MAGIC GRAPHS

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Introduction. In this paper we use number-theoretic properties to classify ordinary graphs that are finite and have no isolated vertices. The classification depends on whether there is an assignment of real values, usually rational integer values, to the edges of the graph, such that the set of assigned values and the set of vertex sums of these values, summed at each vertex over all the edges incident to the vertex, will be a pair of sets with prescribed properties. Then we seek corresponding graph-theoretic properties.

It is possible to describe the problem in terms of a symmetric matrix having specified properties for its row sums, but in this paper we make no use of this interpretation; however, see (3).

It is possible also to interpret the problem for any graph \( G \) by passing to its plane projective dual \( P(G) \) in which vertex-incident edges in \( G \) are replaced by collinear vertices in \( P(G) \). This idea is a bit foreign for graph theory, which does not ordinarily concern itself with collinearity; and also a bit foreign for projective geometry, since the ordinary restriction, that a pair of vertices in \( G \) be joined by at most one edge, results in \( P(G) \) having exactly two lines through each vertex. But the edge-assignment, vertex-sum concepts in \( G \) become rather natural vertex-assignment, line-sum concepts in \( P(G) \). In particular, for the regular complete bipartite graph \( K(n, n) \), the corresponding \( P(K(n, n)) \) is a square grid; and under certain conditions for assignments and sums (see Example 1, §7), we find that our results for \( K(n, n) \) correspond in \( P(K(n, n)) \) to the magic squares of number theory. We borrow the language of the past and ascribe to the original graphs various degrees of supernatural power, such as trivially-magic, zero-magic, semi-magic, pseudo-magic, magic, super-magic, and prime-magic.

Our interest in this topic originated in a problem proposed by J. Sedlacek in (1). In particular, our use of the words "magic" and "prime-magic" agrees with Sedlacek, but we find it useful to introduce the five other classes.

1. The edge space of a graph. Consider a graph \( G \) with vertices \( v_1, v_2, \ldots, v_n \) and undirected edges \( e_1, e_2, \ldots, e_E \), having no multiple edges, no loops, and no isolated vertices (hence \( n \geq 2, E \geq n/2 \)). Let \( \alpha(e) \), or \( \alpha \), indicate a function defined on the edges of \( G \) with values in the real field \( R \). The set \( A(G) \) of all \( \alpha \) is a vector space over \( R \) under the rule

\[
(x_1 \alpha_1 + x_2 \alpha_2)(e) = x_1 \alpha_1(e) + x_2 \alpha_2(e)
\]
for all \( x_1, x_2 \) in \( R \), all \( \alpha_1, \alpha_2 \) in \( A(G) \), and all \( e \) in \( G \). We shall call \( A(G) \) the edge space of \( G \).

In particular, let \( \epsilon_i \) indicate the function \( \epsilon_i(e_j) = d_{ij}, i, j = 1, 2, \ldots, E \), where \( d_{ij} = 0 \) if \( i \neq j \) and \( d_{ii} = 1 \). The set \( \{ \epsilon_i \} \) is a basis for \( A(G) \). For if \( \alpha \) is any function in \( A(G) \), say with \( \alpha(e_i) = a_i \), we can represent \( \alpha \) in the form \( \alpha = a_1 \epsilon_1 + a_2 \epsilon_2 + \ldots + a_E \epsilon_E \); and the representation is unique, for if \( \xi \) is the zero of the vector space with \( \xi(e) = 0 \) for every \( e \) in \( G \), then \( \alpha = \xi \) if and only if every \( a_i = 0 \). Hence in dimension we have \( \dim A(G) = E \).

Each function \( \alpha \) in \( A(G) \) creates a partitioning \( P(\alpha) \) of the edges of \( G \) into classes, where \( e_i \) and \( e_j \) are in the same class if and only if \( \alpha(e_i) = \alpha(e_j) \). Let us use a double subscript notation \( e_{uv} \) in which \( u \) indicates the class and \( v \) indicates an (arbitrary) enumeration within the class. Suppose there are \( w \) classes and that in the \( u \) class there are \( t_u \) edges. Since \( R \) is an ordered field, if \( w > 1 \), we can order the classes and describe \( \alpha \) as follows:

\[
\alpha(e_{uv}) = a_u, \quad 1 \leq u \leq w, \quad 1 \leq v \leq t_u, \quad a_1 < a_2 < \ldots < a_w, \quad t_1 + t_2 + \ldots + t_w = E.
\]

As usual we speak of a partitioning \( P' \) as a proper refinement of a partitioning \( P \) if every class in \( P' \) is a subclass of \( P \) and the number of classes in \( P' \) exceeds the number of classes in \( P \).

**Theorem 1.** If the partitioning \( P(\alpha) \) has \( t_u > 1 \), if \( \alpha' \) has its values in the rational domain and \( \alpha'(e_{u,t_u}) \neq \alpha'(e_{u,1}) \), then \( P(\alpha + xa') \) is a proper refinement of \( P(\alpha) \) when \( x > a_w - a_1 \).

**Proof.** If \( w > 1 \), let \( e_{ps} \) and \( e_{qt} \) be representatives of classes in \( P(\alpha) \). Suppose \( e_{ps} \) and \( e_{qt} \) belong to the same class in \( P(\alpha + xa') \) so that

\[
(\alpha + xa')(e_{ps}) = (\alpha + xa')(e_{qt}).
\]

Then by (1) we have

\[
x(\alpha'(e_{ps}) - \alpha'(e_{qt})) = \alpha(e_{qt}) - \alpha(e_{ps}).
\]

If we assume that \( \alpha'(e_{ps}) > \alpha'(e_{qt}) \), since the values of \( \alpha' \) are rational integers it follows that

\[
x(\alpha'(e_{ps}) - \alpha'(e_{qt})) > x > a_w - a_1 > \alpha(e_{qt}) - \alpha(e_{ps}),
\]

a contradiction. There is a similar contradiction if we assume that

\[
\alpha'(e_{ps}) < \alpha'(e_{qt}).
\]

Hence \( \alpha'(e_{ps}) = \alpha'(e_{qt}) \). Since \( x \neq 0 \), this implies that \( \alpha(e_{qt}) = \alpha(e_{ps}) \) which in turn implies that \( q = p \). In other words, \( P(\alpha + xa') \) is a refinement of \( P(\alpha) \). Furthermore, since \( x \neq 0 \), we have \( a_u + xa'(e_{u,t_u}) \neq a_u + xa'(e_{u,1}) \) so that \( (\alpha + xa')(e_{u,t_u}) \neq (\alpha + xa')(e_{u,1}) \). Hence \( P(\alpha + xa') \) is a proper refinement of \( P(\alpha) \).
If \( w = 1 \), only the matter of a proper refinement is in question, and the argument in the preceding two sentences is adequate with \( u = 1, \tau_1 > 1, x > 0 \).

In the usual applications of Theorem 1, the original \( \alpha \), like \( \alpha' \), will have all its values in the rational domain and \( x \) will be a rational integer; then \( \alpha + x\alpha' \) also will have its values in the rational domain, and \( P(\alpha + x\alpha') \) will be a proper refinement of \( P(\alpha) \) when \( x \geq a_w - a_1 + 1 \).

2. The semi-magic space of a graph. For each vertex \( v_i \) in \( G \) we define the vertex sum \( \sigma^a(v_i) = \sum \alpha(e) \) where the sum \( \sum \) extends over all edges of \( G \) which have \( v_i \) as an end point. The vertex sum is defined for each \( v_i \) in \( G \) since we assumed \( G \) has no isolated vertices. The vertex sum is a covariant of \( G \) under \( A(G) \) since

\[
\sigma^{x_1a_1+x_2a_2}(v_i) = x_1 \sigma^{a_1}(v_i) + x_2 \sigma^{a_2}(v_i).
\]

Let \( S(G) \) be the set of all \( \alpha \) in \( A(G) \) which satisfy the following semi-magic condition:

\[
\sigma^a(v_i) = \sigma(\alpha), \quad 1 \leq i \leq n;
\]

i.e., there is a "constant vertex sum, \( \sigma(\alpha) \)" for all the vertices of \( G \).

The set \( S(G) \) is not empty, since \( \xi \) satisfies (M.1) with \( \sigma(\xi) = 0 \). In fact, \( S(G) \) is a subspace of \( A(G) \) which we shall call the semi-magic space of \( G \). The proof follows readily from (2), for if \( \alpha_1 \) and \( \alpha_2 \) are in \( S(G) \) with

\[
\sigma^{a_1}(v_i) = \sigma(\alpha_1), \quad \sigma^{a_2}(v_i) = \sigma(\alpha_2), \quad 1 \leq i \leq n,
\]

then

\[
\sigma^{x_1a_1+x_2a_2}(v_i) = x_1 \sigma(\alpha_1) + x_2 \sigma(\alpha_2), \quad 1 \leq i \leq n.
\]

Thus \( x_1 \alpha_1 + x_2 \alpha_2 \) is in \( S(G) \) for all \( x_1, x_2 \) in \( R \). Furthermore,

\[
\sigma(x_1 \alpha_1 + x_2 \alpha_2) = x_1 \sigma(\alpha_1) + x_2 \sigma(\alpha_2)
\]

so that the constant vertex sum is a covariant of \( G \) under \( S(G) \).

**Theorem 2.** If \( G \) is connected, then

\[
E - n + 1 \leq \dim S(G) \leq E - n + 2.
\]

**Proof.** Note that \( G_1 = K(1, 2) \), the arc of length 2, has

\[
\dim S(G_1) = 0 = 2 - 3 + 1 = E - n + 1
\]

and that \( G_2 = K(1, 1) = K_2 \), the arc of length 1, has

\[
\dim S(G_2) = 1 = 1 - 2 + 2 = E - n + 1.
\]
Thus the limits in (4) are the best possible. Note that $G$ connected implies $E \geq n - 1$; hence the lower bound in (4) is non-negative.

