intersection problem in Banach spaces. Recently, Li [17] extended the definition of the generalised projection operator $\pi_K : B^* \to K$, where $B$ is a reflexive Banach space with dual space $B^*$ and $K$ is a nonempty, closed and convex subset of $B$ and studied some properties of the generalised projection operator with applications to solving the variational inequality in Banach spaces. Some related works, we refer to [3, 4, 5, 6, 7, 8, 9] and the references therein.

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Motivated and inspired by the above works, in this paper, we introduce and study a new class of generalised $f$-projection operator in Banach spaces, which extends the definition of the generalised projection operators introduced by Alber [1] and Li [17]. Some properties of the generalised $f$-projection operator are given. As an application, we study the existence of solution to a class of variational inequalities.

Let $B$ be a Banach space with dual space $B^*$. The normalised duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \{j(x) \in B^* : \langle j(x), x \rangle = \|j(x)\| \cdot \|x\| = \|x\|^2 = \|j(x)\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $B^*$ and $B$. Without confusion, one understands that $\|j(x)\|$ is the $B^*$-norm and $\|x\|$ is the $B$-norm. Many properties of the normalised duality mapping $J$ can be found in Takahashi [18] or Vainberg [19]. We list some properties below for easy reference:

1. $J$ is a monotone and bounded operator in arbitrary Banach spaces;
2. $J$ is a strictly monotone operator in strictly convex Banach spaces;
3. $J$ is a continuous operator in smooth Banach spaces;
4. $J$ is a uniformly continuous operator on each bounded set in uniformly smooth Banach spaces;
5. $J$ is the identity operator in Hilbert spaces, that is, $J = I_H$;
6. $J(x) = \partial(\|x\|^2/2)$, where $\partial(\|x\|^2/2)$ denotes subdifferential of $\|x\|^2/2$ at $x$.

Let $G : B^* \times B \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2 + f(x),$$

where $\varphi \in B^*$, $x \in B$ and $f : B \to R \cup \{+\infty\}$ is proper, convex, lower semi-continuous, and bounded from below. It is easy to see that

$$G(\varphi, x) \geq (\|\varphi\| - \|x\|)^2 + f(x).$$

From the definitions of $G$ and $f$, it is easy to see that the $G$ has the following properties:

1. $G(\varphi, x)$ is convex and continuous with respect to $\varphi$ when $x$ is fixed;
2. $G(\varphi, x)$ is convex and lower semi-continuous with respect to $x$ when $\varphi$ is fixed;
3. $(\|\varphi\| - \|x\|)^2 + f(x) \leq G(\varphi, x) \leq (\|\varphi\| + \|x\|)^2 + f(x)$.

**Lemma 1.1.** ([15, p. 94].) Let $X$ be a Banach space. The following conditions are equivalent.

1. $X$ is strictly convex;
If \( x, y \in X \) and \( \|x + y\| = \|x\| + \|y\| \), then \( x = 0 \) or \( y = 0 \) or \( y = \alpha x \) for some \( \alpha > 0 \).

**Definition 1.1:** Let \( B \) be a Banach space with dual space \( B^* \). Let \( K \) be a nonempty, closed and convex subset of \( B \). We say that \( \pi^f_K : B^* \to K \) is a generalised \( f \)-projection operator if

\[
\pi^f_K \varphi = \{ u \in K : G(\varphi, u) = \inf_{y \in K} G(\varphi, y) \} \quad \forall \varphi \in B^*.
\]

**Remark 1.1:** If \( f(x) = 0 \) for all \( x \in B \), then the generalised \( f \)-projection operator reduces to the generalised projection operator defined by Alber [1] and Li [17].

### 2. Properties of the Generalised \( f \)-projection \( \pi^f_K \)

The following theorem shows that the operator \( \pi^f_K \) is well defined for reflexive Banach spaces.

**Theorem 2.1:** If \( B \) is a reflexive Banach space with dual space \( B^* \) and \( K \) is a nonempty closed convex subset of \( B \), then \( \pi^f_K \varphi \neq \emptyset \) for all \( \varphi \in B^* \).

