

RESEARCH ARTICLE

The Chern classes and Euler characteristic of the moduli spaces of Abelian differentials

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Abstract

For the moduli spaces of Abelian differentials, the Euler characteristic is one of the most intrinsic topological invariants. We give a formula for the Euler characteristic that relies on intersection theory on the smooth compactification by multi-scale differentials. It is a consequence of a formula for the full Chern polynomial of the cotangent bundle of the compactification.

The main new technical tools are an Euler sequence for the cotangent bundle of the moduli space of multi-scale differentials and computational tools in the Chow ring, such as a description of normal bundles to boundary divisors.

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1. Introduction

The moduli space of Abelian differentials $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ parametrises Riemann surfaces together with meromorphic one-forms whose zeros and poles are of a fixed type $\mu = (m_1, \dots, m_n)$. Interest in these moduli spaces stems from the dynamics of billiards and flat surfaces. This surface dynamics is related to the $\mathrm{SL}_2(\mathbb{R})$ -action on moduli spaces of Abelian differentials. The understanding of this dynamics has advanced rapidly; see [EM18], [EMM15], [Fil16], [EFW18], [EMMW20], to mention just the tip of the iceberg. Recent interest in the moduli space of Abelian differentials also stems from viewing them as a twisted version of the double ramification cycle ([HS21], [BHPSS20]).

In contrast to the dynamics, only a very few aspects of the topology of the moduli spaces of holomorphic or meromorphic Abelian differentials $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ are currently known, such as the connected components ([KZ03], [Boi15]) and partial information about (quotients of) the fundamental group ([CS20]).

This paper provides several insights into the topology and geometry of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. We give an expression for the Chern classes of the cotangent bundle of the compactified moduli spaces of Abelian differentials and a formula to compute the Euler characteristic of these moduli spaces.

The moduli spaces of Abelian differentials can be thought of as relatives of the moduli space of curves $\mathcal{M}_{g,n}$, for which the Euler characteristic was computed in [HZ86] using a cellular decomposition (given by the arc complex) and counting of cells. Our strategy here is quite different. While the Euler characteristic is an intrinsic quantity associated to $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$, our strategy heavily uses the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ constructed in [BCGGM3] and all its properties that make it quite similar to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. Moreover, our strategy is not available to compute the Euler characteristic $\overline{\mathcal{M}}_{g,n}$, as it rather mimics the case of the projective space \mathbb{P}^d : the unprojectivised moduli spaces $\Omega\mathcal{M}_{g,n}(\mu)$ are linear manifolds, and thus the cotangent bundle of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is governed by the Euler sequence, as in the case of \mathbb{P}^d .

Using this strategy, we obtain the complete information about the Chern classes of the (logarithmic) canonical bundle of the compactified moduli spaces of Abelian differentials and thus, for example, the χ - γ -genus. A special case, the formula for the canonical class, is particularly easy to state. We recall that the boundary divisors in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are the divisor D_h of multi-scale differentials whose level graph has one horizontal node and the divisors D_Γ parametrised by level graphs $\Gamma \in \text{LG}_1(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ that have one level below the zero level and no horizontal edges (joining vertices of the same level). Similar to the moduli space of curves, the boundary divisors are nearly (in a sense that we elucidate further down) a product of two lower-dimensional moduli spaces corresponding to the top and bottom level. Those boundary divisors D_Γ come with the integer ℓ_Γ , the least common multiple of the prongs κ_e along the edges; see Section 3.3 for a review of these notions. We let $\xi = c_1(\mathcal{O}(-1))$ be the first Chern class of the tautological bundle on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ (see Section 3.1).

Theorem 1.1. *The first Chern class of the logarithmic cotangent bundle of the projectivised compactified moduli space $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is*

$$c_1(\Omega_{\overline{B}}^1(\log D)) = N \cdot \xi + \sum_{\Gamma \in \text{LG}_1(\overline{B})} (N - N_\Gamma^\top) \ell_\Gamma [D_\Gamma] \in \text{CH}^1(\overline{B}), \tag{1}$$

where $N := \dim(\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ and where $N_\Gamma^\top := \dim(B_\Gamma^\top)$ is the dimension of the unprojectivised top level stratum in D_Γ .

To compute the Euler characteristic, we need to understand the top Chern class, as we recall in Section 2, along with standard terminology from intersection theory. To state a formula for the full Chern character, we need to recall a procedure that also determines adjacency of boundary strata. It is given by undegeneration maps δ_i that contract all the edges except those that cross from level $-i + 1$ to level $-i$; see Section 3.3 and Figure 5 in Section 10. This construction can obviously be generalised so that a larger subset of levels remains: for example, the complement of i , denoted by the undegeneration map δ_i^c . We can now define for any graph $\Gamma \in \text{LG}_L(\overline{B})$ with L levels below zero and without horizontal edges the quantity $\ell_\Gamma = \prod_{i=1}^L \ell_{\delta_i(\Gamma)}$. In the following theorem, we define $N_{\delta_0(\Gamma)}^\top = 0$ so that the term $L = 0$ is well-defined.

Theorem 1.2. *The Chern character of the logarithmic cotangent bundle is*

$$\text{ch}(\Omega_{\overline{B}}^1(\log D)) = e^\xi \cdot \sum_{L=0}^{N-1} \sum_{\Gamma \in \text{LG}_L(\overline{B})} \ell_\Gamma \left(N - N_{\delta_L(\Gamma)}^\top \right) i_{\Gamma*} \left(\prod_{i=1}^L \text{td} \left(\mathcal{N}_{\Gamma/\delta_i^c(\Gamma)}^{\otimes -\ell_{\delta_i(\Gamma)}} \right)^{-1} \right),$$

Table 1. Euler characteristics of some holomorphic strata.

| | | | | | | | | |
|-----------|---------------------|-----------------|-----------------|-------------------|------------------|-----------------|-------------------|--------------------|
| μ | (0) | (2) | (1, 1) | (4) | (3, 1) | (2, 2) | (2, 1, 1) | (1, 1, 1, 1) |
| $\chi(B)$ | $-\frac{1}{12}$ | $-\frac{1}{40}$ | $\frac{1}{30}$ | $-\frac{55}{504}$ | $\frac{16}{63}$ | $\frac{15}{56}$ | $-\frac{6}{7}$ | $\frac{11}{3}$ |
| μ | (6) | (5, 1) | (4, 2) | (3, 3) | (4, 1, 1) | (3, 2, 1) | (2, 2, 2) | (8) |
| $\chi(B)$ | $-\frac{1169}{720}$ | $\frac{27}{5}$ | $\frac{76}{15}$ | $\frac{188}{45}$ | $-\frac{200}{9}$ | $-\frac{96}{5}$ | $-\frac{187}{10}$ | $-\frac{4671}{88}$ |

where $\mathcal{N}_{\Gamma/\delta_i^C(\Gamma)}$ denotes the normal bundle of D_Γ in $D_{\delta_i^C(\Gamma)}$, td is the Todd class and $i_\Gamma : D_\Gamma \hookrightarrow \bar{B}$ is the inclusion map.

We also give closed expressions for the Chern polynomial in Theorem 9.10, both fully factored and as a sum over level graphs.

To compute the Euler characteristics, we can simplify this expression significantly. Moduli spaces of Abelian differentials are not homogeneous spaces, and we should not expect a proportionality between the top Chern class and the Masur-Veech volume form ([Mas82], [Vee82]). For comparison, however, we note that Masur-Veech volumes of holomorphic minimal strata (where $\mu = (2g - 2)$) in each genus are essentially given by the top ξ -power ([Sau18]). For nonminimal holomorphic strata (that is, if all $m_i \geq 0$), this top ξ -power is zero and the Masur-Veech volume is computed by a product of ξ^{2g-1} and ψ -classes ([CMSZ20]). The top ξ -powers of all levels of all strata – and only these – are combined to give the Euler characteristic of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. One thus needs to consider the top ξ -powers for the strata of meromorphic differentials, even if one might be only interested in the holomorphic case. Let $K_\Gamma = \prod_e \kappa_e$ be the product of the prongs over all edges of Γ .

Theorem 1.3. *The orbifold Euler characteristic of the moduli space $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is the dimension-weighted sum over all level graphs $\Gamma \in \text{LG}_L(\bar{B})$ without horizontal nodes*

$$\chi(\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)) = (-1)^d \sum_{L=0}^d \sum_{\Gamma \in \text{LG}_L(\bar{B})} \frac{K_\Gamma \cdot N_\Gamma^T}{|\text{Aut}(\Gamma)|} \cdot \prod_{i=0}^{-L} \int_{B_\Gamma^{[i]}} \xi_{B_\Gamma^{[i]}}^{d_\Gamma^{[i]}} \tag{2}$$

of the product of the top power of the first Chern class $\xi_{B_\Gamma^{[i]}}$ of the tautological bundle at each level, where $d_\Gamma^{[i]} = \dim(B_\Gamma^{[i]})$ and $d = \dim(\bar{B}) = N - 1$.

The stratum $B_\Gamma^{[i]}$ at the level i of a graph Γ is defined in Section 4.1.