We can eliminate $\sigma(a)$ from pairs of equations in (M.1) and obtain an equivalent system in which the last $n - 1$ equations form a subsystem $M$, linear and homogeneous in the $E$ coordinates $a_i$ of $\alpha$. For definiteness we take for $M$ the equations:

$$\sigma^a(v_i) - \sigma^a(v_{i+1}) = 0, \quad i = 1, 2, \ldots, n - 1.$$

(A) If $G$ is connected, then $E \geq n - 1$, so the system $M$ has a rank $r$ satisfying $r \leq \min(E, n - 1) = n - 1$. Hence $r$ of the variables $a_i$ may be expressed in terms of the remaining $E - r$ variables or parameters. In other words, $E - n + 1 \leq E - r = \dim S(G)$. It is worth noting that the coefficients in $M$ are restricted to $-1, 0, +1$; hence the solutions are linear combinations of the parameters with rational coefficients. By a proper selection of the parameters we can obtain solutions in which all the $a_i$ are rational integers, in agreement with the remarks in the Introduction.

(B) If $G$ is connected, we can establish the other part of (4) by an edge-completion argument, holding $n$ fixed and making a finite induction on $E$. We assume the $E$ for $G$ satisfies $n - 1 \leq E < \binom{n}{2}$ and that

$$\dim S(G) \leq E - n + 2, \quad n \geq 3.$$

Let $G'$ be formed from $G$ by inserting any additional edge $e'$. The vertices may be labelled so that $e' = v_{n-1} v_n$. The system $M'$ for $G'$, although involving $E' = E + 1$ coordinates, differs from the system $M$ for $G$ only in having the next-to-last equation contain an extra term $-a_{e'}$. Consequently the rank $r'$ of $M'$ and the rank $r$ of $M$ satisfy $r \leq r' \leq r + 1$. Then

$$E - r \leq E + 1 - r' = E' - r' \leq 1 + E - r;$$

hence

$$\dim S(G) \leq \dim S(G') \leq 1 + \dim S(G).$$

Applying the induction hypothesis, we find that

$$\dim S(G') \leq 1 + \dim S(G) \leq 1 + (E - n + 2) = E' - n + 2.$$

As a basis for the induction we note that a connected $G$ with $E = n - 1$ is a tree $T$. The condition (M.1) dictates that $\alpha(e)$ must be the same for all terminal edges of the tree. This completely determines the assignment for all interior edges. Hence

$$0 \leq \dim S(T) \leq 1 = (n - 1) - n + 2 = E - n + 2.$$

We can pursue the completion argument in (B) and obtain a number of corollaries to Theorem 2. Some preliminary notes and lemmas will be helpful.
When $G$ is connected, we note that $E - n + 1$ is the circuit rank, $C(G)$, with a variety of interpretations.

**Lemma 1.** If $G$ is connected, but not complete, if $G'$ is obtained from $G$ by edge completion, and if $\dim S(G) = C(G)$, then $\dim S(G') = C(G')$.

**Proof.** We note that $C(G') = 1 + C(G)$. Then we apply the hypothesis $\dim S(G) = C(G)$ and the relations (4) and (5) to obtain

$$1 + \dim S(G) = 1 + C(G) = C(G') \leq \dim S(G') \leq 1 + \dim S(G).$$

Hence $\dim S(G') = C(G')$.

With edge completion in mind, we know the basic connected graph is a tree. We may fix attention on one terminal end of a tree by calling this vertex a root. By Euler's relation, the tree has at least one other terminal end, say $v'$, where the requirement $\sigma^a(v') = a \neq 0$, in the following lemma, can be realized by taking $\alpha(v') = a$ for the terminal edge $v'$ incident with $v'$.

**Lemma 2.** If $\alpha$ is in the edge space of a tree $T$ with root $q$, if $\sigma^a(v_i) = a \neq 0$, for all vertices of $T$, except possibly the root $q$, and if $\sigma^a(q) = ka$, then $k$ is an integer and $n = 1 + k \pmod{2}$.

**Proof.** The proof is by induction on $n > 2$. If $n = 2$, the single edge must have the assignment $a$, so $k = 1$ and $n = 1 + k \pmod{2}$. For $n > 3$ assume the lemma is correct for all trees when the number of vertices is less than $n$. Let $qv$ be the root edge assigned the value $ka$. At the stem point $v$, there are (in addition to the root edge), say, $t$ trees $T_1, T_2, \ldots, T_t$, disjoint except for having the common root $v$, and having $n_1, n_2, \ldots, n_t$ vertices, respectively, where every $n_i < n$. Since $n > 3$, we have $t > 1$. The induction hypothesis implies that each tree $T_i$ has a root edge, say $w_i$, which has an edge value $k_i a$, where $k_i$ is an integer and $n_i = 1 + k_i \pmod{2}$. Since the vertex sum at $v$ is $a$, we have $(k + k_1 + \ldots + k_t)a = a$. Since $a \neq 0$, $k + k_1 + \ldots + k_t = 1$. Hence $k$ is an integer and

$$n = 2 + (n_1 - 1) + \ldots + (n_t - 1) = k_1 + \ldots + k_t = 1 + k \pmod{2}.$$

**Lemma 3.** If $T$ is a tree with $n$ vertices and if $n$ is odd, $n > 3$, then $\dim S(T) = 0$.

**Proof.** A tree has $E = n - 1$; hence by (4) we have $0 \leq \dim S(T) \leq 1$. If $\dim S(T) = 1$, there is at least one $\alpha$ in $S(T)$ with $\sigma(\alpha) = a \neq 0$. For a star $T^*$ the condition $\sigma(\alpha) = a$, when enforced at the centre, requires $(n - 1)a = a$; so if $n > 3$ (odd or even), we find $a = 0$, hence $\dim S(T^*) = 0$. If $n = 3$, the only tree is a star.

If $T$ is a tree which is not a star (hence $n > 4$), then there is an interior edge $e$ with vertices $q_1$ and $q_2$. There are subgraphs $T_1$ and $T_2$ of $T$ determined

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by $e$ as follows: there is a tree $T_1$ with $q_1$ as a root and $q_2$ as a stem point with, say, $1 + n_1$ vertices, $n_1 \geq 2$; and there is a tree $T_2$ with $q_2$ as a root and $q_1$ as a stem point with, say, $1 + n_2$ vertices, $n_2 \geq 2$; furthermore, $n = n_1 + n_2$. Suppose $\sigma(e) = k a$. By Lemma 2, the condition $\sigma(e) = a \neq 0$ implies that $k$ is an integer with $k = n_1$ and $k = n_2 \pmod{2}$. Hence $n = n_1 + n_2 = 2k = 0 \pmod{2}$. Thus the case $\dim S(T) = 1$ can arise only if $n$ is even.

**Corollary 2.1.** If $G$ is connected and $n$ is odd, $n \geq 3$, then $\dim S(G) = C(G)$.

*Proof.* Regard $G$ as obtained from a tree $T$ having $n$ vertices by repeated applications of the edge-completion process. Since $n$ is odd, $n \geq 3$, according to Lemma 3 we have $\dim S(T) = 0 = C(T)$. Repeated application of Lemma 1 establishes Corollary 2.1.

**Corollary 2.2.** If $G$ has a vertex of degree $n - 1$, $n \geq 3$, then

$$\dim S(G) = C(G).$$

*Proof.* The hypothesis $\rho(v) = n - 1$ for some vertex $v$ of $G$ implies that $G$ is connected and contains a star graph $T^*$ with $n$ vertices from which $G$ can be obtained by repeated applications of the edge-completion process. Since $n \geq 3$, $\dim S(T^*) = 0 = C(T^*)$; then repeated application of Lemma 1 shows $\dim S(G) = C(G)$.

**Corollary 2.3.** For a complete graph $K_n$, $n \geq 3$, $\dim S(K_n) = \left(\binom{n-1}{2}\right)$.

*Proof.* This is a consequence of Corollary 2.2 since $K_n$ does contain a vertex of degree $n - 1$ and

$$C(K_n) = \binom{n}{2} - n + 1 = \frac{(n-1)(n-2)}{2} = \binom{n-1}{2}.$$

For the discussion of a graph that may not be connected, let $\tau(G)$ indicate the number of components of $G$. Let a typical component be $G_i$ of order $n_i$ with $E_i$ edges (of course, since $G_i$ is connected, $E_i \geq n_i - 1$). Let $\tau'(G)$ indicate the number of components of $G$ for which $\dim S(G_i) = 0$. Let $\sum$ indicate summation with an index $i = 1, 2, \ldots, \tau(G)$.

**Theorem 2'**. (A) If $\tau'(G) = 0$, then $\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i)$ and

$$\max(0, E - n + 1) \leq \dim S(G) \leq E - n + 1 + \tau(G).$$

(B) If $\tau'(G) > 0$, then $\dim S(G) = \tau'(G) - \tau(G) + \sum \dim S(G_i)$ and

$$\max(0, E - n + \tau'(G)) \leq \dim S(G) \leq E - n + \tau(G).$$
Proof. With each $G_i$ the condition (M.1) associates a system of $n_i$ equations involving a common vertex sum $\sigma_i$. Actually (M.1) applies to all of $G$ and demands $\sigma_1 = \sigma_2 = \ldots = \sigma_{\tau(G)}$, but we hold this in abeyance.

(A) If $\tau'(G) = 0$, the systems of equations for the components $G_i$ do not force any $\sigma_i = 0$; hence to satisfy (M.1) for all of $G$, there are exactly $\tau(G) - 1$ added conditions on the edge assignments, resulting, say, from setting $\sigma_i = \sigma_1$, $i = 2, 3, \ldots, \tau(G)$. Consequently, $\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i)$.

Applying Theorem 2 to each component, we have

$$E_i - n_i + 1 \leq \dim S(G_i) \leq E_i - n_i + 2.$$ 

Using $E = \sum E_i$ and $n = \sum n_i$, we obtain $(4')$.

(B) If $\tau'(G) > 0$, suppose $\dim S(G_i) = 0$ for $i = 1, 2, \ldots, \tau'(G)$ and $\dim S(G_j) > 0$ for $\tau'(G) < j \leq \tau(G)$. Of course, each $\sigma_i = 0$, $1 \leq i \leq \tau'(G)$; furthermore, the edge assignments for these components are necessarily all 0.

To satisfy (M.1) for all of $G$, there are exactly $\tau(G) - \tau'(G)$ added conditions on the edge assignments, resulting from setting $\sigma_j = 0$, $\tau'(G) < j \leq \tau(G)$. Consequently, $\dim S(G) = \tau'(G) - \tau(G) + \sum \dim S(G_i)$.