**Proof:** For any given \( \varphi \in B^* \) and \( x \in B \), we have

\[
(\|\varphi\| - \|x\|)^2 + f(x) \leq G(\varphi, x) \leq (\|\varphi\| + \|x\|)^2 + f(x).
\]

Since \( f \) is bounded from below, it follows that, for any given \( \varphi \in B^* \), \( \inf_{y \in K} G(\varphi, y) \) is finite and so there exist a sequence \( \{x_n\} \in K \) such that

\[
\lim_{n \to \infty} G(\varphi, x_n) = \inf_{y \in K} G(\varphi, y).
\]

Let \( \inf_{y \in K} G(\varphi, y) = a \). Then for any given \( \varepsilon > 0 \), there exists \( N > 0 \) such that

\[
|G(\varphi, x_n) - a| < \varepsilon
\]

for all \( n \geq N \). Thus,

\[
(\|\varphi\| - \|x_n\|)^2 + f(x_n) - a < \varepsilon.
\]

Since \( f(x) \) is bounded from below, there exists \( L \in \mathbb{R} \) such that

\[
(\|\varphi\| - \|x_n\|)^2 + L - a < \varepsilon.
\]

On the other hand,

\[
a = \inf_{y \in K} G(\varphi, y) = \inf_{y \in K} \{ \|\varphi\|^2 - 2\langle \varphi, x_n \rangle + \|x_n\|^2 + f(x_n) \}
\]

\[
\geq \inf_{y \in K} \{ \|\varphi\|^2 - 2\langle \varphi, x_n \rangle + \|x_n\|^2 \} + L
\]

\[
\geq \inf_{y \in K} (\|\varphi\| - \|x_n\|)^2 + L
\]

\[
\geq L.
\]
From (2.2) and (2.3), we know that \( \{x_n\} \) is bounded. Since \( B \) is reflexive, there exist a subsequence of \( \{x_n\} \), which after relabelling we again denote by \( \{x_n\} \), and a point \( x_0 \in K \) such that \( \{x_n\} \) converges weakly to \( x_0 \). For each given \( \varphi \), since \( G(\varphi, x) \) is convex and lower-semi-continuous with respect to \( x \), we know that \( G(\varphi, x) \) is weakly lower-semi-continuous with respect to \( x \). Thus, we have

\[
G(\varphi, x_0) \leq \liminf_{n \to \infty} G(\varphi, x_n) = \lim_{n \to \infty} G(\varphi, x_n) = \inf_{y \in K} G(\varphi, y)
\]

and so \( x_0 \in \pi_K^f \varphi \). Therefore, \( \pi_K^f \varphi \neq \emptyset \). This completes the proof. \( \square \)

**Theorem 2.2.** If \( B \) is a reflexive Banach space with dual space \( B^* \) and \( K \) is a nonempty, closed and convex subset of \( B \), then the following properties hold:

1. For any given \( \varphi \in B^* \), \( \pi_K^f \varphi \) is a nonempty, closed, bounded, and convex subset of \( K \);
2. \( \pi_K^f \) is monotone, that is, for any \( \varphi_1, \varphi_2 \in B^* \), \( x_1 \in \pi_K^f \varphi_1 \) and \( x_2 \in \pi_K^f \varphi_2 \),
   \[
   \langle x_1 - x_2, \varphi_1 - \varphi_2 \rangle \geq 0;
   \]
3. If \( B \) is smooth, then for any given \( \varphi \in B^* \), \( x \in \pi_K^f \varphi \) if and only if
   \[
   2\langle \varphi - J(x), x - y \rangle + f(y) - f(x) \geq 0
   \]
   for all \( y \in K \);
4. If \( K \) is a closed convex cone and \( f : K \to R \cup \{+\infty\} \) is positively homogeneous, that is, \( f(tx) = tf(x) \) for all \( t > 0 \) and \( x \in K \), then for any \( \varphi \in B^* \) and \( x_1, x_2 \in \pi_K^f \varphi \), we have \( x_1 \neq \mu x_2 \) for all \( \mu \in (0, +\infty) \) with \( \mu \neq 1 \);
5. If \( K \) is a closed convex cone, \( f : K \to R \cup \{+\infty\} \) is positively homogeneous and \( B \) is strictly convex, then the operator \( \pi_K^f : B^* \to K \) is single-valued.