Table 1 gives the Euler characteristics of some strata of holomorphic differentials. A table of values of top ξ -powers and more examples are provided in Section 10. The evaluation of these formulas is performed by a sage package `diffstrata` that builds on the package `admcycles` for computation in the moduli space of curves ([DSZ21]). Specifically, the evaluation of tautological classes below is performed using the formula for fundamental classes of strata conjectured in [FP18] and [Sch18] and proven recently in [BHPSS20] based on results from [HS21]. The algorithms in this package are explained in [CMZ20].

The Euler sequence.

Next, we outline the ingredients needed to prove these theorems. Recall that for projective space the Euler sequence is the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(V)}^1 \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(-1)^{\oplus \dim(V)} \xrightarrow{\text{ev}} \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0. \tag{3}$$

Over the moduli space $\bar{B} = \mathbb{P}\Xi\bar{\mathcal{M}}_{g,n}(\mu)$, this admits the following generalisation that combines Theorem 6.1 and Theorem 9.2. It states roughly that using the local projective structure induced by period coordinates, in the interior of the stratum, we indeed have a Euler sequence if we replace the

direct sum in the middle of the sequence with a local system. This local system naturally extends across the boundary, but the Euler sequence needs a correction term that we determine explicitly via a local computation using perturbed period coordinates.

Theorem 1.4. *The logarithmic cotangent bundle sits in an exact sequence*

$$0 \longrightarrow \Omega_{\overline{B}}^1(\log D) \left(- \sum_{\Gamma \in \text{LG}_1(\overline{B})} \ell_{\Gamma} D_{\Gamma} \right) \longrightarrow \mathcal{K} \longrightarrow \mathcal{C} \longrightarrow 0, \tag{4}$$

where \mathcal{C} is an explicitly computable sheaf (see Lemma 9.4) supported on the boundary and the vector bundle \mathcal{K} on \overline{B} fits into the Euler exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow (\overline{\mathcal{H}}_{rel}^1)^{\vee} \otimes \mathcal{O}_{\overline{B}}(-1) \xrightarrow{ev} \mathcal{O}_{\overline{B}} \longrightarrow 0. \tag{5}$$

Here $\overline{\mathcal{H}}_{rel}^1$ is the Deligne extension of the local system of relative cohomology.

This theorem directly implies Theorem 1.1. To deduce the other two theorems, we need to exploit further information on the Chow ring of the compactification.

The tautological rings.

In Section 8, we define a notion of a system of *tautological rings* $R^{\bullet}(\overline{\Xi\mathcal{M}}_{g,n}(\mu))$ inside the Chow rings of the compactifications $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of the projectivised strata $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ that have been constructed in [BCGGM3]. This is the smallest system of \mathbb{Q} -subalgebras $R^{\bullet}(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subset \text{CH}^{\bullet}(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ that

- contains the ψ -classes attached to the marked points,
- is closed under the pushforward of the map forgetting a regular marked point (a zero of order zero) and
- is closed under the clutching homomorphisms $\zeta_{\Gamma,*} p^{[i],*}$, defined in Section 4.

For the moduli space of curves $\overline{\mathcal{M}}_g$, the clutching homomorphisms build a boundary divisor from a product of two smaller moduli spaces or from just one for the irreducible boundary divisor that plays the role of our D_h . For multi-scale differentials, the situation is more involved. First, to relate D_{Γ} to a product of moduli spaces, we need to allow spaces of disconnected curves and imposing residue conditions since the levels of Γ have that property. We define such *generalised strata* and their modular compactification in Section 4. Second, the boundary divisors D_{Γ} do not admit maps to such generalised strata since the levels are tied to one another by a datum of the multi-scale differential, the prong-matchings. We need to construct a covering space $c_{\Gamma} : D_{\Gamma}^s \rightarrow D_{\Gamma}$ that removes the stacky structure of D_{Γ} , which has two properties. First, there are projection maps $p^{[i]}$ from D_{Γ}^s to generalised strata; and second, there are clutching maps $\zeta_{\Gamma} : D_{\Gamma}^s \rightarrow \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ that factor as $\zeta_{\Gamma} = i_{\Gamma} \circ c_{\Gamma}$ into the finite map c_{Γ} and a closed embedding i_{Γ} . (The upper index of D_{Γ}^s refers to the use of the simple twist group as in [BCGGM3] in the construction of this covering.) We present the basic structure of the tautological ring to make the striking parallels to $\overline{\mathcal{M}}_{g,n}$ (as in [GP03] or [ACG11]) apparent.

Theorem 1.5. *For each μ , a finite set of additive generators of $R^{\bullet}(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ is given by the classes*

$$\zeta_{\Gamma,*} \left(\prod_{i=0}^{-L} p^{[i],*} \alpha_i \right), \tag{6}$$

where Γ runs over all level graphs for all boundary strata of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ including the trivial graph and where α_i is a monomial in the ψ -classes supported on level i of the graph Γ .

The tautological ring contains the κ -classes and all level-wise tautological line bundle classes $\zeta_{\Gamma, P}^{[i], *}, \xi_{B_{\Gamma}^{[i]}}$ of all level graphs Γ .

An algorithm to perform the multiplication of these generators is given along with the proof of Theorem 1.5 in Section 8. An important technical tool in the proof is the excess intersection formula (see Proposition 8.1), which, like the above formulation of the tautological ring, has large structural similarities with the case of the Deligne-Mumford compactification. This is useful only if the normal bundles to the boundary divisors are known. Contrary to the Deligne-Mumford compactification, the boundary divisors defined by two-level graphs do not self-intersect in the space of multi-scale differentials (see Section 5). This allows us to control the combinatorics of intersections of nonhorizontal boundary components (using the notion of ‘profile’) and to compute their normal bundles as closed substacks instead of needing to compute the normal bundles of the clutching morphisms, which are formally more intricate to define than for the moduli space of curves. Along with the clutching morphisms, we define in Section 4.3 the tautological bundles on the top and bottom level strata of divisors and their first Chern classes ξ^{\top} and ξ^{\perp} . In Section 7, we show:

Theorem 1.6. *The normal bundle \mathcal{N}_{Γ} of a divisor $D_{\Gamma} \in \text{LG}_1(\overline{B})$ has first Chern class*

$$c_1(\mathcal{N}_{\Gamma}) = \frac{1}{\ell_{\Gamma}} (-\xi_{\Gamma}^{\top} - c_1(\mathcal{L}_{\Gamma}^{\top}) + \xi_{\Gamma}^{\perp}) \quad \text{in } \text{CH}^1(D_{\Gamma}), \tag{7}$$

where $\mathcal{L}_{\Gamma}^{\top}$ defined in equation (49) is a line bundle supported on the boundary of D_{Γ} , where the top-level stratum degenerates further.

We define tautological rings $R^{\bullet}(D_{\Gamma})$ of strata using the analogues of the additive generators equation (6), and as a consequence of the preceding theorem, the normal bundle of each D_{Γ} belongs to the tautological ring $R^{\bullet}(D_{\Gamma})$.

Organization and strategy of proof.

After recalling some background on intersection theory in Section 2, we provide the necessary details on the compactification $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ in Section 3. Each level of a level graph gives rise to the notion of generalised strata, defined in Section 4. There we also introduce the covering of boundary strata that allows a decomposition into a product of levels. Section 5 provides a dimension count argument that implies the smoothness of all nonhorizontal boundary strata and is at the heart of a formula for exponentials of sums of over boundary graphs. This formula allows, together with Theorem 1.6, the passage from Theorem 1.4 to Theorem 1.3. Section 6 proves the restriction of Theorem 1.4 to the interior of $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$, and Section 7 proves Theorem 1.6. In Section 8, we prove the properties of the tautological ring announced above. In Section 9, a local calculation at the boundary completes the proof of Theorem 1.4, and computations in the tautological ring allow the passage from Theorem 1.2 to Theorem 1.3.

The strategy used here applies to other linear manifolds for which a compactification similar to that in [BCGGM3] has been constructed. It is already available for meromorphic k -differentials for $k > 0$ (see [CMZ19]) and expected to work for any affine invariant manifold. The proof of the main theorems should carry over with very few adaptations. We hope to address these cases in a sequel.

2. Euler characteristics via logarithmic differential forms

This section connects Euler characteristic to integrals of characteristic classes of the sheaf of logarithmic differential forms. The following proposition is certainly well known but not easy to locate in the literature. We use the occasion to give a self-contained proof; see also [Fio17] and recall some standard exact sequences.

Proposition 2.1. *Let \bar{B} be a compact smooth k -dimensional manifold, let D be a normal crossing divisor, and $B = \bar{B} \setminus D$. Then the Euler characteristic of B can be computed as an integral*

$$\chi(B) = (-1)^k \int_{\bar{B}} c_k(\Omega_B^1(\log D)) \tag{8}$$

over the top Chern class of the logarithmic cotangent bundle.

In all our applications, \bar{B} will be a compact orbifold or proper smooth Deligne-Mumford stack. We work throughout with orbifold Euler characteristics, and since then both sides of equation (8) are multiplicative in the degree of a covering, we can apply Proposition 2.1 verbatim.

2.1. The compact case and the Riemann-Roch theorem

We start with the proof of the special case of the main theorem.