We obtain the first inequality in $(4'')$ by using

$$E_i - n_i + 1 \leq \dim S(G_i), \quad 1 \leq i \leq \tau(G).$$

To obtain the second inequality in $(4'')$ we note from (4) that $\dim S(G_i) = 0$ implies $E_i - n_i + 1 = 0$. We let $\sum'$ indicate summation over $1 \leq i \leq \tau'(G)$ and $\sum''$ indicate summation over $1 + \tau'(G) < j \leq \tau(G)$. Then

$$\dim S(G) = \tau'(G) - \tau(G) + \sum' (E_i - n_i + 1) + \sum'' \dim S(G_j)$$

$$\leq \tau'(G) - \tau(G) + \sum' (E_i - n_i + 1) + \sum'' (E_j - n_j + 2)$$

$$= \tau'(G) - \tau(G) + E - n + \tau'(G) + 2 \tau(G) - 2 \tau'(G)$$

$$= E - n + \tau(G).$$

To show that the limits in $(4')$ and $(4'')$ are the best possible consider the following graphs $T, H, Q$ as possible components of $G$ (see Figure 1). By Corollary 2.2 the star graph $T = K(1, 3)$ has $\dim S(T) = C(T) = 3 - 4 + 1 = 0$. By Corollary 2.2 the graph $H$ consisting of a triangle with a tail of length 1 has

![Figure 1](https://doi.org/10.4153/CJM-1966-104-7)
dim \( S(H) = C(H) = 4 - 4 + 1 = 1 \). By Theorem 2 the quadrilateral \( Q \) has \( 4 - 4 + 1 = 1 \leq \dim S(Q) \leq 2 \); but it is easy to check that \( \dim S(Q) = 2 \), for one pair of opposite edges may be assigned a common value \( a \), and the other pair a common value \( b \), independent of \( a \), and the assignment \( \alpha \) has \( \sigma(\alpha) = a + b \) at every vertex.

Let \( G \) contain \( t_1 \) copies of \( T \), \( t_2 \) copies of \( H \), and \( t_3 \) copies of \( Q \) so that \( \tau(G) = t_1 + t_2 + t_3 \) and \( \tau'(G) = t_1 \).

(A) If \( t_1 = 0 \), then by Theorem 2',
\[
\dim S(G) = 1 - (t_2 + t_3) + t_2 + 2t_3 = 1 + t_3.
\]
Note that
\[
E - n + 1 = 4(t_2 + t_3) - 4(t_2 + t_3) + 1 = 1.
\]
Note that \( t_2 = \tau(G) - t_3 \), so that for any desired \( \tau(G) \geq 1 \), the value of \( t_2 \) is determined by the value of \( t_3 \). Since \( \dim S(G) = 1 + t_3 \) can be made to take every value between \( E - n + 1 = 1 \) and \( E - n + \tau(G) = 1 + \tau(G) \), by taking \( t_3 = 0, 1, 2, \ldots, \tau(G) \), the limits in \((4')\) are the best possible.

(B) If \( t_1 > 0 \), then by Theorem 2',
\[
\dim S(G) = t_1 - (t_1 + t_2 + t_3) + t_2 + 2t_3 = t_3.
\]
Note that
\[
E - n + \tau'(G) = 3t_1 + 4t_2 + 4t_3 - 4(t_1 + t_2 + t_3) + t_1 = 0.
\]
Note that \( t_2 = \tau(G) - t_1 - t_3 \), so that for any desired \( \tau(G) \geq 1 \) and any desired \( \tau'(G) = t_1, 1 \leq t_1 \leq \tau(G) \), the value of \( t_2 \) is determined by \( t_3 \). Since \( \dim S(G) = t_3 \) can be made to take every value between \( E - n + \tau'(G) = 0 \) and \( E - n + \tau(G) = \tau(G) - t_1 \), by taking \( t_3 = 0, 1, 2, \ldots, \tau(G) - t_1 \), the limits in \((4'')\) are the best possible.

It is worth checking that the results in Theorem 2' agree with those in Theorem 2 when \( \tau(G) = 1 \) in both the cases \( \tau'(G) = 0 \) and \( \tau'(G) = 1 \). We note that \( E - n + \tau(G) = C(G) \), the circuit rank of \( G \). If we define \( D(G) = E - n + \tau'(G) \), then we may combine \((4), (4'), (4'')\) into
\[
\max(0, D(G) + d) \leq \dim S(G) \leq C(G) + d,
\]
where \( d = 1 \) or \( d = 0 \) according as \( \tau'(G) = 0 \) or \( \tau'(G) > 0 \).

3. The zero-magic space of a graph. Let \( Z(G) \) be the set of all \( \alpha \) in \( S(G) \) which satisfy the condition
\[
(M.2) \quad \sigma(\alpha) = 0.
\]
The set \( Z(G) \) is a subspace of \( S(G) \) which we shall call the zero-magic space of \( G \). The proof follows from (3) since \( \sigma(\alpha_1) = 0 \) and \( \sigma(\alpha_2) = 0 \) imply that
\[
\sigma(x_1 \alpha_1 + x_2 \alpha_2) = x_1 \sigma(\alpha_1) + x_2 \sigma(\alpha_2) = 0 \quad \text{for all } x_1, x_2 \text{ in } \mathbb{R}.
\]

To the condition (M.1) we add the condition (M.2) and obtain a homogeneous system \( M'' \) equivalent to using the equation \( \sigma^r(v_1) = 0 \) and the previously defined system \( M \). Hence the rank \( r'' \) of \( M'' \) and the rank \( r \) of \( M \) have the relation \( r \leq r'' \leq r + 1 \). Since \( \dim S(G) = E - r \) and \( \dim Z(G) = E - r'' \), we obtain
\[
(6) \quad \dim Z(G) < \dim S(G) \leq 1 + \dim Z(G).
\]

Relation (6) suggests that we define a semi-canonical basis for \( S(G) \) in which either (Case 1) every basis element is in \( Z(G) \) or (Case 2) every basis element, except one, is in \( Z(G) \). For examples, see §5.

4. Trivially-magic, zero-magic, and semi-magic graphs. On the basis of the preceding discussion we describe \( G \) as being
\[
\begin{align*}
\text{(G.1)} & \quad \text{trivially-magic if and only if } \dim S(G) = 0; \\
\text{(G.2)} & \quad \text{zero-magic if and only if } \dim S(G) = \dim Z(G) > 0; \\
\text{(G.3)} & \quad \text{semi-magic if and only if } \dim S(G) > \dim Z(G).
\end{align*}
\]

We illustrate these concepts with some infinite families of appropriate graphs. Other examples will appear in later sections.

We showed in Lemma 3 that all trees of odd order have \( \dim S(T) = 0 \); hence these are trivially-magic.

We shall show that the complete bipartite graph \( G = K(2, t) \) is zero-magic if \( t \geq 3 \). For if the vertices are \( v_1, v_2, w_1, w_2, \ldots, w_t \), we can use the edge-completion process (repeated) to obtain \( G \) from the tree \( T \) having the edges \( v_1 w_1, v_1 w_2, \ldots, v_1 w_t \) and the edge \( w_1 v_2 \) (see Figure 2). In an attempt to

satisfy (M.1) for \( T \), we must assign to all the terminal edges the same value \( a \). The vertex sum condition \( \sigma^r(w_1) = a \) requires \( \alpha(v_1, w_1) = 0 \). Then the condition \( \sigma^r(v_1) = a \) has the form \((t - 1)a = a\). Since \( t \geq 3 \), it follows that \( a = 0 \). Hence \( \dim S(T) = 0 \). It follows from Lemma 1 that \( \dim S(G) = C(G) = t - 1 \).

A set of \( t - 1 \) independent basis elements for \( S(G) \) is given by
\[
\{ \alpha_i = \epsilon_{1i} - \epsilon_{2i} + \epsilon_{3i,i+1} - \epsilon_{1,i+1}, i = 1, 2, \ldots, t - 1 \},
\]
where \( \varepsilon_{1i} \) and \( \varepsilon_{2i} \) are the unit vectors associated with the edges \( v_1 w_i \) and \( v_2 w_i \), respectively. Since every \( \alpha_1 \) is in \( Z(G) \), it follows that \( S(G) = Z(G) \). Hence \( G = K(2, t) \) is zero-magic if \( t \geq 3 \).

The previous example shows there are zero-magic graphs of every dimension \( d > 2 \). For an example of a zero-magic graph of dimension 1, we can use a kite consisting of a quadrilateral with a tail, where the tail is an arc of odd length (for an outline of the argument see Figure 3).

\[
\begin{align*}
\text{Analysis:} & \quad \text{at } v \\
2a = a & \quad a = 0
\end{align*}
\]

\[
\begin{align*}
\text{Conclusion:} & \quad S(G) = Z(G) \\
\dim S(G) = 1 &
\end{align*}
\]

\text{Figure 3.}

It is clear that \((G.3)\) can hold if and only if there is an \( \alpha \) in \( S(G) \) for which \( \sigma(\alpha) \neq 0 \). In fact we may change this condition to \( \sigma(\alpha) > 0 \), for from \((3)\) we have \( \sigma(-\alpha) = -\sigma(\alpha) \).

We can apply this test to conclude that any regular graph is semi-magic; for if we use the \( \alpha \) that assigns to every edge the value \( a > 0 \), we find that \((M.1)\) is satisfied with \( \sigma(\alpha) = \rho a > 0 \), where \( \rho \) is the common vertex degree. For example, a triangle \( K_3 \) is semi-magic with \( \dim S(K_3) = 1 \); and the graph \( G \) consisting of \( t_3 \) copies of the quadrilateral \( Q \) (see the example following Theorem 2') is semi-magic with \( \dim S(G) = 1 + t_3 \).

Let us turn from examples to a partial characterization of the trivially-magic graphs. Theorem 2 shows that a necessary condition for a connected graph to be trivially-magic is that the graph be a tree. For if \( G \) is connected, \( n - 1 \leq E \); and if \( \dim S(G) = 0 \), then \((4)\) demands that \( E - n + 1 \leq 0 \), so that \( E \leq n - 1 \). Hence \( E = n - 1 \) and \( T \) is a tree.

Lemma 3 shows that a sufficient condition for a tree to be trivially-magic is that the tree be of order \( n \) where \( n \) is odd, \( n \geq 3 \).