**Proof:** (f1) For any point \( \varphi \in B^* \), Theorem 2.1 implies that \( \pi_K^f \varphi \) is nonempty. Since \( f \) is bounded from below and \( G(\varphi, x) \geq (\|\varphi\| - \|x\|)^2 + f(x) \), it is easy to see that \( \pi_K^f \varphi \) is bounded. Next we prove that \( \pi_K^f \varphi \) is closed. Suppose \( \{x_n\} \in \pi_K^f \varphi \) and \( x_n \to x_0 \) as \( n \to \infty \). By property (viii) of the functional \( G \), we have

\[
G(\varphi, x_0) \leq \liminf_{n \to \infty} G(\varphi, x_n) = \lim_{n \to \infty} G(\varphi, x_n) = \inf_{y \in K} G(\varphi, y).
\]

Thus, \( x_0 \in \pi_K^f \varphi \) and so \( \pi_K^f \varphi \) is closed. Finally, we prove that \( \pi_K^f \varphi \) is convex. Suppose \( x_1, x_2 \in \pi_K^f \varphi \) and \( 0 \leq t \leq 1 \). From property (viii) of the functional \( G \), we have

\[
G(\varphi, tx_1 + (1 - t)x_2) \leq tG(\varphi, x_1) + (1 - t)G(\varphi, x_2)
= t \inf_{y \in K} G(\varphi, y) + (1 - t) \inf_{y \in K} G(\varphi, y)
= \inf_{y \in K} G(\varphi, y)
\]
and so $tx_1 + (1-t)x_2 \in \pi_K^f \varphi$. This implies that $\pi_K^f \varphi$ is convex.

($f_2$) For any $\varphi_1, \varphi_2 \in B^*$, $x_1 \in \pi_K^f \varphi_1$ and $x_2 \in \pi_K^f \varphi_2$, from definition $\pi_K^f$, we have

$$||\varphi_1||^2 - 2\langle \varphi_1, x_1 \rangle + ||x_1||^2 + f(x_1) \leq ||\varphi_1||^2 - 2\langle \varphi_1, x_2 \rangle + ||x_2||^2 + f(x_2)$$

and

$$||\varphi_2||^2 - 2\langle \varphi_2, x_2 \rangle + ||x_2||^2 + f(x_2) \leq ||\varphi_2||^2 - 2\langle \varphi_2, x_1 \rangle + ||x_1||^2 + f(x_1).$$

It follows from ($2.4$) and ($2.5$) that $\pi_K^f$ is monotone.

($f_3$) We first prove that $x \in \pi_K^f \varphi$ implies that

$$2\langle \varphi - J(x), x - y \rangle + f(y) - f(x) \geq 0, \quad \forall y \in K.$$ 

In fact, for any $y \in K$ and $t \in (0,1]$, it follows from the definition of $\pi_K^f$ that

$$G(\varphi, x) \leq G(\varphi, x + t(y - x)).$$

Thus,

$$||\varphi||^2 - 2\langle \varphi, x \rangle + ||x||^2 + f(x) \leq ||\varphi||^2 - 2\langle \varphi, x + t(y - x) \rangle$$

$$+ ||x + t(y - x)||^2 + f(x + t(y - x))$$

and so

$$2\langle \varphi, t(y - x) \rangle + ||x||^2 + f(x) \leq ||x + t(y - x)||^2 + f(x + t(y - x))$$

$$\leq ||x + t(y - x)||^2 + (1-t)f(x) + tf(y).$$

It follows that

$$2\langle \varphi, t(y - x) \rangle + ||x||^2 \leq ||x + t(y - x)||^2 + t(f(y) - f(x)).$$

Now from the properties of the normalised duality mapping, we have

$$\langle J(x + t(y - x)), -t(y - x) \rangle \leq \frac{1}{2} ||x||^2 - \frac{1}{2} ||x + t(y - x)||^2.$$ 

By (2.6), we get

$$2\langle J(x + t(y - x)), y - x \rangle \geq f(x) - f(y) + 2\langle (\varphi, y - x) \rangle.$$ 

Since $B$ is smooth, we know that $J$ is demi-continuous. Letting $t \to 0$ in the above inequality, we have

$$2\langle J(x) - \varphi, y - x \rangle + f(y) - f(x) \geq 0.$$
Conversely, suppose

\[ 2\langle J(x) - \varphi, y - x \rangle + f(y) - f(x) \geq 0 \text{ for all } y \in K. \]

Then

\[ \|y\|^2 - \|x\|^2 \geq 2\langle J(x), y - x \rangle - 2\langle \varphi, y - x \rangle + f(x) - f(y) \]

which implies that \( G(\varphi, x) \leq G(\varphi, y) \) for all \( y \in K \), that is, \( x \in \pi_1^K\varphi \).