Proposition 2.2. *If $B = \bar{B}$ is smooth, compact and k -dimensional, then*

$$\chi(B) = \int_{\bar{B}} c_k(T_B). \tag{9}$$

We start by recalling some intersection theory. Let \mathcal{E} be a holomorphic vector bundle on B . Denote by $c_i := c_i(\mathcal{E}) \in \text{CH}^i(B)$ the i th Chern class of E . Recall that $c_0 = 1$ and $c_i = 0$ for $i > \text{rk } E =: r$. The total Chern class of \mathcal{E} is the formal sum $c(\mathcal{E}) = 1 + c_1 + \dots + c_r$ in $\text{CH}(B)$. Splitting formally $c(\mathcal{E}) = \prod_{i=1}^r (1 + \alpha_i)$ into the Chern roots, the Chern character is defined as the formal power series

$$\text{ch}(E) = \sum_{i=1}^r \exp(\alpha_i) = \sum_{s \geq 0} \frac{1}{s!} \sum_{i=1}^r \alpha_i^s = \text{rk}(E) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \dots$$

Furthermore, the Todd class is defined as

$$\text{td}(E) = \prod_{i=1}^r \frac{\alpha_i}{1 - \exp(-\alpha_i)} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \dots$$

The Grothendieck-Riemann-Roch theorem in the case of a map $f : X \rightarrow Y$, and for the special case that the higher direct images $R^i f_* \mathcal{E}$ vanish, states that

$$\text{ch}(f_* \mathcal{E}) \cdot \text{td}(T_Y) = f_*(\text{ch}(\mathcal{E}) \cdot \text{td}(T_X)). \tag{10}$$

Proof of Proposition 2.2. For a topological proof, see, for example, [BT82, Proposition 11.24]. Using the notations already set up, we give a quick proof if moreover B is Kähler. From the Borel-Serre identity ([Ful98, Example 3.2.5]) on a k -dimensional manifold,

$$c_k(T_B) = \text{ch}\left(\sum_{j=1}^k (-1)^j \Omega_B^j\right) \cdot \text{td}(T_B)$$

and the application

$$\int_{\bar{B}} \text{ch}((-1)^j \Omega_B^j) \cdot \text{td}(T_B) = \sum_{\ell \geq 0} (-1)^{\ell+j} h^\ell(B, \Omega_B^j)$$

of the Grothendieck-Riemann-Roch theorem for the map from B to a point, we get

$$\int_{\overline{B}} c_k(T_B) = \sum_{\ell, j \geq 0} (-1)^{\ell+j} h^\ell(B, \Omega_B^j) = \chi(B)$$

by the Hodge decomposition. □

2.2. The noncompact case and log differential forms

We suppose throughout that $D = \cup_{j=1}^s D_j$ is a reduced normal crossing divisor: that is, with distinct irreducible components D_i intersecting each other transversally. In this situation, $\Omega_{\overline{B}}^1(\log D)$ is defined to be the vector bundle of rank n with the following local generators. In a neighbourhood U of a point where (say) the first $r \leq s$ divisors meet and x_1, \dots, x_k is a local coordinate system with $D_j = \{x_j = 0\}$, then the logarithmic cotangent bundle is defined by

$$\Omega_{\overline{B}}^1(\log D)(U) = \left\langle \frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, dx_{r+1}, \dots, dx_k \right\rangle \tag{11}$$

as an $\mathcal{O}_{\overline{B}}(U)$ -module. There is a fundamental exact sequence for log differential forms, namely

$$0 \rightarrow \Omega_{\overline{B}}^1 \rightarrow \Omega_{\overline{B}}^1(\log D) \rightarrow \oplus_{j=1}^s (i_j)_* \mathcal{O}_{D_j} \rightarrow 0, \tag{12}$$

where $i_j : D_j \rightarrow \overline{B}$ is the inclusion map. More details can be found, for example, in [EV92, Proposition 2.3].

Proof of Proposition 2.1. We first reduce to the case that D has simple normal crossings: that is, to the case that the D_j are all smooth. This can always be achieved by an étale covering. Since both sides of equation (8) are multiplied by the degree under such a covering, we can assume simple normal crossings. Our goal is to prove

$$\int_{\overline{B}} c_k \left(\Omega_{\overline{B}}^1 \left(\log \sum_{i \geq 2} D_i \right) \right) = \int_{\overline{B}} c_k \left(\Omega_{\overline{B}}^1(\log D) \right) - \int_{D_1} c_{k-1} \left(\Omega_{D_1}^1 \left(\log \left(\sum_{i \geq 2} D_i \cap D_1 \right) \right) \right).$$

The claim follows then from the additivity $\chi(B) + \chi(D) = \chi(\overline{B})$ of the Euler characteristic, Proposition 2.2 and an application of the preceding identity to $B_j = \overline{B} \setminus \cup_{i=j}^s D_j$.

We consider the inclusion of the boundary divisor D_1 and deduce from the ideal sheaf sequence that $c((i_1)_* \mathcal{O}_{D_1}) = (1 - [D_1])^{-1}$ and that $c(\mathcal{N}_{D_1}) = 1 + i_1^*[D_1]$. Moreover the normal bundle sequence $0 \rightarrow \mathcal{T}_{D_1} \rightarrow i_1^* \mathcal{T}_{\overline{B}} \rightarrow \mathcal{N}_{D_1} \rightarrow 0$ implies

$$c(\Omega_{D_1}^1) = i_1^* \left(c(\Omega_{\overline{B}}^1) \cdot \frac{1}{1 - [D_1]} \right). \tag{13}$$

On the other hand, the sequence in equation (12) gives

$$c(\Omega_{\overline{B}}^1(\log D)) = c(\Omega_{\overline{B}}^1) \cdot \frac{1}{1 - [D_1]} \cdot \prod_{j=2}^s \frac{1}{1 - [D_j]} \tag{14}$$

and also

$$c \left(\Omega_{D_1}^1 \left(\log \left(\sum_{i \geq 2} D_i \cap D_1 \right) \right) \right) = c(\Omega_{D_1}^1) \cdot \prod_{j=2}^s \frac{1}{1 - [D_1 \cap D_j]}. \tag{15}$$

Hence, comparing with equation (13), we get

$$c\left(\Omega_{D_1}^1\left(\log\left(\sum_{i \geq 2} D_i \cap D_1\right)\right)\right) = i_1^*c\left(\Omega_B^1(\log D)\right). \tag{16}$$

Finally, from equation (14) and the appropriate version of the sequence in equation (12), we also get

$$c\left(\Omega_B^1(\log D)\right) = \frac{1}{1 - [D_1]}c\left(\Omega_B^1\left(\log \sum_{i \geq 2} D_i\right)\right). \tag{17}$$

The claim now follows by multiplying this last expression with $1 - [D_1]$, integrating and taking the k th coefficient, using that $\int_B [D_1] \cdot c_{k-1}(\Omega_B^1(\log D)) = \int_{D_1} i_1^*c_{k-1}(\Omega_B^1(\log D))$. \square

3. The moduli space of multi-scale differentials

We recall here from [BCGGM3] basic properties of the moduli space of multi-scale differentials $\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ and its projectivisation $\mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ that compactifies the moduli space $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ of projectivised meromorphic differentials. Throughout, we suppose that $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$ is the type of a differential: that is, $\sum_{j=1}^n m_j = 2g - 2$. We usually abbreviate $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ and $\overline{B} = \mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu)$.

3.1. Enhanced level graphs

To define strata and the ambient space in the meromorphic case, we assume that there are r positive m 's, s zeroes and l negative m 's, with $r + s + l = n$: that is, we have $m_1 \geq \dots \geq m_r > m_{r+1} = \dots = m_{r+s} = 0 > m_{r+s+1} \geq \dots \geq m_n$. Note that $m_j = 0$ is allowed, representing an ordinary marked point. A pointed flat surface is usually denoted by (X, ω, \mathbf{z}) , where $\mathbf{z} = (z_1, \dots, z_n)$ are the marked points corresponding to the zeros, ordinary marked points and poles of ω . The sections over $\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ corresponding to those marked points are denoted by \mathcal{Z}_i . We denote the polar part of μ by $\tilde{\mu} = (m_{r+s+1}, \dots, m_n)$. The strata of meromorphic differentials are then naturally defined inside the twisted Hodge bundle

$$K\overline{\mathcal{M}}_{g,n}(\tilde{\mu}) = f_*\left(\omega_{\mathcal{X}/\overline{\mathcal{M}}_{g,n}}\left(-\sum_{j=r+s+1}^n m_j \mathcal{Z}_j\right)\right).$$

The strata are smooth complex substacks $\Omega\mathcal{M}_{g,n}(\mu)$ of dimension $N = 2g - 1 + n$ in the holomorphic case $l = 0$ and $N = 2g - 2 + n$ in the meromorphic case.

To each boundary point in $D = \overline{\Xi\mathcal{M}}_{g,n}(\mu) \setminus \Omega\mathcal{M}_{g,n}(\mu)$, there is an associated *enhanced level graph*, and D is stratified by the type of this associated graph. Here a *level graph* is defined to be a stable graph $\Gamma = (V, E, H)$, with half-edges in H that are either paired to form edges E or correspond to the n marked points, together with a total order on the vertices (with equality permitted). The graph Γ is supposed to be connected here; from Section 8 on its components are in bijection with the components of the flat surfaces the generalised stratum parametrises. For convenience, we usually define the total order using a *level function* $\ell : V(\Gamma) \rightarrow \mathbb{Z}$, usually normalized to take values in $\{0, -1, \dots, -L\}$. We usually write $H_m = H \setminus E$ for the half-edges corresponding to the marked points. Moreover, an *enhancement* (in [FP18] or [CMSZ20], this number is called a *twist*) is an assignment of a number $\kappa_e \geq 0$ to each edge e , so that $\kappa_e = 0$ if and only if the edge is horizontal. The triple $(\Gamma, \ell, \{\kappa_e\}_{e \in E(\Gamma)})$ is called an *enhanced level graph*. We denote the closure of the boundary stratum parametrising multi-scaled differentials (as defined below) compatible with $(\Gamma, \ell, \{\kappa_e\})$ by $D_{(\Gamma, \ell, \{\kappa_e\})}$ or usually simply by D_Γ .