If \( T \) is a tree of even order, then \( T \) is either trivially-magic with \( \dim S(T) = 0 \), as in the case of stars, or \( T \) is semi-magic with \( \dim S(T) = 1 > \dim Z(T) \), as in the case of arcs. For \( T \) has \( E - n + 1 = 0 \), hence \((4)\) limits \( \dim S(T) \) to the two values 0 or 1. If \( \dim S(T) = 1 \), then \( \dim Z(T) = 0 \); for if \( \alpha \) is in \( S(T) \) with \( \sigma(\alpha) = a \), then the terminal edges of \( T \) must have the assignment \( a \); hence it is possible to satisfy \((M.2)\) only by taking \( a = 0 \). We have not been able to find a neat graph-theoretic distinction between these cases. As an example, the 23 trees of order 8 are separated into 14 trivially-magic and 9 semi-magic.
If $G$ has more than one component, we can use the notation of Theorem 2' and show that $G$ is trivially-magic if and only if $\tau(G) > 0$ and $\dim S(G) = 1$, $\tau(G) < j \leq \tau(G)$. For if $\tau(G) = 0$, every component $G_i$ of $G$ has $\dim S(G_i) > 1$, and part (A) of Theorem 2' shows that

$$\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i) > 1.$$ 

If $\tau(G) > 0$, of course $\dim S(G) > 1$, but if any $G_j$ has $\dim S(G_j) > 2$, then part (B) of Theorem 2' shows that

$$\dim S(G) = \tau'(G) - \tau(G) + \sum \dim S(G_j) > 1.$$ 

In both cases (G.1) is not satisfied. Thus if $G$ is to be trivially-magic, the conditions stated are necessary. Conversely, if the conditions are satisfied, it is easy to use the formula for $\dim S(G)$ in (B) to check that $\dim S(G) = 0$, so that $G$ is trivially-magic.

**Theorem 3.** A graph is semi-magic if and only if each component is semi-magic.

**Proof.** (A) If $G$ has components $G_1, G_2, \ldots, G_n$, then the corresponding vector space $A(G)$ has disjoint components $A_1, A_2, \ldots, A_n$. If one component, say $G_i$, is not semi-magic, then $G$ is not semi-magic. For condition (M.1) on $G$ can only be satisfied by having (M.1) hold on $G_i$. Since $G_i$ is not semi-magic, $G_i$ must be trivially-magic or zero-magic. In either case if $\alpha$ is in $S(G)$, then $\sigma(\alpha) = 0$ on the vertices of $G_i$, hence $\sigma(\alpha) = 0$ for all the vertices of $G$. Thus $S(G) = Z(G)$ and $G$ is not semi-magic.

(B) If each component $G_i$ is semi-magic, there exists an associated $\alpha^i$ in $S(G_i)$ for which $\sigma(\alpha^i) > 0$, $1 \leq i \leq \tau$. Let $M$ be the product of all the $\sigma(\alpha^i)$, or if the $\sigma(\alpha^i)$ are all rational integers, let $M$ be the least common multiple of $\sigma(\alpha^1), \sigma(\alpha^2), \ldots, \sigma(\alpha^\tau)$. Define $M_i = M/\sigma(\alpha^i)$. Define $\alpha_i$ for $G$ by $\alpha_i(e) = \alpha^i(e)$ for all $e$ in $G_i$, and $\alpha_i(e) = 0$ for all $e$ in $G$, but not in $G_i$. Finally, define

$$\alpha = M_1 \alpha_1 + M_2 \alpha_2 + \ldots + M_\tau \alpha_\tau.$$ 

Note that each edge and each vertex of $G$ appears in exactly one component. Hence if $v_k$ appears in the component $G_i$ we find from (2) that

$$\sigma(\alpha) = \sum^k \alpha(e) = M_1 \sum^k \alpha_1(e) = M_1 \sum^k \alpha^i(e) = M_i \sigma(\alpha^i) = M.$$ 

Since this result is independent of the choice of $v_k$, we know that $\alpha$ is in $S(G)$. Since $\sigma(\alpha) = M > 0$, it follows that $G$ is semi-magic.

We say that $U$ is a skeleton for $G$ if $U$ is a proper subgraph of $G$ which includes every vertex of $G$ and the components of $U$ are not isolated vertices.

**Theorem 4.** A sufficient condition that $G$ be semi-magic is that $G$ has a skeleton $U$ all of whose components are semi-magic.
Proof. Part (B) of Theorem 3 shows that $U$ is semi-magic under an $\alpha^U$ of the type defined in (7). Since the edges of $U$ are edges of $G$, it is possible to define an $\alpha$ for $G$ by using $\alpha(e) = \alpha^U(e)$ if $e$ is in $U$, and $\alpha(e) = 0$ if $e$ is in $G$, but not in $U$. Since every vertex of $G$ is in $U$ and since $U$ is semi-magic under $\alpha^U$, it follows easily that $G$ is semi-magic under $\alpha$ with $\sigma(\alpha) = \sigma(\alpha^U) > 0$.

Corollary 4.1. A sufficient condition that $G$ be semi-magic is that $G$ has a skeleton $U$ all of whose components are regular.

Proof. A regular graph $U_i$ of degree $p_i$ is semi-magic under an assignment $\alpha^i(e) = 1$ for each $e$ of $U_i$, for this implies $\sigma(\alpha^i) = p_i > 0$. The method of (7) and Theorem 4 now applies.

For example, consider the graph $G$ with skeleton $U$ in Figure 4. Here $U$ has

![Figure 4](https://doi.org/10.4153/CJM-1966-104-7 Published online by Cambridge University Press)

...two regular components: $U_1$ with $p_1 = 2$ and $U_2$ with $p_2 = 1$. With $M = 2$, $M_1 = 1$, $M_2 = 2$, we apply (7) to find $\alpha = (e_1 + e_2 + e_3) + 2e_5$ with $\sigma(\alpha) = 2$.

The condition in Theorem 4 is not a necessary one for $G$ to be semi-magic. For consider the graph $G$ in Figure 5, which has several proper skeletons, as shown. However, no one of these is semi-magic. Nevertheless, $G$ itself is

![Figure 5](https://doi.org/10.4153/CJM-1966-104-7 Published online by Cambridge University Press)

semi-magic. By Corollary 2.1, $\dim S(G) = C(G) = 1$. We find $S(G)$ generated by $\alpha = 3e_1 - e_2 - e_3 + 2e_4 + 2e_5$ for which $\sigma(\alpha) = 2 > 0$; hence $G$ is semi-magic.

5. Pseudo-magic graphs. We describe a graph $G$ (with $E \geq 2$) as being pseudo-magic if and only if there exists an $\alpha$ in $S(G)$ which has the property of distinctness:
Theorem 5. A necessary and sufficient condition that $G$ (with $E \geq 2$) be pseudo-magic is that

\[(C.1) \quad \text{for each pair of edges in } G, e_i \neq e_j, \text{ there exists an } \alpha_{ij} \text{ in } S(G), \text{ with values in the rational integers, such that } \alpha_{ij}(e_i) \neq \alpha_{ij}(e_j).\]

Proof. We use the notation of Theorem 1.

(A) Suppose (C.1) is satisfied. Then there exists $\alpha = \alpha_{12}$ in $S(G)$ with $\alpha(e_1) \neq \alpha(e_2)$; hence $P(\alpha)$ has $2 \leq w < E$. If $w < E$, then there exists $u$ with $t_u > 1$ and $\alpha(e_{u1}) = \alpha(e_{ut_1})$. If we take $e_i = e_{u1}$ and $e_j = e_{ut_1}$, then (C.1) implies the existence of $\alpha' = \alpha_{ij}$ with values in the rational domain such that $\alpha'(e_{u1}) \neq \alpha'(e_{u1})$. According to Theorem 1 the partitioning $P(\alpha + x\alpha')$ is a proper refinement of $P(\alpha)$ if we choose $x > a_w - a_1 + 1$. Since $S(G)$ is a subspace of $A(G)$, it follows that $\alpha^* = \alpha + x\alpha'$ is in $S(G)$, and with $x > a_w - a_1 + 1$ we have $w < w^*$. If $w^* < E$, we can repeat this procedure. In a finite number of steps we can reach $\alpha^{**}$ in $S(G)$ with $w^{**} = E$. But this is equivalent to the condition (M.3); hence $G$ is pseudo-magic.

(B) Suppose (C.1) is not satisfied. If there exists a pair of edges $e_i \neq e_j$ in $G$ such that every $\alpha$ in $S(G)$ with values in the rational domain has $\alpha(e_i) = \alpha(e_j)$, we can show that $\alpha(e_i) = \alpha(e_j)$ for every real-valued $\alpha$ in $S(G)$. Hence (M.3) fails and $G$ is not pseudo-magic. If $\dim S(G) = 0$, there is no problem, for then $\alpha$ is the only function in $S(G)$. If $\dim S(G) > 0$, the $\alpha$ in $S(G)$ are determined by solving (M.1) for the coordinates $a_{k\alpha}$ of $\alpha$. As remarked in part (A) of the proof of Theorem 2, if the independent parameters are $a_{ks}', s = 1, 2, \ldots, E - r = \dim S(G)$, then the dependent coordinates are expressed as $a_k = \sum d_{ks} a_{ks}', k = 1, 2, \ldots, r$, where the $d_{ks}$ are rational. Let $D$ be the least common denominator of the set of $d_{ks}$. Among the $\alpha$ in $S(G)$ whose values are in the rational domain are the $\alpha''$ determined by taking every $a_{ks}'$ to be an integer multiple of $D$. If $\alpha(e_i) = \alpha(e_j)$ for all $\alpha$ with values in the rational domain, it cannot be that both $a_i$ and $a_j$ are independent parameters, for then there are functions of the type $\alpha''$ which have $\alpha''(e_i) \neq \alpha''(e_j)$. If $a_i$, say, is dependent and $a_j$ independent, or if both $a_i$ and $a_j$ are dependent, the hypothesis that $\alpha(e_i) = \alpha(e_j)$ for all $\alpha$ with values in the rational domain includes the infinitely many $\alpha''$. Hence, in both cases, $a_i$ and $a_j$ must be identical linear expressions in the independent parameters. Therefore $\alpha(e_i) = \alpha(e_j)$ for all real-valued $\alpha$ in $S(G)$.

For example, $K_3$ is semi-magic, but not pseudo-magic. Since $n$ is odd, $\dim S(K_3) = C(K_3) = 1$, and $S(K_3)$ is generated by $\alpha_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$. Thus for every $\alpha = xa_1$ in $S(K_3)$ we find that $\alpha(e_1) = \alpha(e_2) = \alpha(e_3) = x$. Hence

\[(M.3) \quad \text{for all pairs of edges in } G, e_i \neq e_j \text{ implies } \alpha(e_i) \neq \alpha(e_j).\]
(M.3) fails. However, \( \sigma(\alpha) = 2x > 0 \) when \( x > 0 \); hence (G.3) holds (see Figure 6).

