ON THE NONEXISTENCE OF L^p SOLUTIONS OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

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The asymptotic behavior of the solutions of ordinary nonlinear differential equations will be considered here. The growth of the solutions of a differential equation will be discussed by establishing criteria to determine when the differential equation does not possess a solution that is an element of the space $L^p(0, \infty)$ $(p \geqq 1)$.

The first theorem below gives a sufficient condition which guarantees that the solutions of a certain differential equation are not in $L^p(0, \infty)$. This theorem is an extension of a result originally due to Wintner [**8**], where a second-order linear differential equation was considered. This result was successfuly extended to nonlinear second-order differential equations by Suyemoto and Waltman [**6**] and Burlak [**2**]. Our extension is to an nth order nonlinear differential equation, namely,

$$y^{(n)}+g(t,y)=0, \quad y^{(n)}=\frac{d^n y}{dt^n}, \tag{1}$$

where

$$|g(t,y)| \leqq f(t)|y|^r \tag{2}$$

with $n \geqq 1$, $r \geqq 1$, and $f(t)$ continuous on $[0, \infty)$.

We shall assume throughout that $g(t, y)$ is sufficiently smooth to guarantee the existence of solutions of (1). The word " solution ", for the remainder of this note, will mean a non-trivial (i.e., not identically zero) solution that exists on the interval $[0, \infty)$.

THEOREM 1. *Let $y(t)$ be any solution of equation* (1) *with condition* (2) *imposed*; *then $y(t)$ is not in $L^{2r}(0, \infty)$ provided that*

$$\int_0^\infty t^{2n-1} f^2(t)\, dt < \infty. \tag{3}$$

Proof. The proof is similar to that given in the above references and requires the following propositions.

(I) If equation (1) has a solution $y(t)$ in $L^{2r}(0, \infty)$ and (3) is satisfied, then $y^{(n-1)}(t)$, $y^{(n-2)}(t), \ldots, y^{(1)}(t)$, $y(t)$ approach zero as t approaches infinity.

(II) If equation (1) has a solution $y(t)$ in $L^{2r}(0, \infty)$ and (3) is satisfied, then $y(t)$ is in $L^2(0, \infty)$.

Proposition (I) will now be established. From equation (1), we obtain

$$y^{(n-1)}(t)-y^{(n-1)}(0) = -\int_0^t g(s, y(s))\, ds. \tag{4}$$

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Considering the integral in equation (4), using inequality (2) and the Schwarz Inequality, we obtain

$$\left|\int_0^t g(s, y(s))\,ds\right| \leqq \int_0^t |g(s, y(s))|\,ds$$
$$\leqq \int_0^t f(s)\,|y(s)|^r\,ds$$
$$\leqq \left[\int_0^t f^2(s)\,ds\right]^{\frac{1}{2}}\left[\int_0^t |y(s)|^{2r}\,ds\right]^{\frac{1}{2}}.$$

If $y(t)$ is a solution of (1) in $L^{2r}(0, \infty)$, then, from (4), it follows that $\lim_{t\to\infty} y^{(n-1)}(t)$ exists, since the integral is majorized by

$$\left[\int_0^\infty f^2(s)\,ds\right]^{\frac{1}{2}}\left[\int_0^\infty |y(s)|^{2r}\,ds\right]^{\frac{1}{2}}.$$

If $\lim_{t\to\infty} y^{(n-1)}(t) = A_{n-1} \neq 0$, then $\int_0^\infty |y(t)|^{2r}\,dt$ diverges, which contradicts the hypothesis of proposition (I); thus $A_{n-1} = 0$.