\((f_4)\) Assume \( x_1, x_2 \in \pi_1^K\varphi \) and \( x_1 = \mu x_2 \) for some real number \( \mu > 0 \) with \( \mu \neq 1 \).

Then \( G(\varphi, x_1) = G(\varphi, x_2) \) and so

\[ 2\langle \varphi, x_2 - x_1 \rangle = \|x_2\|^2 + f(x_2) - \|x_1\|^2 - f(x_1). \]

Replacing \( x_1 \) by \( \mu x_2 \) in above equality, we have

\[ 2(1 - \mu)\langle \varphi, x_2 \rangle = (1 - \mu^2)\|x_2\|^2 + (1 - \mu)f(x_2). \]

Since \( \mu \neq 1 \), we obtain

\[ 2\langle \varphi, x_2 \rangle = (1 + \mu)\|x_2\|^2 + f(x_2). \]

Let

\[ x_3 = (x_2 + x_1)/2 = ((1 + \mu)/2)x_2. \]

It follows from \((f_1)\) that \( x_3 \in \pi_1^K\varphi \) and so \( G(\varphi, x_2) = G(\varphi, x_3) \). Similarly, we can get

\[ 2\langle \varphi, x_2 \rangle = \left(1 + \frac{1}{2}(1 + \mu)\right)\|x_2\|^2 + f(x_2). \]

Now \((2.7)\) and \((2.8)\) imply that \( 1 + \mu = 1 + (1 + \mu)/2 \) and so \( \mu = 1 \), which is a contradiction to \( \mu \neq 1 \). Thus, \((f_4)\) is true.

\((f_5)\) Suppose there exists \( \varphi \in B^* \) such that \( \pi_1^K\varphi \) is not a singleton. Then for any \( x_1, x_2 \in \pi_1^K\varphi \) and \( x_1 \neq x_2 \), we have \( G(\varphi, x_1) = G(\varphi, x_2) \). This implies

\[ 2\langle \varphi, x_2 - x_1 \rangle = \|x_2\|^2 + f(x_2) - \|x_1\|^2 - f(x_1). \]

By property \((f_1)\), for any \( t \in [0, 1] \), we know that \( x_1 + t(x_2 - x_1) \in \pi_1^K\varphi \). Since \( G(\varphi, x_1 + t(x_2 - x_1)) = G(\varphi, x_1) \),

\[ 2t\langle \varphi, x_2 - x_1 \rangle = \|x_1 + t(x_2 - x_1)\|^2 + f(x_1 + t(x_2 - x_1)) - \|x_1\|^2 - f(x_1). \]

Combining \((2.9)\) and \((2.10)\), we have

\[ t(\|x_2\|^2 + f(x_2) - \|x_1\|^2 - f(x_1)) \]

\[ = \|x_1 + t(x_2 - x_1)\|^2 + f(x_1 + t(x_2 - x_1)) - \|x_1\|^2 - f(x_1) \]

\[ \leq \|x_1 + t(x_2 - x_1)\|^2 + t\left((x_2 - f(x_1))\right) - \|x_1\|^2. \]
This implies that
\[
(2.11) \quad t\|x_2\|^2 + (1 - t)\|x_1\|^2 \leq \|x_1 + t(x_2 - x_1)\|^2
\]
and so
\[
\|x_1 + t(x_2 - x_1)\|^2 \leq (t\|x_2\| + (1 - t)\|x_1\|)^2
\]
\[
= t^2\|x_2\|^2 + 2t(1 - t)\|x_1\|\|x_2\| + (1 - t)^2\|x_1\|^2
\]
\[
\leq t\|x_2\|^2 + (1 - t)\|x_1\|^2
\]
\[
\leq \|x_1 + t(x_2 - x_1)\|^2.
\]

Thus,
\[
t\|x_2\| + (1 - t)\|x_1\| = \|x_1 + t(x_2 - x_1)\|.
\]

Taking \(t = 1/2\) in the above equation, we get
\[
\|x_2\| + \|x_1\| = \|x_1 + x_2\|.
\]

From (2.11), we know that if \(x_1 = 0\), then \(x_2 = 0\). Hence \(x_1 \neq 0\) and \(x_2 \neq 0\). Since \(B\) is strictly convex, according to Lemma 1.1, there exists some \(\alpha > 0\) such that \(x_1 = \alpha x_2\), which contradicts (f4). This completes the proof. \(\square\)

From (f3), it is easy to prove the following result.