Similarly, \( K_4 \) is semi-magic, but not pseudo-magic. By Corollary 2.3, \( \dim S(K_4) = 3 \). If we let \( \epsilon_{ij} \) indicate the unit vector associated with the edge \( e_{ij} = v_i v_j \), then a basis for \( S(K_4) \) is given by

\[
\alpha_1 = \epsilon_{12} + \epsilon_{34}, \quad \alpha_2 = \epsilon_{13} + \epsilon_{24}, \quad \alpha_3 = \epsilon_{14} + \epsilon_{23}.
\]

Thus for every \( \alpha = a\epsilon_1 + b\epsilon_2 + c\epsilon_3 \) in \( S(K_4) \) we find \( \alpha(\epsilon_{12}) = a = \alpha(\epsilon_{34}) \). Hence (M.3) fails. However, \( \sigma(\alpha) = a + b + c > 0 \) when, say, \( a > 0, b = c = 0 \); hence (G.3) holds (see Figure 6).

In a later section we shall show that \( K_n \) is a special kind of semi-magic and pseudo-magic graph when \( n \geq 5 \).

Graphs may be pseudo-magic and not semi-magic. We showed that the complete bipartite graph \( K(2, t) \) is zero-magic when \( t > 3 \). We can now show that \( K(2, t) \) is pseudo-magic for \( t > 3 \). For a direct demonstration we define

\[
\alpha(v_i w_i) = b_i = -\alpha(v_2 w_i), \quad i = 1, 2, \ldots, t - 1,
\]

where \( 0 < b_1 < b_2 < \ldots < b_{t-1} \), and

\[
\alpha(v_2 w_i) = b_1 + b_2 + \ldots + b_{t-1} = -\alpha(v_1 w_i).
\]

Then we readily check that (M.2) and (M.3) are satisfied.

It is clear that the selection of the \( \alpha_{ij} \) in (C.1) may be made easier by the use of the skeletons described in Theorem 4. For example, to show that the "fan" \( F_4 \) (see Figure 7) is pseudo-magic, we use \( \alpha_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + 2\epsilon_5, \quad \alpha_2 = \epsilon_5 + \epsilon_7 + \epsilon_9, \quad \alpha_3 = \epsilon_5 + \epsilon_7 + \epsilon_9, \quad \alpha_4 = \epsilon_2 - \epsilon_4 + \epsilon_7 - \epsilon_8, \) which are based on skeletons, and \( \alpha_4 = \epsilon_2 - \epsilon_4 + \epsilon_7 - \epsilon_8, \) which is in \( Z(F_4) \), and hence in \( S(F_4) \). Since the set \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) is extensive enough to guarantee (C.1), it follows that \( F_4 \) is pseudo-magic. We find that \( \alpha = 2\alpha_1 + 6\alpha_2 + 4\alpha_3 + \alpha_4 \) has property (M.1) with \( \sigma(\alpha) = 14 \) and property (M.3) with \( w = E = 9 \) distinct edge values.
To motivate the next section we note that the most general \( \alpha \) in \( S(F_4) \), namely, \( \alpha = x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3 + x_4 \alpha_4 \) (for by Corollary 2.2, 
\[
\dim S(F_4) = C(F_4) = 4
\]
and \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are independent), has the property \( \alpha(e_2) = x_4, \alpha(e_4) = -x_4 \). Hence it is impossible to find an \( \alpha \) in \( S(F_4) \) for which every edge has a positive assignment.

6. Magic graphs. We describe \( G \) (with \( E \geq 2 \)) as being magic if and only if there exists an \( \alpha \) in \( S(G) \) which has the property (M.3) of distinctness and the property of "positiveness":

\[
(M.4) \quad \text{for every edge } e \text{ in } G, \alpha(e) > 0.
\]

We agree that \( K_2 \) with \( E = 1 \) is also magic.

**Theorem 6.** If \( G \) is pseudo-magic, a sufficient condition that \( G \) be magic is that 
\[
(C.2) \quad \text{for each edge } e \text{ in } G \text{ there exists a skeleton } R \text{ that contains } e \text{ and whose components are semi-magic and have property } (M.4) \text{ under functions that have values in the rational domain.}
\]

**Proof.** Since \( G \) is assumed pseudo-magic, there exists an \( \alpha \) in \( S(G) \) with \( w = E \) and \( a_1 < a_2 < \ldots < a_E \). We may assume that \( 0 < a_E \); otherwise \( G \) is already known to be magic under \( -\alpha \). If \( 0 < a_1 \), then \( G \) is already magic under \( \alpha \). If \( a_1 = 0 \) and \( \alpha(e_1) = a_1 \), let \( R \) be the skeleton, provided by the hypothesis (C.2), which contains \( e_1 \). Using the components of \( R \), construct as in (7) the function \( \alpha^R = M_1 \alpha_1^R + \ldots + M_t \alpha_t^R \). From (C.2) each \( \alpha_t^R \) has values that are positive integers, each \( M_t \) is a positive integer, and hence \( \alpha^R(e) \) is a positive integer for each \( e \) in \( R \). Extend \( \alpha^R \) to an \( \alpha' \) for \( G \) by the usual plan: \( \alpha'(e) = \alpha^R(e) \) if \( e \) is in \( R \); and \( \alpha'(e) = 0 \) if \( e \) is in \( G \), but not in \( R \). Consider the function \( \alpha^* = \alpha + xa' \) with \( x > a_E - a_1 + 1 \). According to Theorem 1, \( w^* = w = E \). According to Theorem 4, \( G \) is semi-magic under \( \alpha' \); since \( \sigma(\alpha) > 0 \) and

\[
\sigma(\alpha^*) = \sigma(\alpha) + x\sigma(\alpha') > \sigma(\alpha') > 0,
\]

it follows that \( G \) is semi-magic under \( \alpha^* \). Thus \( G \) is pseudo-magic under \( \alpha^* \).

If \( i > 1 \), then \( \alpha^*(e_i) = \alpha(e_i) + xa'(e_i) \geq \alpha(e_i) = a_i \). If \( e_i \) belongs to the \( j \)th component of \( R \), \( \alpha^*(e_i) = \alpha(e_i) + xM_j \alpha_j^R(e_i) \); hence \( \alpha^*(e_i) \geq a_i + 1 \). Thus the minimum \( a_i^* \) for \( \alpha^* \) has the property \( a_i^* \geq \min (a_2, a_1 + 1) \). In a finite number of steps, we can find an \( \alpha^{**} \) under which \( G \) is pseudo-magic with \( w^{**} = E \) and for which the minimum \( a_i^{**} > 0 \); hence \( G \) is magic under \( \alpha^{**} \).

Let \( U^* \) indicate the set of all the skeletons \( U \) of \( G \) which have the property that each component of \( U \) is regular. If \( U \) is in \( U^* \), let \( \beta \) be the corresponding function described in the proof of Corollary 4.1.
Corollary 6.1. Sufficient conditions that \( G \) be magic are that both

\[(U.1) \text{ for each pair of edges in } G, e_i \neq e_j, \text{ there exists a skeleton } U_{ij} \text{ in } U^* \text{ such that } \beta_{ij}(e_i) \neq \beta_{ij}(e_j); \text{ and}\]

\[(U.2) \text{ for each edge } e_i \text{ in } G \text{ there exists a skeleton } U_i \text{ in } U^* \text{ such that } U_i \text{ contains } e_i.\]

Proof. From Corollary 4.1 we see that \( (U.1) \) corresponds to \( (C.1) \) and we can apply Theorem 5 to see that \( G \) is pseudo-magic. From Corollary 4.1 we see that \( (U.2) \) implies that the components of \( U_i \) have the property \( (M.4) \) under the function \( \beta_i; \) hence \( (U.2) \) corresponds to \( (C.2) \) and we can apply Theorem 6 to see that \( G \) is magic.

7. Applications of Theorem 6.

Example 1. A complete bipartite graph \( K(n, n) \) is magic for \( n \geq 3. \)

Proof. The graph \( K(n, n) \) consists of two sets of vertices \( v_1, v_2, \ldots, v_n \) and \( w_1, w_2, \ldots, w_n \) with every \( v \) joined to every \( w \), but no \( v \) joined to another \( v \) and no \( w \) joined to another \( w \). Because of the great symmetry of the graph, the condition \( (U.1) \) of Corollary 6.1 can be checked by examination of only two cases.

Case 1. We produce a skeleton \( U \) in \( U^* \) containing \( v_1 w_1 \) but not containing \( v_2 w_2 \); hence \( \beta(v_1 w_1) = 0 < \beta(v_1 w_1) \). We use

\[ U = (v_1 w_1) \oplus (v_2 w_2) \oplus (v_3 w_3) \oplus (v_4 w_4) \oplus \ldots \oplus (v_n w_n) \]

which is possible since \( n \geq 3. \)

Case 2. We produce a skeleton \( U \) in \( U^* \) containing \( v_1 w_1 \), but not containing \( v_1 w_2 \), by using

\[ U = (v_1 w_1) \oplus (v_2 w_2) \oplus (v_3 w_2) \oplus \ldots \oplus (v_n w_n). \]

To satisfy condition \( (U.2) \) we may use the last-described skeleton, for \( v_1 w_1 \) is a typical edge.

By Corollary 6.1 the proof for Example 1 is now complete.

If we interpret \( \alpha(v_i w_j) \) to be the entry in the \( i \)th row and \( j \)th column of a square table, then \( \alpha^r(v_i) \) is the row sum in the \( i \)th row of the table and \( \alpha^c(w_j) \) is the column sum in the \( j \)th column of the table. Hence if \( K(n, n) \) is magic under \( \alpha \), then the table is one of the weakly-magic squares of number theory and Example 1 shows (by graph-theoretic methods!) the existence of weakly-magic squares of all orders \( n \geq 3. \) However, the magic squares of number
Exercise 1. Show that the complete bipartite graph $K(n, m)$ with $n > m > 1$ is semi-magic if and only if $n = m$.

Example 2. A complete graph $K_n$ is magic for $n = 2$ and all $n > 5$.