Theorem 3.1 ([BCGGM3]). *There is a proper smooth Deligne-Mumford stack $\mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ that contains the projectivised stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ as an open dense substack with the following properties.*

- (i) The boundary $\mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu) \setminus \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is a normal crossing divisor.
- (ii) The codimension of a boundary stratum D_Γ in $\mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ is equal to the number of horizontal edges plus the number L of levels below zero.

In particular, the *boundary divisors* consist of the divisor D_h parametrising level graphs on just one level and with just one horizontal edge and the (‘vertical’) boundary divisors indexed by two-level graphs without horizontal edges. (For holomorphic types μ , the divisor D_h is irreducible and parametrises graphs with a nonseparating edge; but for general μ , this divisor is reducible.) We give local coordinates near the boundary divisors in Section 6.2.

Note that the boundary strata D_Γ may be empty for some enhanced level graphs. Deciding nonemptiness is the same as the realisability question that was addressed in [MUW21] purely in terms of graphs. The general version, taking into account the residue conditions, is stated in the algorithmic part of [CMZ20]. Note that these boundary strata may also be disconnected; see the discussion of prong-matching equivalence classes below.

Recall that the construction of $\mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ in [BCGGM3] gives a morphism $\mathbb{P}\overline{\Xi\mathcal{M}}_{g,n}(\mu) = \overline{B} \rightarrow \mathbb{P}\left(f_*\omega_{\overline{X}/\overline{\mathcal{M}}_{g,n}}(-\sum_{j=r+s+1}^n m_j \mathcal{Z}_j)\right)$ to the projectivised twisted Hodge bundle over the Deligne-Mumford compactification. The line bundle $\mathcal{O}_{\overline{B}}(-1)$ is the pullback of the tautological bundle from there.

3.2. Twisted differentials and multi-scale differentials

The space $\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ is a moduli stack for families of a certain collection of differentials, called multi-scale differentials, and this modular interpretation will be used, for example, in Section 4 to define clutching maps and projection maps at the boundary. However, we will refer to $\overline{\Xi\mathcal{M}}_{g,n}(\mu)$ as a moduli space to stick to the commonly used terminology. We recall the definition of a single multi-scale differentials, referring for full details of the definition for families to [BCGGM3]. We will recall further details where needed.

When referring to prongs, we fix a direction in S^1 throughout, say the horizontal direction. Suppose that a differential ω has a zero of order $m \geq 0$ at $q \in X$. The differential ω selects inside the real projectivised tangent space $P_q = T_p X / \mathbb{R}_{>0}$ a collection of $\kappa = m + 1$ horizontal (outgoing) *prongs* at q , the tangent vectors $\mathbb{R}_{>0} \cdot \zeta_\kappa^i \partial / \partial z$ in a chart where $\omega = z^m dz$ is in standard form and ζ_κ is a primitive κ th root of unity. We denote them by $P_q^{\text{out}} \subset P_q$. The prongs are equivalently the tangent vectors to the outgoing horizontal rays. Dually, if ω has a pole of order $m \leq -2$, then ω has $\kappa = -m - 1$ horizontal (incoming) *prongs* at q , denoted by $P_q^{\text{in}} \subset P_q$, the tangent vectors $\mathbb{R}_{>0} \cdot -\zeta_\kappa^i \partial / \partial z$ in a chart where $\omega = z^m dz$.

We start with an auxiliary notion of differentials from [BCGGM1]. Given a pointed stable curve (X, \mathbf{z}) , a *twisted differential* is a collection of differentials η_v on each component X_v of X : that is, *compatible with a level structure* on the dual graph Γ of X – it vanishes as prescribed by μ at the marked points z and satisfies the matching order condition at vertical nodes, the matching residue condition at horizontal nodes and the global residue condition of [BCGGM1]. We usually group the differentials on the components of level i of X to form the collection $\eta_{(i)}$ and refer to a twisted differential by $\boldsymbol{\eta} = (\eta_{(i)})$.

A *multi-scale differential of type μ* on a stable curve X consists of an enhanced level structure $(\Gamma, \ell, \{\kappa_e\})$ on the dual graph Γ of X , a twisted differential of type μ compatible with the enhanced level structure and a prong-matching for each node of X joining components of nonequal levels. Here the *compatibility with the enhanced level structure* requires that at each of the two points q^\pm glued to form the node corresponding to the edge $e \in E(\Gamma)$ the number of prongs of the differential $\boldsymbol{\eta}$ is equal to κ_e . Moreover, a *prong-matching* is an order-reversing isometry $\sigma_q : P_{q^-} \rightarrow P_{q^+}$ that induces a cyclic order-reversing bijection $\sigma_q : P_{q^-}^{\text{in}} \rightarrow P_{q^+}^{\text{out}}$ between the incoming prongs at q^- and the outgoing prongs at q^+ .

Finally, we state the equivalence relation on multi-scale differentials used to construct $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. Multi-scale differentials only retain the information on the lower levels up to projectivisation. This rescaling of the lower levels is roughly given by a multiplicative torus $T^{L(\Gamma)}$. More precisely, the group $(\mathbb{C}^*)^{L(\Gamma)}$ acts by rescaling plainly the differentials on each level. The universal cover $\mathbb{C}^{L(\Gamma)} \rightarrow (\mathbb{C}^*)^{L(\Gamma)}$ acts by rescaling the differentials on each level and simultaneously by fractional Dehn twists on the prong-matching. In fact, a subgroup acts trivially: the twist group Tw_Γ that we describe in detail in Section 3.4. So the action factors through the action of the quotient $T_\Gamma = \mathbb{C}^{L(\Gamma)} / \text{Tw}_\Gamma$ called the *level rotation torus*, and two multi-scale differentials are defined to be equivalent if they differ by the action of T_Γ .

The projectivised space $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ parametrises projectivised multi-scale differentials, where \mathbb{C}^* acts by simultaneously rescaling the differentials on all levels and leaving the prong-matchings untouched.

3.3. Divisors, degeneration, undegeneration

We let $\text{LG}_L(B)$ be the set of all enhanced $(L + 1)$ -level graphs without horizontal edges. Recall that boundary divisors of \overline{B} are D_h and D_Γ for $\Gamma \in \text{LG}_1(B)$. For later use, we define

$$D = D_h + \sum_{\Gamma \in \text{LG}_1(B)} D_\Gamma \tag{18}$$

to be the total boundary divisor. The structure of the normal crossing boundary of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is encoded by *undegenerations*. Given a nonhorizontal level graph Γ with $L + 1$ levels, the associated boundary stratum D_Γ is contained in the intersection of L boundary divisors D_{Γ_i} for $i = 1, \dots, L$, and we can describe this inclusion as follows. View the i th level passage as a horizontal line just above level $-i$. Contract in Γ all edges that do not cross this horizontal line to obtain a contraction map $\delta_i : \Gamma \rightarrow \Gamma_i$ of enhanced level graphs, where Γ_i obtains a two-level structure with the top level corresponding to the components above the horizontal line and the bottom level those below that line. We call this the i th undegeneration of Γ . This can be generalised for any subset $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, L\}$ and results in the undegeneration map

$$\delta_{i_1, \dots, i_n} : \text{LG}_L(B) \rightarrow \text{LG}_n(B),$$

which contracts all the passage levels of a nonhorizontal level graph D_Γ except for the passages between levels $-i_k + 1$ and $-i_k$, for those $i_k \in I$. For notational convenience, we define $\delta_I^{\mathbb{C}} = \delta_{I\mathbb{C}}$.

A *degeneration* of level graphs is simply the inverse of an undegeneration. It is convenient to have a symbol to express this dual process, and we write

$$\Gamma \rightsquigarrow \widehat{\Delta} \quad \text{or} \quad \Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta} \tag{19}$$

for a general undegeneration, respectively, specifically for an undegeneration where the i th level is split into two levels.

Remark 3.2. With the convention used here and in all of the rest, the levels of a level graph with $L + 1$ levels are indexed by negative integers $\{0, -1, \dots, -L\}$, while the level passages are indexed by positive integers $\{1, \dots, L\}$. This implies for example that $\Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta}$ is equivalent to $\Gamma = \delta_{(-i+1)}^{\mathbb{C}}(\widehat{\Delta})$.

Note that the map of graphs δ_I is only well-defined up to post-composition by automorphisms of the enhanced level graph $\delta_I(\Gamma)$. Taking this into account will be important for intersection theory; see Section 8.1 and [CMZ20, Section 7].

3.4. Prong-matchings and their equivalence classes

In this section, we illustrate the amount of combinatorial information encoded in the notion of prong-matching, given that we also have to take into account the action of the level rotation torus. We start with a recurrent example.