**Theorem 2.3.** Let \(A\) be an arbitrary operator acting from the reflexive and smooth Banach space \(B\) to \(B^*\), and \(\xi \in B^*\). Then the point \(x^* \in K \subset B\) is a solution of the variational inequality
\[
(Ax - \xi, y - x) + f(y) - f(x) \geq 0, \quad \forall y \in K
\]
if and only if \(x^*\) is a solution of the following inclusion
\[
x \in \pi_K(J(x) - \frac{1}{2}(Ax - \xi)).
\]

3. Applications

As an application of our results, in this section, we shall study the existence of solutions to the following variational inequality problem: Find \(x^* \in K\) such that
\[
(3.1) \quad (Ax^*, y - x) + f(y) - f(x^*) \geq 0, \quad \forall y \in K,
\]
where \(K\) is a nonempty, closed and convex subset of the Banach space \(B\), and \(A : K \to B^*\) and \(f : K \to R \cup \{+\infty\}\) are two mappings.
DEFINITION 3.1: (KKM mapping) Let \( K \) be a nonempty subset of a linear space \( X \). A set-valued mapping \( G : K \to 2^X \) is said to be a KKM mapping if for any finite subset \( \{y_1, y_2, \cdots, y_n\} \) of \( K \), we have

\[
\text{co}\{y_1, y_2, \cdots, y_n\} \subseteq \bigcup_{i=1}^{n} G(y_i),
\]

where \( \text{co}\{y_1, y_2, \cdots, y_n\} \) denotes the convex hull of \( \{y_1, y_2, \cdots, y_n\} \).

LEMMA 3.1. (FanKKM Theorem [20].) Let \( K \) be a nonempty convex subset of a Hausdorff topological vector space \( X \) and let \( G : K \to 2^X \) be a KKM mapping with closed values. If there exists a point \( y_0 \in K \) such that \( G(y_0) \) is a compact subset of \( K \), then \( \bigcap_{y \in K} G(y) \neq \emptyset \).

THEOREM 3.1. Let \( K \) be a nonempty, closed and convex subset of a reflexive and smooth Banach space \( B \) with dual space \( B' \). Let \( A : K \to B^* \) be a continuous mapping and \( f : K \to \mathbb{R} \cup \{+\infty\} \) be proper, convex, lower semi-continuous, and bounded from below. Let there exist an element \( y_0 \in K \) such that

\[
(3.2) \quad \{x \in K : 2\langle Jx - \frac{1}{2}Ax, y_0 - x \rangle + \|x\|^2 + f(x) \leq \|y_0\| + f(y_0)\}
\]

is a compact subset of \( K \). Then the variational inequality (3.1) has a solution.

PROOF: From Theorem 2.3, we only need to prove that the following inclusion has a solution,

\[
x \in \pi_K'(J(x) - \frac{1}{2}Ax).
\]

Define a set-valued mapping \( W : K \to 2^K \) as follows:

\[
W(y) = \{x \in K : G\left(Jx - \frac{1}{2}Ax, x\right) \subseteq G\left(Jx - \frac{1}{2}Ax, y\right)\}.
\]

Clearly, for each given \( y \in K \), \( W(y) \) is nonempty. Next we prove that \( W(y) \) is closed for each given \( y \in K \). Suppose \( \{x_n\} \subseteq W(y) \) and \( x_n \to x_0 \) as \( n \to \infty \). Then,

\[
G\left(Jx_n - \frac{1}{2}Ax_n, x_n\right) \subseteq G\left(Jx_n - \frac{1}{2}Ax_n, y\right)
\]

and so

\[
-2\langle Jx_n - \frac{1}{2}Ax_n, x_n \rangle + \|x_n\|^2 + f(x_n) \leq -2\langle Jx_n - \frac{1}{2}Ax_n, y \rangle + \|y\|^2 + f(y).
\]

Since \( J \) and \( A \) are continuous and \( f \) is lower-semi-continuous,

\[
-2\langle Jx_0 - \frac{1}{2}Ax_0, x_0 \rangle + \|x_0\|^2 + f(x_0) \leq -2\langle Jx_0 - \frac{1}{2}Ax_0, y \rangle + \|y\|^2 + f(y).
\]
Hence,
\[ G\left(Jx_0 - \frac{1}{2}Ax_0, x_0\right) \leq G\left(Jx_0 - \frac{1}{2}Ax_0, y\right), \]
which implies that \( x_0 \in W(y) \).