Proof. We have already discussed the cases $n = 2, 3, 4$, discovering that $K_2$ is magic, while $K_3$ and $K_4$ are not pseudo-magic, and hence not magic. Consider $K_n$ with $n > 5$. To satisfy (U.1) consider two cases.

Case 1. Suppose $p, q, u, v$ are four distinct vertices in $K_n$. We can produce a skeleton $U$ in $U^*$ which includes the edge $pq$, but does not include the edge $uv$; hence $\beta(uv) = 0 < \beta(pq)$. Select a fifth distinct vertex $t$, which is possible since $n > 5$. In $K_n$ there is an arc from $v$ to $t$, direct if $n = 5$, and passing through all the other $n - 5$ vertices if $n > 5$. For $U$ we can use the circuit $((tupqv . . . t))$ containing $pq$ and not containing $uv$.

Case 2. Suppose $p, q, v$ are three distinct vertices in $K_n$. We can produce a skeleton $U$ in $U^*$ which includes the edge $pq$, but does not include the edge $qv$. Select a fourth distinct vertex $t$. In $K_n$ there is an arc from $v$ to $t$, passing through the other $n - 4$ vertices. For $U$ we can use the circuit $((tqpv . . . t))$ containing $pq$ and not containing $qv$.

To satisfy (U.2) we note that $K_{n-2}$ is regular and is not an isolated vertex since $n > 5$. As a typical case we consider $e = pq$ and use the skeleton $U = (pq) \odot K_{n-2}$ where $K_{n-2}$ is based on the other $n - 2$ vertices.

It follows from Corollary 6.1 that $K_4$ is magic for all $n > 5$.

If each vertex $v_0, v_1, . . . , v_k$ of an arc $(k > 1)$ is connected to another vertex $v$, we describe the resulting graph as a fan with $k$ blades, denoted by $F_k$ (see Figure 7 for $F_4$). We note that $F_k$ has $n = k + 2$ and $E = 2k + 1$. By Corollary 2.2, $\dim S(F_k) = C(F_k) = k$.

Example 3. A fan $F_k$ is magic if and only if $k$ is odd and $k > 3$.

Proof. (A) We note that $F_1$ is the triangle $K_3$, which is not magic. We set $k = 2K + 1$ and give two proofs that $F_k$ is magic for $K > 1$.

(A.1) For an existence proof we use Corollary 6.1. Consider the following series of skeletons, each in $U^*$ (see Figure 8):

$$U_i = (v_0 v_1) \odot . . . \odot (v_{2i-2} v_{2i-1}) \odot ((v_{2i} v_{2i+1} v)) \odot (v_{2i+2} v_{2i+3})$$

$$\odot . . . \odot (v_{2K v_{2K+1}}), \quad 0 < i < K, K > 1;$$

$$U'_i = (v_0 v_1) \odot . . . \odot (v_{2i-2} v_{2i-1}) \odot ((v_{2i} v_{2i+1} v_{2i+2} v_{2i+3} v))$$

$$\odot (v_{2i+4} v_{2i+9}) \odot . . . \odot (v_{2i + 2K + 1}), \quad 0 < i < K - 1, K > 1.$$
It is not difficult to check that this set of skeletons is extensive enough to guarantee (U.1) and (U.2). In fact, the associated sets \( \{ \beta_i \} \) and \( \{ \beta'_i \} \) form a basis for \( S(F_k) \).

\begin{figure}[h]
\centering
\includegraphics{fig8}
\caption{Figure 8.}
\end{figure}

(A.2) For a constructive proof we describe a specific \( \alpha \) (see Figure 9) in which the edge assignments are positive (M.4) and readily checked to be distinct (M.3):

\begin{align*}
\alpha(v_i) &= i, \quad 1 \leq i \leq 2K; \\
\alpha(v_0) &= 3K + 1; \\
\alpha(v_{2K+1}) &= 4K + 1; \\
\alpha(v_{2i}) &= 2K + (K - i) + 1, \quad K \geq i \geq 1; \\
\alpha(v_{2i+1}) &= 2K^2 + 4K + 1 + (K - i), \quad K \geq i \geq 0.
\end{align*}

The first set runs consecutively from 1 to \( 2K \); the second, from \( 2K + 1 \) to \( 3K \); the third, from \( 2K^2 + 4K + 1 \) to \( 2K^2 + 5K + 1 \). It is obvious that the other two values are distinct from these. It remains to check (M.1) by verifying that the vertex sums are all rearrangements of \( 2K^2 + 8K + 2 \), viz.:

\begin{align*}
\sigma^\alpha(v_0) &= (3K + 1) + (2K^2 + 5K + 1) = (4K + 1) + (2K^2 + 4K + 1) \\
&= \sigma^\alpha(v_{2K+1}); \\
\sigma^\alpha(v_i) &= (3K - i + 1) + (2i) + (2K^2 + 5K + 1 - i), \quad 1 \leq i \leq K; \\
\sigma^\alpha(v_{i+1}) &= (2K^2 + 5K + 1 - i) + (2i + 1) + (3K - i), \quad 0 \leq i \leq K - 1; \\
\sigma^\alpha(v) &= (3K + 1) + (4K + 1) + (1 + 2 + \ldots + 2K) \\
&= 7K + 2 + K(2K + 1) = 2K^2 + 8K + 2.
\end{align*}

\begin{figure}[h]
\centering
\includegraphics{fig9}
\caption{Figure 9.}
\end{figure}
(B) There is a hint that \( F_{2^K} \) is not magic for \( K > 1 \), since there is no skeleton \( U \) which includes \( vv_{2i-1}, 1 \leq i \leq K \); hence (U.1) and (U.2) cannot be satisfied. However, we recall that the conditions in Corollary 6.1 are sufficient, not necessary, so we resort to an algebraic proof (underneath whose thin disguise we are really finding a basis for \( S(F_{2^K}) \)).

Let \( \alpha \) be any function in \( S(F_{2^K}) \); hence \( \alpha \) satisfies (M.1).

To certain selected edges we make the arbitrary assignments:

\[
\alpha(v_0) = a, \quad \alpha(v_0v_1) = b, \quad \alpha(v_1v_1) = x_0, \quad 1 \leq i \leq 2K - 1;
\]

for we can show that all the other assignments are determined by the condition (M.1) which requires that every vertex sum be equal to \( \sigma^\alpha(v_0) = a + b \). For example, \( \sigma^\alpha(v) = a + b \) requires

\[
\alpha(v_{2^K}) = b - \sum_{i=1}^{2K-1} x_i.
\]

By (limited) induction we can show that we must have

\[
(8) \quad \alpha(v_{2i-1}v_{2i}) = a + \sum_{i=1}^{2i-1} (-1)^i x_i, \quad 1 \leq i \leq K.
\]

For when \( i = 1 \), we require

\[
\sigma^\alpha(v_1) = a + b = \alpha(v_0v_1) + \alpha(v_1v_1) + \alpha(v_1v_2) = b + x_1 + \alpha(v_1v_2);
\]

hence \( \alpha(v_1v_2) = a - x_1 \), in agreement with the formula (8). From the requirements \( \sigma^\alpha(v_{2i}) = \sigma^\alpha(v_{2i+1}) \) we find that

\[
\alpha(v_{2i-1}v_{2i}) + \alpha(v_{2i}v_{2i+1}) = \alpha(v_{2i}v_{2i+1}) + \alpha(v_{2i+1}v_{2i+2})
\]

hence \( \alpha(v_{2i+1}v_{2i+2}) = \alpha(v_{2i-1}v_{2i}) + x_{2i} - x_{2i+1} \). If we use the induction hypothesis (8) for \( 1 \leq i < K \), we find that

\[
\alpha(v_{2i+1}v_{2i+2}) = a + \sum_{i=1}^{2i+1} (-1)^i x_i,
\]

which is the correct form for (8) in the case \( i + 1 \).

In particular, from (8) we have

\[
\alpha(v_{2K-1}v_{2K}) = a + \sum_{i=1}^{2K-1} (-1)^i x_i.
\]

Hence

\[
\sigma^\alpha(v_{2K}) = \alpha(v_{2K-1}v_{2K}) + \alpha(v_{2K}v_{2K}) = a + b - 2 \sum_{i=1}^{K} x_{2^{i}-1}.
\]

Requiring \( \sigma^\alpha(v_{2K}) = a + b \) demands

\[
\sum_{i=1}^{K} x_{2^{i}-1} = 0.
\]
Since this is impossible using only positive real values for the $x_{2^n-1}$, there is no $\alpha$ in $S(F_{2K})$ which satisfies (M.4). Hence $F_{2K}$ is not magic for all $K \geq 1$.

**Exercise 2.** Consider a rectangular lattice $L(s, t)$ with $n = (s + 1)(t + 1)$ and $E = s(t + 1) + t(s + 1)$. Note that $C(L) = st$ and that $st$ linearly independent $\alpha$ in $Z(L)$ are suggested by the st cells of $L$, of the type $\alpha = \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4$.

1. Use the method of Corollary 4.1 to show that $L(1, b)$ is semi-magic, but not pseudo-magic, for all $b \geq 1$; and that $\dim S(L) = 1 + C(L)$.

2. Use Lemma 1 and Theorem 5 to show that $L(2a, 2b)$ is zero-magic and pseudo-magic for all $a \geq 1, b \geq 1$; and that $\dim S(L) = C(L)$.

3. Use Corollary 6.1 to show that $L(a, 2b + 1)$ is magic for all $a \geq 2, b \geq 1$; and that $\dim S(L) = 1 + C(L)$.

If each vertex $v_1, v_2, \ldots, v_k$ of a circuit ($k \geq 3$) is connected to another vertex $v$, we call the resulting graph a *wheel with $k$ spokes*, denoted by $W_k$. If one spoke is removed from $W_k$, say $v$ and $v_k$ are not joined, we call the resulting graph a *basket*, denoted by $B_k$.

**Example 4.** A basket $B_k$ is magic for $k = 4$ and all $k \geq 6$.

**Proof.** (A) If $k = 2K + 1, K \geq 3$, define $\alpha$ as follows (see Figure 10):

$$
\alpha(v_i) = i, \quad 1 < i < 2K - 2;
\alpha(v_{2K}) = 2K - 1; \quad \alpha(v_{2K-1}) = 4K - 2; \quad \alpha(v_{2K} v_{2K-1}) = K^2 - K;
\alpha(v_{2i+1} v_{2i}) = K^2 + K - 1 - i, \quad K - 1 > i > 1;
\alpha(v_1 v_{2K+1}) = K^2 + K - 1;
$$

$$
\alpha(v_{2i} v_{2i-1}) = K^2 + 2K + 1 - i, \quad K - 1 > i > 1;
\alpha(v_{2K+1} v_{2K}) = K^2 + 2K - 1.
$$

It is easy to check that (M.3) holds for $K \geq 5$ with the listed ordering; luckily (M.3) holds for $K = 3$ and $K = 4$ with a reordering of the second, third, and fourth items. Finally, (M.1) holds with $\sigma(\alpha) = 2K^2 + 3K - 2$.