Case of a level graph $\Gamma \in \text{LG}_1(\mathbb{B})$: that is, a divisor D_Γ different from D_h . Such an enhanced level graph has $|E(\Gamma)|$ edges, each of which carries the information of the prongs; consequently there are $K_\Gamma = \prod_{e \in E(\Gamma)} \kappa_e$ prong-matchings. However, this does not imply that locally D_Γ is a degree K_Γ -cover of the product of the moduli spaces corresponding to the upper and lower levels. Instead, the effect of projectivisation of the lower level on prong-matchings must be considered. This effect is given by the action of the level rotation group $R_\Gamma \cong \mathbb{Z} \subset \mathbb{C}$ in the universal cover of the level rotation torus. This group R_Γ acts diagonally, turning the prong-matching at each edge by one (in a fixed direction). The stabiliser of a prong-matching is the twist group Tw_Γ referred to above. It is isomorphic to $\ell_\Gamma \mathbb{Z}$ as a subgroup of R_Γ , where

$$\ell_\Gamma = \text{lcm}(\kappa_e : e \in E(\Gamma)). \tag{20}$$

Orbits of R_Γ are also called *equivalence classes of prong-matchings*. For two-level graphs, there are $g_\Gamma := K_\Gamma/\ell_\Gamma$ such equivalence classes.

For a general level graph Δ the situation is more complicated and the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ acquires a nontrivial quotient stack structure that can be computed as follows. As above, there are $K_\Delta = \prod_{e \in E(\Delta)} \kappa_e$ prong-matchings. Now the level rotation group is $R_\Delta \cong \mathbb{Z}^L$, where the i th factor twists by one all prong-matchings that cross the horizontal line above level $-i$. The stabiliser of a prong-matching is still called the twist group Tw_Δ . However, this group is no longer a product of the level-wise factors. In fact, for each $i \in \mathbb{N}$, the twist group of the level-undegeneration $D_{\delta_i(\Delta)}$ is a subgroup of Tw_Δ , and we call the sum of these subgroups the *simple Twist group* Tw_Δ^s . The generic stack structure of D_Δ is given by the product of the action of the group $\text{Aut}(\Delta)$ of enhanced level graphs automorphisms and a cyclic group of order

$$e_\Delta = [\text{Tw}_\Delta : \text{Tw}_\Delta^s].$$

The number of prong-matching equivalence classes is

$$g_\Delta := |R_\Delta - \text{orbits on the set } K_\Delta| = K_\Delta / [R_\Delta : \text{Tw}_\Delta]. \tag{21}$$

These indices can easily be computed using the elementary divisor theorem.

We discuss a simple case below where $\text{Tw}_\Delta \neq \text{Tw}_\Delta^s$ in preparation for the examples in Section 10. Finally, we generalise for later use the lcm defined above. We define $\ell_\Delta = \prod_{i=1}^L \ell_{\Delta,i}$ and we use from now on the notation

$$\ell_{\Delta,i} = \text{lcm}\left(\kappa_e : e \in E(\Gamma)_{\substack{>-i \\ \leq -i}}\right) = \ell_{\delta_i(\Delta)} \tag{22}$$

as an abbreviation of the one defined in the introduction, where $E(\Gamma)_{\substack{>-i \\ \leq -i}}$ are the edges starting at level $-i + 1$ or above and ending at level $-i$ or below.

Example 3.3. In the graph Δ in Figure 1 (left), there are three edges e_1, e_2 and e_3 with enhancements a, b and c . The group $R_\Delta \cong \mathbb{Z}^2$ acts on $\mathbb{Z}/a \oplus \mathbb{Z}/b \oplus \mathbb{Z}/c$ by mapping

$$(1, 0) \mapsto (1, 1, 0) \quad \text{and} \quad (0, 1) \mapsto (0, 1, 1).$$

Consequently, there are $\text{gcd}(a, b, c)$ orbits: that is, that many equivalence classes of prong-matchings near such a boundary point. The index of the twist group Tw_Δ in R_Δ is thus $abc/\text{gcd}(a, b, c)$. On the

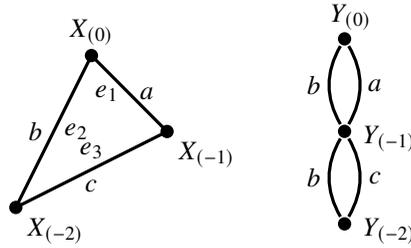


Figure 1. The triangle level graph and a graph with the same undegenerations.

other hand, as a consequence of the discussion in the divisor case, the index of the simple twist group Tw_Δ^s in R_Δ is $ab/\text{gcd}(a, b) \cdot bc/\text{gcd}(b, c)$. Since Δ has no level graphs automorphisms – that is, $\text{Aut}(\Delta)$ is trivial – we conclude that in this case D_Δ is a quotient stack by a group of order

$$e_\Delta = \frac{\text{gcd}(a, b, c) \text{lcm}(a, b) \text{lcm}(b, c)}{abc}. \tag{23}$$

4. Clutching and projection to generalised strata

In this section, we define *generalised strata*, where we allow disconnected surfaces and residues constrained to a residue space \mathfrak{R} . This is similar to a discussion in [Sau19]. More precisely, we show in Section 4.1 how the construction of [BCGGM3] carries over to this generalised context to give a compactification $\mathbb{P}\overline{\Omega\mathcal{M}}_{\mathbf{g}, \mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ of generalised strata.

The reason for dealing with generalised strata is to be able to work with objects (like line bundles and Chow rings) on the individual levels of a boundary stratum, and those might be disconnected and have residue conditions imposed by the GRC. We construct in Section 4.2 for each boundary stratum D_Γ a finite covering $D_\Gamma^s \rightarrow D_\Gamma$ that admits projections $p_\Gamma^{[i]} : D_\Gamma^s \rightarrow B_\Gamma^{[i]}$, where $B_\Gamma^{[i]}$ are the generalised strata at level i of D_Γ .

4.1. The compactification of generalised strata

We start with the definition of strata in the generality that we need. First, we allow for disconnected surfaces. Throughout, $\boldsymbol{\mu}_i = (m_{i,1}, \dots, m_{i,n_i}) \in \mathbb{Z}^{n_i}$ is the type of a differential: that is, we require that $\sum_{j=1}^{n_i} m_{i,j} = 2g_i - 2$ for some $g_i \in \mathbb{Z}$ for $i = 1, \dots, k$. For a tuple $\mathbf{g} = (g_1, \dots, g_k)$ of genera and a tuple $\mathbf{n} = (n_1, \dots, n_k)$ together with $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k)$, we define the disconnected stratum

$$\Omega\mathcal{M}_{\mathbf{g}, \mathbf{n}}(\boldsymbol{\mu}) = \prod_{i=1}^k \Omega\mathcal{M}_{g_i, n_i}(\boldsymbol{\mu}_i). \tag{24}$$

The projectivised stratum $\mathbb{P}\Omega\mathcal{M}_{\mathbf{g}, \mathbf{n}}(\boldsymbol{\mu})$ is the quotient by the diagonal action of \mathbb{C}^* , not the quotient by the action of $(\mathbb{C}^*)^k$.

Next, we prepare for global residue conditions. Let $H_p \subseteq \cup_{i=1}^k \{(i, 1), \dots, (i, n_i)\}$ be the set of marked points such that $m_{i,j} < -1$. Now consider vector spaces \mathfrak{R} of the following special shape, modeled on the global residue condition from [BCGGM1]. Let λ be a partition of H_p with parts denoted by $\lambda^{(k)}$ and a subset $\lambda_{\mathfrak{R}}$ of the parts of λ such that

$$\mathfrak{R} := \left\{ r = (r_{i,j})_{(i,j) \in H_p} \in \mathbb{C}^{H_p} \quad \text{and} \quad \sum_{(i,j) \in \lambda^{(k)}} r_{i,j} = 0 \quad \text{for all} \quad \lambda^{(k)} \in \lambda_{\mathfrak{R}} \right\}.$$

The subspace of surfaces with residues in \mathfrak{R} will be denoted by $\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$, and we will refer to them as *generalised strata*, too.

To compute dimensions, for example, it is convenient to define the *residue subspace*

$$R = \prod_{i=1}^k R_i \subseteq \prod_{i=1}^k \mathbb{C}^{l_i} \tag{25}$$

of differentials of the generalised stratum $\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$, where l_i is the number of negative entries in $\boldsymbol{\mu}_i$. Here R_i is the vector subspace cut out by the residue theorem in the i th component in the space generated by the vectors $r_{i,j}$ for each (i, j) with $m_{i,j} \leq -1$. When writing $\mathfrak{R} \cap R$, we consider the intersection inside $\prod_{i=1}^k \mathbb{C}^{l_i}$.

Remark 4.1. The dimension of the generalised stratum $\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ is

$$\dim(\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})) = \left(\sum_{i=1}^k 2g_i + n_i - 1 \right) - (l - \dim(\mathfrak{R} \cap R)), \tag{26}$$

where $l = \sum l_i$ is the total number of poles: that is, marked points with $m_{i,j} < 0$.