Suppose that it has been established that $\lim_{t\to\infty} y^{(n-j)}(t) = 0$, where j is some fixed integer such that $1 \leqq j \leqq n-1$. From equation (1), we obtain

$$y^{(n-j)}(t) = (-1)^{j+1}\int_t^\infty \int_{t_1}^\infty \cdots \int_{t_{j-1}}^\infty g(t_j, y(t_j))\,dt_j\,dt_{j-1}\ldots dt_1.$$

Integration of the above equation, inequality (2) and the Schwarz Inequality lead to the inequality

$$\int_0^\infty |y^{(n-j)}(t)|\,dt \leqq \int_0^\infty \int_t^\infty \int_{t_1}^\infty \cdots \int_{t_{j-1}}^\infty |g(t_j, y(t_j))|\,dt_j\,dt_{j-1}\ldots dt_1\,dt$$
$$\leqq \int_0^\infty t_j^j\,|g(t_j, y(t_j))|\,dt_j$$
$$\leqq \int_0^\infty s^j f(s)\,|y(s)|^r\,ds$$
$$\leqq \left[\int_0^\infty s^{2j} f^2(s)\,ds\right]^{\frac{1}{2}}\left[\int_0^\infty |y(s)|^{2r}\,ds\right]^{\frac{1}{2}}. \qquad (5)$$

The last integrals in (5) are finite by hypothesis. Therefore, since $\int_0^\infty |y^{(n-j)}(s)|\,ds$ converges, $\lim_{t\to\infty} y^{(n-j-1)}(t)$ exists; let this limit be A_{n-j-1}. Again, under the hypothesis of (I), it is necessary that $A_{n-j-1} = 0$. This establishes proposition (I).

To show that (II) holds, let $y(t)$ be a solution of (1) in $L^{2r}(0, \infty)$. Using (I), we may write

$$y(t) = (-1)^{n-1}\int_t^\infty\int_{t_1}^\infty\cdots\int_{t_{n-2}}^\infty\int_{t_{n-1}}^\infty g(t_n, y(t_n))\,dt_n\,dt_{n-1}\ldots dt_1.$$

Proceeding as in (5) above, we obtain the inequality

$$|y(t)| \leqq \int_t^\infty\int_{t_1}^\infty\cdots\int_{t_{n-2}}^\infty\int_{t_{n-1}}^\infty f(t_n)\,|y(t_n)|^r\,dt_n\,dt_{n-1}\ldots dt_1.$$

Therefore

$$\begin{aligned}\int_0^\infty |y(t)|^2\,dt &\leqq \int_0^\infty\left[\int_t^\infty\int_{t_1}^\infty\cdots\int_{t_{n-2}}^\infty\int_{t_{n-1}}^\infty f(t_n)\,|y(t_n)|^r\,dt_n\,dt_{n-1}\ldots dt_1\right]^2 dt\\ &\leqq \int_0^\infty\left[\int_t^\infty t_n^{2(n-1)}f^2(t_n)\,dt_n\right]\left[\int_t^\infty |y(t_n)|^{2r}\,dt_n\right]dt\\ &\leqq \left[\int_0^\infty |y(s)|^{2r}\,ds\right]\left[\int_0^\infty s^{2n-1}f^2(s)\,ds\right] < \infty. \qquad (6)\end{aligned}$$

The above inequality shows that the existence of a solution $y(t)$ in $L^{2r}(0, \infty)$ implies that $y(t)$ is in $L^2(0, \infty)$; that is, proposition (II) is verified.

To complete the proof of Theorem 1, suppose that there exists a solution $y(t)$ of (1) which is in $L^{2r}(0, \infty)$. By virtue of (I), $\lim_{t\to\infty} y(t) = 0$; therefore there exists a $t = T$ such that, for all $t \geqq T$, $|y(t)| < 1$, and $y(t) \neq 0$ on $[T, \infty)$. For $r \geqq 1$ and $t \geqq T$, we have $|y(t)|^{2r} \leqq |y(t)|^2$ and

$$\int_t^\infty |y(t)|^{2r}\,dt \leqq \int_t^\infty |y(t)|^2\,dt.$$

An argument similar to that used in (6) with $t \geqq T$ leads to

$$\begin{aligned}\int_t^\infty |y(s)|^2\,ds &\leqq \left[\int_t^\infty |y(s)|^{2r}\,ds\right]\left[\int_t^\infty s^{2n-1}f^2(s)\,ds\right]\\ &\leqq \left[\int_t^\infty |y(s)|^2\,ds\right]\left[\int_t^\infty s^{2n-1}f^2(s)\,ds\right]. \qquad (7)\end{aligned}$$

Since

$$\int_t^\infty |y(s)|^2\,ds > 0,$$

from (7) we have

$$1 \leqq \int_t^\infty s^{2n-1}f^2(s)\,ds.$$

However, this contradicts (3) and concludes the proof of the theorem.