(B) If $k = 2K, K \geq 4$, define $\alpha$ as follows (see Figure 10):

$$
\alpha(v_i) = i, \quad 1 < i < 2K - 3;
\alpha(v_{2K-1}) = 2K - 2; \quad \alpha(v_{2K-1} v_{2K-2}) = 2K - 1; \quad \alpha(v_{2K} v_{2K-2}) = 3K - 3;
\alpha(v_{2i+1} v_{2i}) = 4K - 3 - i, \quad K - 2 > i > 1; \quad \alpha(v_1 v_{2K}) = 4K - 3;
\alpha(v_{2i} v_{2i-1}) = 2K^2 + 1 - i, \quad K - 1 > i > 1;
\alpha(v_{2K} v_{2K-1}) = 2K^2 - 4K + 1.
$$

It is easy to check that (M.3) and (M.4) are satisfied because $K \geq 4$; and (M.1) is satisfied with $\sigma(\alpha) = 2K^2 - 2$. 

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(C) For the cases $B_4$ and $B_8$ we produce individual solutions (see Figure 10).

(D) It is easy to show that $B_4$ and $B_8$ are not magic by finding bases for $S(B_4)$ and $S(B_8)$ (see Figure 10).

(A) $B_4$

(B) $B_8$

(C) $B_4$

(D) $B_8$

8. Completion. If $G$ is not complete, let $G^*$ be formed from $G$ by inserting one new edge $e^*$. Thus $G^*$ has $n^* = n$ and $E^* = E + 1$.

**Theorem 7.** If $G$ is magic and if $G^*$ has a skeleton $R^*$ which contains $e^*$ and the components of $R^*$ are semi-magic and have property (M.4) under functions which have values in the rational domain, then $G^*$ is magic.

*Proof.* If $\alpha$ is in $A(G)$, let $\alpha^*$ be defined in $A(G^*)$ by $\alpha^*(e^*) = 0$ and $\alpha^*(e) = \alpha(e)$ when $e$ is in $G$, but $e \neq e^*$. Given that $G$ is magic under $\alpha$, certainly $G^*$ is pseudo-magic under $\alpha^*$ for the vertex sums are the same as for $\alpha$, and the edge assignments are distinct with $\psi^* = E + 1 = E^*$. The only discrepancy, keeping $G^*$ from being magic under $\alpha^*$, is the assignment $\alpha^*(e^*) = 0$. The assumptions on $R^*$ and $e^*$ are almost like (C.2) in Theorem 6. The same argument applies, but only one step is required to arrive at an appropriate $\alpha^{**} = \alpha^* + xa'$ under which $G^*$ is magic. Since the minimum edge assignment for $\alpha^*$ is 0 and the maximum edge assignment is $a_R$ (the same as for $\alpha$), a suitable condition on $x$ is $x \geq a_R - 0 + 1 = a_R + 1$.

**Corollary 7.1.** If $G$ is magic and if $G^*$ has a skeleton $R^*$ which contains $e^*$ and the components of $R^*$ are regular, then $G^*$ is magic.

*Proof.* Combine Corollary 4.1 and Theorem 7.
Example 5. A wheel $W_t$ is magic for $t > 4$.

Proof. The wheel $W_t$ can be obtained from the basket $B_t$ by edge completion, restoring the edge $e^* = v_t v$. The Hamilton circuit $((v_t v_1 v_2 \ldots v_t))$ is a regular skeleton of $W_t$ which includes the edge $e^*$. By Corollary 7.1 and Example 4 the proof is complete for $t = 4$ and all $t > 6$. For the case $t = 5$ we use the scheme in Figure 11.

If a graph is magic, it is sometimes amusing to seek an $\alpha$, with each $\alpha(e)$ a positive integer, satisfying (M.1) and (M.3), and having a minimal value for $\sigma(\alpha)$. The examples given for $B_5$, $B_6$, and $W_6$ result from such a search.

A proper skeleton $F$ of $K_n$, $n > 5$, will be called a magic base for $K_n$ if every skeleton $G$ of $K_n$ which contains $F$ is magic. We say that $F$, a magic base for $K_n$, is irreducible if each skeleton of $F$, obtained by deleting an edge, is not magic. The following Theorem 8 demonstrates the existence of magic bases, chosen for ease of description and irreducibility, but not unique.

Theorem 8. For $n = 5$ and all $n > 7$, the basket $B_{n-1}$ is an irreducible magic base for $K_n$. The wheel $W_6$ is an irreducible magic base for $K_6$.

Proof. (A) Example 4 shows that $B_{n-1}$ is magic for $n = 5$ and all $n > 7$. Let $H^*$ be a graph obtained from $B_{n-1}$ by the addition of an edge $e^*$. We claim that $H^*$ has a skeleton $R^*$ which contains $e^*$ and whose components are regular (see the cases listed below). By Corollary 7.1 the graph $H^*$ is magic. If $G$ is any skeleton of $K_n$ which includes $B_{n-1}$, but does not include $e^*$, and if $G^*$ is formed from $G$ by adding $e^*$, then $G^*$, like $H^*$, has the same skeleton $R^*$ containing $e^*$. By finite induction, we can add edge after edge and make repeated applications of Corollary 7.1 to conclude that any skeleton of $K_n$ which contains $B_{n-1}$ is magic. Hence $B_{n-1}$ is a magic base for $K_n$ for $n = 5$ and all $n > 7$.

We use the previously established notation in Example 4 in which $B_{n-1}$ has the centre $v$, the circuit $((v_1 v_2 \ldots v_{n-1} v_1))$, and all the spokes $v v_i$ except $vv_{n-1}$.

Case 1. $e^* = vv_{n-1}$, $R^* = ((v_1 v_2 \ldots v_{n-2} v_{n-1} v))$.

Case 2. $e^* = v_{n-1} v_i$, $i \neq 1$, $i \neq n - 2$,

$$R^* = ((v_i v_{i+1} \ldots v_{n-2} v_{n-1} v_i) \oplus (v_1 v_2 \ldots v_{i-1} v_{i+1})).$$
Case 3. \( e^* = v_j v_i, 1 < i < j - 1 < n - 3 \),
\[
R^* = ((v_1 v_{i+1} \ldots v_{j-1} v_j v_i)) \oplus ((v_1 v_2 \ldots v_{i-1} v_{i+1} \ldots v_{n-1} v_1)).
\]

Case 4. \( e^* = v_i v_1, i \neq 2, i \neq n - 1 \),
\[
R^* = ((v_1 v_i v_{i+1} \ldots v_{n-2} v_{n-1} v_1)) \oplus ((v_2 v_3 \ldots v_{i-1} v_i)).
\]

Case 5. \( e^* = v_{n-2} v_i, i \neq n - 3, i \neq n - 1, i \neq 1 \),
\[
R^* = ((v_1 v_{n-2} v_{n-1} v_1 v_2 \ldots v_{i-1} v_i)) \oplus ((v_{i+1} v_{i+2} \ldots v_{n-3} v_1)).
\]

(B) In the case \( n = 6 \), the proof is similar. In Example 5 it was shown that \( W_5 \) is magic. The description of an appropriate skeleton \( R^* \) is easier than in part (A), for one case suffices.

(C) The proof that these magic bases are irreducible can be accomplished by a series of algebraic arguments. For example, consider \( W_5 \). Deleting an outer edge produces \( F_4 \), which was shown to be not magic in Example 3. Deleting a spoke produces \( B_5 \) shown to be not magic in Figure 10.

Exercise 3. If \( n \) is odd, \( n > 5 \), show that the fan \( F_{n-2} \) is an irreducible magic base for \( K_n \). (Comparison with Theorem 8 justifies the remark that \( K_n \) may have more than one irreducible magic base.)

Exercise 4. The graph \( G \) with two components \( K_2 \) and \( K_n, n > 5 \), is magic; but \( G^* \) is not magic. (Hence the edge-completion process does not always maintain the magic property.)

9. Super-magic graphs. We say that \( G \) is super-magic if and only if there exists an \( a \) in \( S(G) \) which satisfies the condition
\[
\text{(M.5) the set \{a(e_i)\} consists of consecutive positive integers.}
\]

We can appeal to the literature (cf. 2) concerning the magic squares of number theory and use the correspondence explained in Example 1 to assert that \( K(n, n) \) is super-magic for all \( n > 3 \). Before discussing other examples we note some necessary conditions.

If the integers in (M.5) are indicated as \( a, a + 1, \ldots, a + (E - 1) \), then the conditions (M.1) and (M.5) require
\[
(9) \quad n \sigma(a) = 2 \sum a(e_i) = 2(Ea + E(E - 1)/2), \quad \text{for at least one } a > 1.
\]

The relation (9) imposes congruence conditions on the parameter \( a \) which must be satisfied if \( \sigma(a) \) is to be an integer.

Suppose \( \rho' \) is the maximum degree in \( G \). The condition (M.1) at a vertex of degree \( \rho' \) requires
\[
(10') \quad \rho' a + \rho' (\rho' - 1)/2 \leq \sigma(a),
\]
since the minimum sum of \( \rho' \) integers in the set defined by (M.5) must not exceed \( \sigma(a) \).

Suppose \( \rho'' \) is the minimum degree in \( G \). The condition (M.1) at a vertex of degree \( \rho'' \) requires
\[
(10'') \quad \rho'' (a + E - 1) - \rho'' (\rho'' - 1)/2 \geq \sigma(a),
\]
since the maximum sum of \( \rho'' \) integers in the set defined by (M.5) must not be less than \( \sigma(a) \).
For example, consider the basket \( B_4 \). The relation (9) requires \( 5\sigma(\alpha) = 2(7a + 21) \); hence \( a \equiv 2 \) (mod 5). The condition (10") with \( \rho' = 2 \) requires \( 13 > 4a \). Since \( a > 1 \), the only possibility is \( a = 2 \). By trial we find that \( B_4 \) is super-magic (see Figure 12) and in the sense of isomorphism there is only one solution.