We claim that the construction in [BCGGM3] can be carried out for disconnected surfaces and surfaces with an assigned residue subspace. We only have to replace in the definition of the twisted differentials $(X = (X_v)_{v \in V(G)}, \eta = (\eta_v)_{v \in V(G)})$ compatible with an enhanced level graph Γ the global residue condition by the following condition. We construct a new *auxiliary level graph* $\tilde{\Gamma}$ by adding a new vertex $v_{\lambda^{(k)}}$ to Γ at level ∞ for each element $\lambda^{(k)} \in \lambda_{\mathfrak{R}}$ and converting a tuple $(i, j) \in \lambda^{(k)}$ into an edge from the marked point (i, j) to the vertex $v_{\lambda^{(k)}}$.

- **\mathfrak{R} -global residue condition (\mathfrak{R} -GRC).** The tuple of residues at the poles in H_p belongs to \mathfrak{R} , and for every level $L < \infty$ of $\tilde{\Gamma}$ and every connected component Y of the subgraph $\tilde{\Gamma}_{>L}$, one of the following conditions holds:
 - i) The component Y contains a marked point with a prescribed pole that is *not* in $\lambda_{\mathfrak{R}}$.
 - ii) The component Y contains a marked point with a prescribed pole $(i, j) \in H_p$, and there is an $r \in \mathfrak{R}$ with $r_{(i,j)} \neq 0$.
 - iii) Let q_1, \dots, q_b denote the set of edges where Y intersects $\tilde{\Gamma}_{=L}$. Then

$$\sum_{j=1}^b \text{Res}_{q_j^-} \eta_{v^-(q_j)} = 0,$$

where $v^-(q_j) \in \tilde{\Gamma}_{=L}$.

This differs from the global residue condition in [BCGGM1] only in the subdivision of cases in i) and ii). As for the normal GRC (see [MUW21]), the \mathfrak{R} -GRC also has an algorithmic graph theoretic description; see [CMZ20].

Proposition 4.2. *There is a proper smooth Deligne-Mumford stack $\mathbb{P}\overline{\mathcal{E}\mathcal{M}}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ containing $\mathbb{P}\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ as an open dense substack with the following properties:*

- (i) *The boundary $\mathbb{P}\overline{\mathcal{E}\mathcal{M}}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu}) \setminus \mathbb{P}\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ is a normal crossing divisor.*
- (ii) *A multi-scale differential defines a point in $\mathbb{P}\overline{\mathcal{E}\mathcal{M}}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ if and only if it is compatible with an enhanced level graph Γ that satisfies the \mathfrak{R} -GRC.*
- (iii) *The codimension of a boundary stratum D_{Γ} in $\mathbb{P}\overline{\mathcal{E}\mathcal{M}}_{\mathbf{g},\mathbf{n}}^{\mathfrak{R}}(\boldsymbol{\mu})$ is equal to the number of horizontal edges plus the number L of levels below zero.*

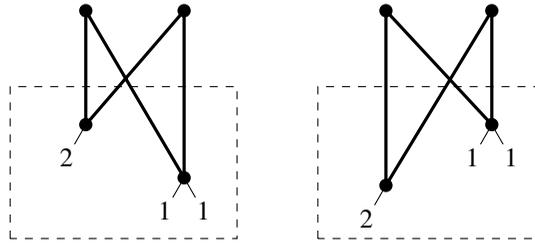


Figure 2. Auxiliary level graphs $\tilde{\Gamma}_1$ (left) and $\tilde{\Gamma}_2$ (right) for the boundary strata Γ_1 and Γ_2 (in the dashed boxes).

Proof. The residue spaces matter only for the existence of modification differentials needed for gluing. In [BCGGM1, Lemma 4.6], their existence for each component Y as in the global residue condition was shown in case iii). This lemma also covers case i) since we can impose a residue at that marked point to ensure that the total sum equals zero. Since \mathfrak{R} is a vector space, this can still be done in case ii).

The smoothness and the normal crossing divisor properties follow from the same reasoning as in [BCGGM3]. We leave the straightforward verification of those many hidden claims of the proposition to the reader. \square

Again, as in the usual situation, the divisors D_Γ of generalised strata may also be disconnected or empty.

Example 4.3. To illustrate the \mathfrak{R} -global residue condition, we consider the generalised stratum $\bar{B} = \mathbb{P}(\Omega\mathcal{M}_{0,3}(-2, -2, 2) \times \Omega\mathcal{M}_{0,4}(-2, -2, 1, 1))^{\mathfrak{R}}$, where the special legs are given by the first two marked points of the first component and the first two marked points of the second components,

$$H_p = \{(1, 1), (2, 1), (1, 2), (2, 2)\},$$

and the residue space is given by the partition

$$\lambda_{\mathfrak{R}} = \{(1, 1), (1, 2)\}, \{(2, 1), (2, 2)\}.$$

This means

$$\mathfrak{R} = \{r_{(1,1)} + r_{(2,1)} = 0, r_{(1,2)} + r_{(2,2)} = 0\} \subset \mathbb{C}^4,$$

and R is the subspace defined by the residue theorem on each of the two components, namely

$$R = \{r_{(1,1)} + r_{(1,2)} = 0, r_{(2,1)} + r_{(2,2)} = 0\}.$$

By Remark 4.1, the above generalised stratum has dimension 1. We want to show that the \mathfrak{R} -GRC implies that there is only one 2-level boundary divisor in the compactification defined in Proposition 4.2. This divisor is given by the 2-level graph with the 4-marked component on level 0 and the other component on level -1 .

The only two possible level graphs that could occur are the 2-level graph Γ_1 described above and the 2-level graph Γ_2 , where the two components are inverted. Consider the auxiliary level graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ needed to check the \mathfrak{R} -GRC given in Figure 2. It is easy to see that condition (iii) of the \mathfrak{R} -GRC implies that the graph Γ_1 is illegal since both residues on the genus 0 component with the single zero of order 2 on the top level are zero, and this cannot happen.

4.2. Level projections and clutching

Consider a boundary stratum D_Γ given by an enhanced level graph Γ . It parametrises multi-scale differentials, a differential on each level together with a prong-matching. However, there are no well-defined projection morphisms to the generalised strata on each level. For example, D_Γ might have generically trivial quotient stack structure, and the generalised strata on its levels might have everywhere trivial stack structure, and yet special points of D_Γ have nontrivial quotient structure. A graph Γ with two edges and two levels degenerating to a triangle (Figure 1, left) provides an example. This is due to the fact that the equivalence relation in the notion of multi-scale differentials involves the twist group, which in the presence of edges across multiple levels intertwines what happens at the levels. Our goal here is to define a cover of D_Γ that has such projection maps.

To define the generalised strata at the levels of D_Γ , we let $(\mathbf{g}^{[i]}, \mathbf{n}^{[i]}, \boldsymbol{\mu}^{[i]})$ for $i = 0, \dots, -L$ be the discrete parameters genus, number of points and type at level i , and we let $\mathfrak{R}^{[i]}$ be the residue condition imposed at level i . These residue conditions are constructed via the \mathfrak{R} -GRC described before. Our goal is:

Proposition 4.4. *There exists a stack D_Γ^s , called the simple boundary stratum of type Γ that admits a finite map $c_\Gamma : D_\Gamma^s \rightarrow D_\Gamma$ and finite forgetful maps*

$$p_\Gamma^{[i]} : D_\Gamma^s \rightarrow B_\Gamma^{[i]} := \mathbb{P}\overline{\mathcal{M}}_{\mathbf{g}^{[i]}, \mathbf{n}^{[i]}}^{\mathfrak{R}^{[i]}}(\boldsymbol{\mu}^{[i]}) \tag{27}$$

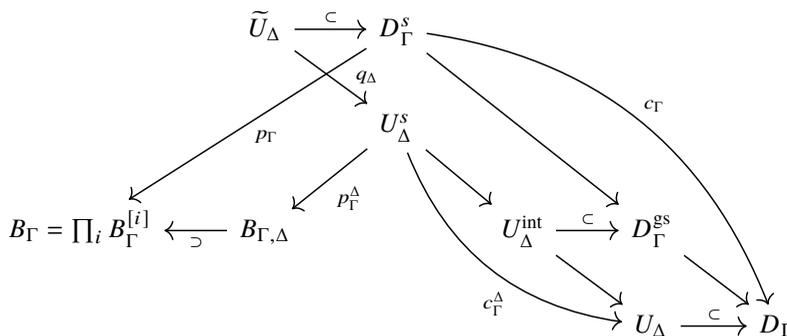
for each $i = 0, \dots, -L$.

We denote by $p_\Gamma = \prod_{i=0}^{-L} p_\Gamma^{[i]}$ the product of all level projections. In the case that D_Γ is a divisor, we will also denote the two projections by

$$p_\Gamma^\top : D_\Gamma^s \rightarrow B_\Gamma^\top = \mathbb{P}\overline{\mathcal{M}}_{\mathbf{g}^\top, \mathbf{n}^\top}^{\mathfrak{R}^\top}(\boldsymbol{\mu}^\top) \quad \text{and} \quad p_\Gamma^\perp : D_\Gamma^s \rightarrow B_\Gamma^\perp = \mathbb{P}\overline{\mathcal{M}}_{\mathbf{g}^\perp, \mathbf{n}^\perp}^{\mathfrak{R}^\perp}(\boldsymbol{\mu}^\perp).$$

With the help of the finite coverings c_Γ and the inclusion of the boundary strata $\mathfrak{i}_\Gamma : D_\Gamma \rightarrow \overline{B}$, we have now the *clutching maps* $\zeta_\Gamma = \mathfrak{i}_\Gamma \circ c_\Gamma$ at our disposal to define the generators of the tautological ring appearing in Theorem 1.5.