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Remark 1. It is clear that the above theorem can be extended to a differential equation of the type

$$y^{(n)}+g(t, y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}) = 0,$$

where g satisfies

$$|g(t, y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)})| \leqq f(t)|y(t)|^r;$$

that is, g is a bounded function of $y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}$. The condition (3) is, in a sense, the best possible, as it cannot be replaced by

$$\int_0^\infty t^{2n-1+\varepsilon} f^2(t)\,dt < \infty \quad \text{and} \quad \int_0^\infty t^{2n-1} f^{2+\varepsilon}(t)\,dt < \infty$$

with $\varepsilon > 0$ (see [8]).

For the remainder of this note we shall be primarily concerned with obtaining a sufficient condition for some of the solutions of a nonoscillatory differential equation not to be in $L^p(0, \infty)$ $(p \geqq 1)$. In [4] Kurss used the Sturm Comparison Theorems in an eigenvalue problem to determine the nonexistence of solutions of a linear differential equation in the Hilbert space $L^2(0, \infty)$. In what follows we shall use a nonlinear analogue of the Sturm Comparison Theorem due to Ševelo and Štelik [5] to establish the nonexistence of (L^p)-solutions of a nonlinear second-order differential equation (cf. Theorem 4 below). Also, we shall give a numerical comparison theorem for a pair of nonlinear differential equations (cf. Theorem 3 below).

Consider the differential equations

$$[p(t)u']' + f_1(t, u, u') = 0, \tag{8}$$

$$[p(t)v']' + f_2(t, v, v') = 0, \tag{9}$$

where $p'(t)$ is continuous on $[0, \infty)$, $p(t) \geqq p_0 > 0$, p_0 is constant, and the $f_i(t, z, z')$ $(i = 1, 2)$ are continuous and satisfy conditions sufficient to guarantee the existence and uniqueness of solutions of (8) and (9) for $t \in [0, \infty)$.

The following result can be found in [5].

THEOREM 2. *If* $f_2(t, v, v')/v - f_1(t, u, u')/u \geqq 0$ *for all* $u', v', t \in [0, \infty)$, $u \in U = (-\bar{u}, \bar{u})$, $v \in V = (-\bar{v}, \bar{v})$ $(0 < \bar{u} \leqq \infty, 0 < \bar{v} \leqq \infty)$, *then between any two consecutive zeros of an arbitrary solution* $u(t) \in U$ *of equation* (8) *there is situated at least one zero of each solution* $v(t) \in V$ *of equation* (9).

Remark 2. Subject to the hypotheses of the above theorem, we observe that, if equation (9) has a nonoscillatory solution $v(t)$ in V, then all solutions $u(t)$ in U of equation (8) are non-oscillatory. This is true because, if some solution of (8) is oscillatory, then, by Theorem 2, all solutions of (9) are oscillatory.

THEOREM 3. *Let* $u(t)$ *and* $v(t)$ *be solutions of* (7) *and* (8), *respectively, such that*

$$u(t_0) = v(t_0) \neq 0, \quad u'(t_0) = v'(t_0).$$

If

$$f_2(t,v,v')/v > f_1(t,u,u')/u \tag{10}$$

for all $t \in [t_0, \infty), u', v', u, v$, *where* $0 < |v| \leqq |u|$, *then* $|u(t)| > |v(t)|$ *provided that* $v(t)$ *is not zero.*