For the wheel \( W_4 \) the relation (9) requires \( 5\sigma(\alpha) = 2(8a + 28) \); hence \( a \equiv 4 \) (mod 5). The condition (10') with \( \rho' = 4 \) requires \( 4a \leq 26 \). The only possibility is \( a = 4 \). We find that \( W_4 \) is super-magic in essentially one way (see Figure 12). In similar fashion we find that \( W_5 \) is super-magic in essentially one way (see Figure 12).

In contrast, we can show that \( W_n \) is not super-magic for \( n > 6 \), for (9) and (10') can be combined to require

\[
(n + 1)(na + n(n - 1)/2) \leq 2(2na + 2n(2n - 1)/2),
\]

which is equivalent to \( n^3 - 8n^2 + 3n + 2n(n - 3)a \leq 0 \). But using the condition \( a > 1 \), we find that

\[
n^3 - 8n^2 + 3n + 2n(n - 3)a > n(n(n - 6) - 3) > 0
\]

when \( n > 7 \). In the case \( n = 6 \), the congruence condition (9) requires \( a \equiv 5 \) (mod 7), and with \( a \geq 5 \), the condition (10') again provides a contradiction.

\[B_4\] \hspace{1cm} \[W_4\] \hspace{1cm} \[W_5\]

**Figure 12.**

If \( G \) is regular of degree \( \rho \), the function \( \beta \), with \( \beta(e) = 1 \) for every \( e \) in \( G \), belongs to \( S(G) \) with \( \sigma(\beta) = \rho \). If \( a \) is in \( S(G) \) and has property (M.5), then \( a + x\beta \) is in \( S(G) \) and has property (M.5) with the corresponding set ranging from \( a + x \) to \( a + x + E - 1 \), providing \( x \) is an integer and \( a + x \geq 1 \). Consequently, if \( G \) is regular, then \( G \) is either not super-magic, or is super-magic for all \( a \geq 1 \). Thus in using (9) we may assume \( a = 1 \) and test

\[(9') \quad n\sigma(\alpha) = E(E + 1) .\]

For example, for \( K_n \) we have \( E = n(n - 1)/2 \) and find that (9') reduces to

\[(9'') \quad 4\sigma(\alpha) = (n - 1)(n^2 - n + 2) ;\]

hence \( \sigma(\alpha) \) is not an integer when \( n \equiv 0 \) (mod 4).

We can show that \( K_6 \) is not super-magic, for there are only a few combinations to try, and none is successful.

In another paper (3) we show that \( K_n \) is super-magic when \( n > 5 \) and \( n \not\equiv 0 \) (mod 4). Here we are content with showing solutions for \( K_6 \) and \( K_7 \) (see Figure 13).
MAGIC GRAPHS

Figure 13.
Exercise 5. For the graphs corresponding to the five Platonic solids, show that only the octahedron is super-magic.

10. Prime-magic graphs. We say that $G$ is prime-magic if and only if there exists an $\alpha$ in $S(G)$ which satisfies the condition

$$(\text{M.6}) \quad \text{the set } \{\alpha(e_i)\} \text{ consists of distinct positive rational integers which are primes.}$$

Except for the trivial $K_2$, the simplest prime-magic graphs are the fan $F_3$ and the complete graph $K_5$ (see Figure 14).

There are infinitely many graphs that are magic, but not prime-magic. In Example 3 we showed that the fan $F_{2K+1}$ is magic for $K \geq 1$; but we can show that $F_{2K+1}$ is not prime-magic for $K \geq 2$. Since there is only one positive even prime, and $F_{2K+1}$ has two vertices of degree 2, properties (M.1) and (M.6) require that $\sigma(\alpha)$ be even. But the sum of three odd primes is odd, and $F_{2K+1}$, for $K \geq 2$, has more than two vertices of degree 3; hence properties (M.1) and (M.6) require that $\sigma(\alpha)$ be odd, a contradiction. This answers, in the negative, a conjecture of Sedlacek that every magic graph is prime-magic.

Perhaps a qualified conjecture was intended—it may be that every regular magic graph is prime-magic. Sedlacek showed that the cube and octahedron are prime-magic. We can use the ideas in the following theorem to show that the icosahedron is prime-magic; but we have not yet been able to show that the dodecahedron is more than magic. The difficulty surrounding any general statement about prime-magic graphs is obvious—the irregular distribution of the primes.

For relatively prime positive integers $a$ and $b$, define a set of integers:

$$\Pi(a, b) = \{k | k \geq 0, ak + b \text{ prime}\}.$$ 

According to Dirichlet, the set $\Pi(a, b)$ is infinite.

Theorem 9. If $G$ is regular and if $G$ is magic under an $\alpha$ whose values are in $\Pi(a, b)$, then $G$ is prime-magic.

Proof. Given $G$ is magic under $\alpha(e)$. Let $\beta$ be the function with $\beta(e) = 1$ for all $e$ in $G$. Define $\alpha'$ in $A(G)$ by $\alpha' = a\alpha + b\beta$. Since $G$ is regular, say of degree $\rho$,
\( \sigma'(v) = a\sigma(v) + pb \) for every vertex of \( G \). Since \( \alpha \) satisfies (M.1), it follows that \( \alpha' \) is in \( S(G) \), with \( \sigma(\alpha') = a\sigma(\alpha) + pb \). Since \( \alpha \) satisfies (M.3) and since \( a \neq 0 \), if \( e_i \neq e_j \), then

\[
\alpha'(e_i) = a\alpha(e_i) + b \neq a\alpha(e_j) + b = \alpha'(e_j);
\]

hence \( \alpha' \) satisfies (M.3). Finally, by hypothesis each \( \alpha(e) \) is in \( \Pi(a, b) \); hence \( \alpha'(e) = a\alpha(e) + b \) is a positive prime. Thus \( G \) is prime-magic.

For example, if we note that \( K_6 \) is magic under an \( \alpha \) whose values \( 1, 2, 3, 4, 5, 9, 11, 12, 13, 15 \) are all in \( \Pi(30, 7) \), then we can apply Theorem 9 to obtain the solution in Figure 14.

For a more elaborate example, consider the graph \( G \) of the truncated tetrahedron. By the method of Corollary 6.1 it is easy to find that \( G \) is magic under the function \( \gamma \) shown in the first part of Figure 15. Comparison with \( \Pi(6, -1) \) shows that only values that need to be changed are 11 and 55. However, the simplest skeleton, with regular components involving these two edges, involves at least four other edges. We try the skeleton \( U \), with associated \( a\delta \), shown in the second part of the figure. To avoid integers of the type \( x \equiv 1 \pmod{5} \) which are not in \( \Pi(6, -1) \) when \( x > 1 \), we find we must use \( a \equiv 4 \pmod{5} \). The choice \( a = 29 \) results in \( \alpha = \gamma + 29\delta \) having all values in \( \Pi(6, -1) \); and \( G \) is magic under \( \alpha \). Applying Theorem 9, we obtain the function \( \alpha' = 6\alpha - \beta \) under which \( G \) is prime-magic.

\[\text{Figure 15.}\]
It is possible for a graph to be both super-magic and prime-magic, witness $K(3, 3)$—shown in dual form below:

$$
\begin{array}{ccc}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2 \\
\end{array}
\quad
\begin{array}{ccc}
13 & 73 & 53 \\
107 & 29 & 3 \\
19 & 37 & 83 \\
\end{array}
$$

It is claimed (2, p. 211) that $K(n, n)$ is prime-magic for $3 \leq n \leq 12$; however, the examples given used 1 as a prime, which does not seem quite fair to this writer.

For other combinations we note that $B_4$ and $W_4$ are super-magic, but not prime-magic; $K_5$ is prime-magic, but not super-magic; and $W_{2k}$ is magic, but neither super-magic, nor prime-magic, for $K \geq 3$.

**Summary.** For each undirected, finite graph $G$, without multiple edges, without loops, and without isolated vertices, we have defined, over the real field, vector spaces $A(G), S(G), Z(G), \xi$ which in the sense of inclusion stand in the relation

$$
A(G) \supseteq S(G) \supseteq Z(G) \supseteq \xi.
$$

(The first equality is required only in the case $K_2$ where $E = 1$.) Concerning dimension we have shown that $\dim A(G) = E$,

$$
\dim Z(G) \leq \dim S(G) \leq 1 + \dim Z(G),
$$

and

$$
\max(0, D(G) + d) \leq \dim S(G) \leq C(G) + d.
$$

We have classified the graphs of admissible type as being trivially-magic, zero-magic, semi-magic, pseudo-magic, magic, super-magic, and prime-magic where the categories are related in the sense of inclusion as shown in Figure 16.

**Figure 16.**

Except possibly for the prime-magic case, the classes are infinite as the following examples show:

trivially-magic: stars $K(1, r), r \geq 2$;
zero-magic, not pseudo-magic: kites with tailing arcs of odd length $Q_{2t+1}$, $t \geq 0$;
zero-magic and pseudo-magic: complete bipartite $K(2, t)$, $t \geq 3$;
semi-magic, not pseudo-magic: arcs of odd length $C_{2t+1}$, $t \geq 1$;
semi-magic and pseudo-magic, not magic: fans $F_{2t}$, $t \geq 2$;
magic, not super-magic, not prime-magic: wheels $W_{2t}$, $t \geq 3$;
super-magic: complete bipartite $K(n, n)$, $n \geq 3$; complete $K_n$, $n > 5$, $n \equiv 0 \pmod{4}$.

For the prime-magic case, the classes are at least non-empty:
super-magic, not prime-magic: basket $B_4$, wheel $W_4$;
super-magic and prime-magic: $K(3, 3)$, $K(4, 4)$, octahedron;
prime-magic, not super-magic: fan $F_5$, $K_5$.

 Added in proof.

**Corollary 2.4.** If $G$ is a connected graph containing a circuit of odd order, then $\dim S(G) = C(G)$.

**Proof.** Suppose the connected graph $G$ contains a circuit $H$ of odd order. Either $G$ itself contains only the one circuit $H$, or $G$ can be obtained by edge completion from a proper subgraph $U$ containing only the one circuit $H$. Because $H$ is of odd order an algebraic argument using Lemma 2 shows that $\dim S(U) = 1$. Since $C(U) = 1$, Lemma 1 shows that $\dim S(G) = C(G)$.

**References**


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