The strategy of the proof of the proposition is to construct D_Γ^s as a cover dominating the local covers of neighbourhoods of more degenerate boundary strata, following the strategy already used in [Mum83]. We do not attempt to analyse whether D_Γ^s is smooth, but the covering we construct is branched at worst over the boundary divisors (hence locally over the coordinate axis, since the boundary is normal crossing); so D_Γ^s has at worst Cohen-Macaulay singularities (Proposition 2.2 in [Mum83]), which allows us to perform intersection theory as in [Mum83]. The objects of the construction are summarised in the following diagram that we now explain.



For any level graph Δ that is a degeneration of Γ , we let $U_\Delta \subset D_\Gamma$ be the open subset parametrising multi-scale differentials compatible with an undegeneration of Δ . In particular, $U_\Gamma \subset D_\Gamma$ is the complement of all boundary strata where Γ degenerates further. These U_Δ are covered by open subsets of the projectivised Dehn space [BCGGM3, Section 12], which is given as a set by

$$\mathbb{P}\mathcal{E}D_\Delta = \left(\coprod_{\Gamma \rightsquigarrow \Pi \rightsquigarrow \Delta} (\mathfrak{B}_{\text{pm}}(\Pi)/T_\Pi^s)/\text{Tw}_\Delta \right) / \mathbb{C}^*, \tag{28}$$

where $\mathfrak{B}_{\text{pm}}(\Pi)$ is the space of prong-matched twisted differentials compatible with a Tw_Π^s -marking [BCGGM3, Sections 5 and 8], where the (simple) twist groups are defined in Section 3.4 and $T_\Pi^s = \mathbb{C}^{L(\Pi)}/\text{Tw}_\Pi^s$ is the simple level rotation torus, a finite cover of the level rotation torus defined there. The complex structure of this Dehn space is provided in [BCGGM3, Sections 5 and 8], which at the same time exhibits this space as the quotient stack of the projectivised simple Dehn space

$$\mathbb{P}\mathcal{E}D_\Delta^s = \left(\coprod_{\Gamma \rightsquigarrow \Pi \rightsquigarrow \Delta} (\mathfrak{B}_{\text{pm}}(\Pi)/T_\Pi^s)/\text{Tw}_\Delta^s \right) / \mathbb{C}^*. \tag{29}$$

We define $U_\Delta^s \rightarrow U_\Delta$ to be the cover induced locally by the covering $\mathbb{P}\mathcal{E}D_\Delta^s \rightarrow \mathbb{P}\mathcal{E}D_\Delta$ and moreover by labelling the edges of the ‘ambient’ graph Γ , killing automorphisms that permute these. The first step is well-defined since we work on strata undegenerating Δ , and the passage to the simple twist group defines a cover independently of the chosen auxiliary Teichmüller marking.

Since the edges of Γ are labelled for points in U_Δ^s , and since the equivalence relation in equation (29) is defined level by level, we may decompose the differentials parametrised by U_Δ^s according to the levels of Γ . In this way, we obtain maps $p_\Gamma^{\Delta, [i]} : U_\Delta^s \rightarrow B_\Gamma^{[i]}$ such that the product map $p_\Gamma^\Delta = \prod_i p_\Gamma^{\Delta, [i]}$ is a finite cover of an open subset $B_{\Gamma, \Delta}$ of the product of level strata $B_\Gamma = \prod_i B_\Gamma^{[i]}$.

The last step is to define a covering dominating all the c_Γ^Δ . For technical reasons, we first define the ‘generically simple’ intermediate space D_Γ^{gs} that removes the stack structure over the open subset U_Γ (if there is) just as above, by passage to the simple covering and marking edges. The maps $U_\Delta^s \rightarrow D_\Gamma$ factor through this D_Γ^{gs} , defining an ‘intermediate’ open substack U_Δ^{int} . Finally, we take D_Γ^s to be the normalization of D_Γ^{gs} in a Galois field extension of the function field of D_Γ^{gs} that contains all the extensions defined by $U_\Delta^s \rightarrow D_\Gamma^{\text{gs}}$. (If D_Γ happens to be reducible, we perform the construction on each connected component. Actually, the U_Δ^s still have a stack structure due to automorphisms of the underlying stable curves. The details for how to construct the covering with this caveat are in [Mum83, Section 2b].) This space comes with a forgetful map $c_\Gamma : D_\Gamma^s \rightarrow D_\Gamma$ that factors as $c_\Gamma = c_\Gamma^\Delta \circ q_\Delta : \tilde{U}_\Delta \rightarrow U_\Delta$ over the preimages of U_Δ . We may now define $p_\Gamma^{[i]} = p_\Gamma^{\Delta, [i]} \circ q_\Delta$, since the \tilde{U}_Δ for all degenerations $\Gamma \rightsquigarrow \Delta$ cover D_Γ^s . This completes the *proof of Proposition 4.4*.

4.3. Push-pull comparison

Let $\Gamma \in \text{LG}_L(\bar{B})$ be a level graph. Several recursive computations in the sequel are performed on the level strata $B_\Gamma^{[i]}$, and we want to transfer the result via $p^{[i]}$ -pullback and c_Γ -pushforward to D_Γ . This section provides the basic relations in this push-pull procedure. The degree of c_Γ seems difficult to compute. In applications, we only need the following relative statement.

Lemma 4.5. *The ratio of the degrees of the projections in Proposition 4.4 is*

$$\frac{\text{deg}(p_\Gamma)}{\text{deg}(c_\Gamma)} = \frac{K_\Gamma}{|\text{Aut}(\Gamma)| \ell_\Gamma}. \tag{30}$$

Proof. The degrees can be computed at the generic point, where both maps factor through q_Γ . Using the notation introduced in Section 3.4, we find that the degree of p_Γ is the number of equivalence classes of

prong-matchings, which is $K_\Gamma/[R_\Gamma : Tw_\Gamma]$ by (21). The degree of c_Γ is the index $[Tw_\Gamma : Tw_\Gamma^s] \cdot |\text{Aut}(\Gamma)|$. The claimed equality

$$\frac{\text{deg}(p_\Gamma)}{\text{deg}(c_\Gamma)} = \frac{1}{|\text{Aut}(\Gamma)|} \frac{K_\Gamma}{[R_\Gamma : Tw_\Gamma^s]} = \frac{K_\Gamma}{|\text{Aut}(\Gamma)| \ell_\Gamma} \tag{31}$$

follows from the definition of the simple twist group. □

Next we compare codimension 1 boundary classes on the strata $D_\Gamma \in \text{LG}_L(B)$ and on their level strata $B_\Gamma^{[i]}$ to pull back tautological relations. We use the symbol $[D_\Gamma]$ to denote the fundamental class of the substack of \bar{B} parametrising multi-scale differentials compatible with a degeneration of Γ . Let $i \in \mathbb{Z}_{\leq 0}$.

Consider a graph $\Delta \in \text{LG}_1(B_\Gamma^{[i]})$ defining a divisor in $B_\Gamma^{[i]}$. We aim to compute its pullback to D_Γ^s and the push forward to D_Γ and to \bar{B} . Recall that in D_Γ^s the edges of Γ have been labeled once and for all (we write Γ^\dagger for this labeled graph) and that the level strata $B_\Gamma^{[i]}$ inherit these labels. Consequently, there is a unique graph $\widehat{\Delta}^\dagger$ that is a degeneration of Γ^\dagger and such that extracting the levels i and $i - 1$ of $\widehat{\Delta}^\dagger$ equals Δ . The resulting unlabeled graph will simply be denoted by $\widehat{\Delta}$. (Recall from Remark 3.2 that $\delta_{(-i+1)}^C(\widehat{\Delta}) = \Gamma$.) On the other hand, the procedure of glueing in and forgetting labels is not injective. For a fixed labeled graph Γ^\dagger , we denote by $J(\Gamma^\dagger, \widehat{\Delta})$ the set of $\Delta \in \text{LG}_1(B_\Gamma^{[i]})$ such that $\widehat{\Delta}$ is the result of that procedure. Obviously the graphs in $J(\Gamma^\dagger, \widehat{\Delta})$ differ only by the labeling of their half-edges.

Lemma 4.6. *The cardinality of $J(\Gamma^\dagger, \widehat{\Delta})$ is determined by*

$$|J(\Gamma^\dagger, \widehat{\Delta})| \cdot |\text{Aut}(\widehat{\Delta})| = |\text{Aut}(\Delta)| \cdot |\text{Aut}(\Gamma)|.$$

Proof. Consider the map $\varphi : \text{Aut}(\widehat{\Delta}) \rightarrow \text{Aut}(\Gamma)$ induced by the undegeneration $\delta_{(-i+1)}^C$ of the $(-i + 1)$ th level passage of $\widehat{\Delta}$. For an element in the kernel, the graph Γ is fixed, so we may as well label it. Thanks to these labels, extraction of the levels i and $i - 1$ now defines a graph $\Delta \in \text{LG}_1(B_\Gamma^{[i]})$, and the restriction map $\text{Ker}(\varphi) \rightarrow \text{Aut}(\Delta)$ is an isomorphism. To determine the cokernel of φ , we use the labels given by Γ^\dagger and a degeneration $\widehat{\Delta}^\dagger$ labeled except for the edges interior to that pair of levels. After restriction to the levels i and $i - 1$, the elements in the image of φ act trivially. The resulting bijection of $\text{Coker}(\varphi)$ and $J(\Gamma^\dagger, \widehat{\Delta})$ proves the result. □

We now determine the multiplicities of the push-pull procedure. Recall from equation (22) the definition of $\ell_{\Gamma,j}$ for $j \in \mathbb{Z}_{\geq 1}$.