Proof. The proof follows that given by Tricomi [7, p. 103] and is a consequence of the identity

$$p(t)[vu' - uv'](t) = \int_{t_0}^{t} u(s)v(s)\left[\frac{f_2(s, v(s), v'(s))}{v(s)} - \frac{f_1(s, u(s), u'(s))}{u(s)}\right] ds.$$

THEOREM 4. *Let inequality* (10) *hold for all* $t \in [0, \infty)$, u', v', u, *and* v; *furthermore, suppose that equation* (9) *is nonoscillatory. If some solution of* (9) *is not in* $L^p(0, \infty)$ $(p \geqq 1)$, *then some solution of* (8) *is not in* $L^p(0, \infty)$.

Proof. If $v(t)$ is a solution of the nonoscillatory equation (9) which is not in $L^p(0, \infty)$, then, by Remark 2 with $f_1 = f_2$, $v(t)$ is nonoscillatory. Hence there exists a $t = t_0$ such that $|v(t)| > 0$ for $t \geqq t_0$. By virtue of Theorem 3, it follows that the solution $u(t)$ of equation (8) that satisfies the initial conditions

$$u(t_0) = v(t_0), \quad u'(t_0) = v'(t_0)$$

also satisfies the inequality

$$|u(t)| > |v(t)|.$$

Therefore $u(t)$ is not in $L^p(t_0, \infty)$ and consequently not in $L^p(0, \infty)$.

Remark 3. The above theorem can be used as an (L^p)-existence theorem for equation (9). We illustrate this by the following result of Bellman [1].

If all the solutions of

$$u'' + a(t)u = 0 \tag{11}$$

belong to $L^2(0, \infty)$, then all the solutions of

$$v'' + \{a(t) + b(t)\}v = 0 \tag{12}$$

belong to $L^2(0, \infty)$ provided that $|b(t)| \leqq c_1$ $(t > 0)$.

It follows from Theorem 4 that the same conclusion is now obtained without the boundedness condition $|b(t)| \leqq c_1$ $(t > 0)$, but under the different condition that (12) is nonoscillatory. Actually, the equation (12) may be generalized to a nonlinear equation; for example,

$$v'' + a(t)v + \sum_{j=1}^{n} b_j(t)v^{2j-1} = 0, \tag{12'}$$

where $b_j(t) \geqq 0$ for all $j = 1, 2, \ldots, n$, and $b_k(t) > 0$ for some $k = 1, 2, \ldots, n$.

However, even for the linear case, the above result is not easy to apply. To observe this, we consider the following question: When does the nonoscillatory linear equation (11) possess all solutions in $L^2(0, \infty)$? (The fact that (11)—and not merely (12)—must be nonoscillatory results from Remark 2 above.)

As a partial answer, we observe that $a(t)$ cannot be bounded [1, p. 138, Problem 8]. Also, $a(t)$ cannot be of constant sign; for if $a(t) < 0$ on $[0, \infty)$, consider the equations $v'' = 0$ and $u'' + a(t)u = 0$. The hypotheses of Theorem 4 are satisfied and, consequently, all solutions of $u'' + a(t)u = 0$ cannot be in $L^2(0, \infty)$. On the other hand, if $a(t) > 0$ on $[0, \infty)$, $\int^{\infty} a(t)\,dt < \infty$ is necessary for nonoscillation [3, p. 367]; and, as remarked above, $a(t)$ cannot be bounded. However, if $a(t) > 0$, then all solutions of (11) are either oscillatory or monotone; hence, in the case under consideration, all solutions must be monotone. But, if $\int^{\infty} a(t)\,dt < \infty$, then all solutions cannot be bounded [1, p. 121, Problem 2]; therefore all solutions cannot be in $L^2(0, \infty)$.

Thus, if equation (11) is to be nonoscillatory and have all solutions in $L^2(0, \infty)$, then it is necessary that $a(t)$ be oscillatory and $\limsup_{t\to\infty} |a(t)| = \infty$. In this situation it is an open question if the hypotheses are compatible with those on f_1 and f_2 in Theorems 2, 3 and 4.

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