Next we prove that the map \( W : K \to 2^K \) is a KKM mapping in \( K \). In fact, suppose \( y_1, y_2, \ldots, y_n \in K \) and \( 0 < \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \). Let \( v = \sum_{i=1}^{n} \lambda_i y_i \). By property (viii) of \( G \), we have
\[
G(Jv - \frac{1}{2}Av, v) = G\left(Jv - \frac{1}{2}Av, \sum_{i=1}^{n} \lambda_i y_i\right) \\
\leq \sum_{i=1}^{n} \lambda_i G\left(Jv - \frac{1}{2}Av, y_i\right).
\]
This implies that
\[ G\left(Jv - \frac{1}{2}Av, v\right) \leq \max_{1 \leq i \leq n} G\left(Jv - \frac{1}{2}Av, y_i\right). \]
Hence there exists at least one number \( j = 1, 2, \ldots, n \), such that
\[ G\left(Jv - \frac{1}{2}Av, v\right) \leq G\left(Jv - \frac{1}{2}Av, y_j\right), \]
that is, \( v \in W(y_j) \). Thus, \( W \) is a KKM mapping.

If \( x \in W(y_0) \), then \( G(Jx - (1/2)A, x) \leq G(Jx - (1/2)A, y_0) \). From the definition of \( G \), we have
\[
\left\|Jx - \frac{1}{2}A\right\|^2 - 2\left\langle Jx - \frac{1}{2}A, x\right\rangle + \|x\|^2 + f(x) \\
\leq \left\|Jx - \frac{1}{2}A\right\|^2 - 2\left\langle Jx - \frac{1}{2}A, y_0\right\rangle + \|y_0\|^2 + f(y_0).
\]
Simplifying the above inequality, we have
\[
2\left\langle Jx - \frac{1}{2}A, y_0 - x\right\rangle + \|x\|^2 + f(x) \leq \|y_0\|^2 + f(y_0).
\]
We get that
\[
W(y_0) = \left\{ x \in K : 2\left\langle Jx - \frac{1}{2}A, y_0 - x\right\rangle + \|x\|^2 + f(x) \leq \|y_0\|^2 + f(y_0) \right\}.
\]
By condition (3.2), we know that \( W(y_0) \) is compact. It follows from Lemma 3.1 that \( \bigcap_{y \in K} W(y) \neq \emptyset \) and so there exits at least one \( x^* \in \bigcap_{y \in K} W(y) \), that is,
\[
G\left(Jx^* - \frac{1}{2}A, x^*\right) \leq G\left(Jx^* - \frac{1}{2}A, y\right), \quad \forall y \in K.
\]
From the definition of the generalised $f$-projection operator $\pi_K^f$, we have

$$x^* \in \pi_K^f \left( Jx^* - \frac{1}{2} Ax^* \right).$$

This completes the proof.

**THEOREM 3.2.** Let $B$ be a reflexive and smooth Banach space with dual space $B^*$ and $K$ be a nonempty, closed and convex subset that contains the origin $\theta$ of $B$. Let $A : K \to B^*$ be a continuous mapping and $f : K \to \mathbb{R} \cup \{+\infty\}$ be proper, convex, lower semi-continuous, and bounded from below. If the set

$$(3.3) \quad \{ x \in K : (Ax, x) + f(x) \leq \|x\|^2 + f(\theta) \}$$

is compact, then variational inequality (3.1) has a solution.

**PROOF:** Taking $y_0 = \theta$ in condition (3.2) and noticing that $(Jx, x) = \|x\|^2$, it follows from condition (3.3) that all conditions of Theorem 3.1 hold. Thus Theorem 3.1 implies that the conclusion of Theorem 3.2 hold. This completes the proof. $\square$

From Theorem 3.2, it is easy to have the following result.

**THEOREM 3.3.** Let $B$ be a reflexive and smooth Banach space with dual space $B^*$ and $K$ be a nonempty, closed and convex cone of $B$. Let $A : K \to B^*$ be a continuous mapping and $f : K \to \mathbb{R} \cup \{+\infty\}$ be proper, convex, lower semi-continuous and bounded from below. If

$$\{ x \in K : (Ax, x) + f(x) \leq \|x\|^2 + f(\theta) \}$$

is compact, then variational inequality (3.1) has a solution.

**REFERENCES**


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