Proposition 4.7. *For a fixed $\Delta \in \text{LG}_1(B_\Gamma^{[i]})$, the divisor classes of $D_{\widehat{\Delta}}$ and the clutching of D_Δ are related by*

$$\frac{|\text{Aut}(\widehat{\Delta})|}{|\text{Aut}(\Delta)||\text{Aut}(\Gamma)|} \cdot c_\Gamma^*[D_{\widehat{\Delta}}] = \frac{\ell_\Delta}{\ell_{\widehat{\Delta},-i+1}} \cdot p_\Gamma^{[i],*}[D_\Delta] \tag{32}$$

in $\text{CH}^1(D_\Gamma^s)$ and consequently by

$$\frac{|\text{Aut}(\widehat{\Delta})|}{|\text{Aut}(\Gamma)|} \cdot \ell_{\widehat{\Delta},-i+1} \cdot [D_{\widehat{\Delta}}] = \frac{|\text{Aut}(\Delta)|}{\text{deg}(c_\Gamma)} \cdot \ell_\Delta \cdot c_{\Gamma,*}(p_\Gamma^{[i],*}[D_\Delta]) \tag{33}$$

in $\text{CH}^1(D_\Gamma)$.

Proof. It suffices to show the first equation, and then the second follows by taking $c_{\Gamma,*}$. Since the two sides are supported on the same set, it suffices to verify the multiplicities. Since near the divisors under

consideration both sides are pullbacks via $q_{\widehat{\Delta}}$, this can be done by computing the ramification orders of the finite maps $c_{\widehat{\Gamma}}^{\Delta}$ and $p_{\widehat{\Gamma}}^{\Delta}$ over the divisor $D_{\widehat{\Delta}}$ and over $\widetilde{D}_{\Delta} = D_{\Delta} \times \prod_{j \neq i} B_{\Gamma}^{[j]}$, respectively.

We start with $c_{\widehat{\Gamma}}^{\Delta}$. There, passing to the equivalence relation by the torus T_{Γ}^s gives a covering of degree $[\text{Tw}_{\Gamma} : \text{Tw}_{\Gamma}^s]$, both at a generic point and over $D_{\widehat{\Delta}}$. Adding the markings on the edges of Γ gives $|\text{Aut}(\Gamma)|$ additional choices at a generic point. Over $D_{\widehat{\Delta}}$, only the automorphism in the image of the map φ (as in the proof of Lemma 4.6) can be rigidified by adding the marking. This image has cardinality $|\text{Aut}(\widehat{\Delta})|/|\text{Aut}(\Gamma)|$, and thus the ramification order is the reciprocal of the factor on the left-hand side of equation (32).

Next we consider the map $p_{\widehat{\Gamma}}^{\Delta}$. Since in $\prod_j B_{\Gamma}^{[j]}$ and thus also on \widetilde{D}_{Δ} the half-edges that form the edges of Γ are labelled, graph automorphisms do not contribute to branching. However, after adding the prong-matching for Γ , the orbits of the $(-i + 1)$ st component of the integer subgroup $\mathbb{Z}^{L+1} \subset \mathbb{C}^{L+1}$ of the level rotation torus change. In \widetilde{D}_{Δ} (and in D_{Δ}), the orbit has size ℓ_{Δ} , while in D_{Γ}^s , the orbit has size $\ell_{\Delta, -i+1}$ since the prongs of edges of $\widehat{\Delta}$ are acted on, too. Since this component of the level rotation torus is not present at a generic point and since all other components have the same effect at a generic point and over \widetilde{D}_{Δ} , we conclude that the ramification order is the reciprocal of the factor on the right-hand side of equation (32). □

Next we compare various versions of the ξ -class on boundary strata. A first definition is by a local description. Consider a level $i \in \{0, \dots, -L\}$ of a boundary stratum D_{Γ} , and recall that it is a moduli space of multi-scale differentials compatible with a degeneration of Γ . We define the line bundle $\mathcal{O}_{\Gamma}^{[i]}(-1)$ on D_{Γ} as follows. On open sets where Γ does not degenerate further, it is generated by the i th component $\eta_{(i)}$ of the multi-scale differential. If Γ degenerates to Γ_1 , the level i splits up into an interval i to $i - k$ of levels, and then the local generator of $\mathcal{O}_{\Gamma}^{[i]}(-1)$ is the multi-scale component $\eta_{(i)}$ for the top of these levels. We let $\xi_{\Gamma}^{[i]} = c_1(\mathcal{O}_{\Gamma}^{[i]}(-1))$ and write ξ_{Γ}^{\top} for the top-level contribution.

Remark 4.8. Since stable differentials on a boundary stratum are zero on all levels apart from the top one, we have $\xi_{\Gamma}^{\top} = \xi|_{D_{\Gamma}}$.

Proposition 4.9. *The first Chern classes of the tautological bundles on the levels of a boundary divisor are related by*

$$c_{\Gamma}^* \xi_{\Gamma}^{[i]} = p_{\Gamma}^{[i],*} \xi_{B_{\Gamma}^{[i]}} \quad \text{in } \text{CH}^1(D_{\Gamma}^s). \tag{34}$$

Proof. Comparing local generators, we obtain a collection of isomorphisms

$$c_{\Gamma}^{\Delta,*} \mathcal{O}_{B_{\Gamma}^{[i]}(-1)} \cong (p_{\Gamma}^{\Delta,[i]})^* \mathcal{O}_{\Gamma}^{[i]}(-1)$$

compatible with restrictions to undegenerations. The q_{Δ} -pullback of this collection of maps gives the isomorphism on D_{Γ}^s , and then we take the first Chern class. □

We will continue the study of the tautological ring in Sections 7 and 8, using local descriptions near the boundary introduced along with Section 6.

5. The structure of the boundary

In this section, we show that the nonhorizontal boundary divisors D_{Γ} are smooth (as stacks). More generally, we show that if a collection of nonhorizontal divisors intersects, then there is a unique order on this collection such that the i th divisor appear as the i th 2-level undegeneration of an intersection point.

In the sequel, it will be convenient to assume that the 2-level graphs have been numbered once and for all, say as $\text{LG}_1(\mathbf{B}) = \{\Gamma_1, \dots, \Gamma_M\}$. Note that the intersection of two divisors, say D_{Γ_1} and D_{Γ_2} , consists a priori of the sublocus D_{12} of unions of D_{Λ} , for $\Lambda \in \text{LG}_2(\mathbf{B})$ with $\delta_1(\Lambda) = \Gamma_1$ and $\delta_2(\Lambda) = \Gamma_2$,

and the sublocus D_{21} , which is the union of D_Λ for $\Lambda \in \text{LG}_2(B)$ with $\delta_1(\Lambda) = \Gamma_2$ and $\delta_2(\Lambda) = \Gamma_1$. The notation generalises to any number of levels. We define the suborbifold

$$D_{i_1, \dots, i_L} \subseteq \bigcap_{j=1}^L D_{\Gamma_{i_j}} \tag{35}$$

consisting of all D_Λ , with $\Lambda \in \text{LG}_L(B)$ such that $\delta_j(\Lambda) = \Gamma_{i_j}$ for all $j = 1, \dots, L$, and we refer to this by the ordered set $[i_1, \dots, i_L]$, called the *profile* of the boundary stratum. We denote by $\mathcal{P} = \mathcal{P}(B)$ the set of profiles of B and by \mathcal{P}_L those of length L . The language of profiles is used mainly in this section and then again in Theorem 9.10, while elsewhere we usually work with the set of level graphs. The sage package `diffstrata` makes full use of the notion of profiles and the following proposition.

Proposition 5.1. *If $\cap_{j=1}^L D_{\Gamma_{i_j}}$ is not empty, there is a unique ordering $\sigma \in \text{Sym}_L$ on the set $I = \{i_1, \dots, i_L\}$ of indices such that*

$$D_{\sigma(I)} = \bigcap_{j=1}^L D_{\Gamma_{i_j}} .$$

Moreover, if $i_k = i_{k'}$ for a pair of indices $k \neq k'$, then $D_{i_1, \dots, i_L} = \emptyset$.

Remark 5.2. In general, the intersection of boundary divisors $D_{\sigma(I)}$ is not irreducible: that is, it consists of boundary strata associated to different enhanced level graphs; see for example the 3-level graphs in Figure 1.

The preceding proposition also gives a useful relation. Suppose two divisors D_{Γ_1} and D_{Γ_2} meet in a boundary stratum D_Δ . Two situations may occur. Either $\delta_1(\Delta) = \Gamma_1$ and $\delta_2(\Delta) = \Gamma_2$ or vice versa. In the first situation, Δ arises from degenerating the lower level of Γ_1 . We phrase this by saying that Γ_2 goes under Γ_1 and write $\Gamma_2 < \Gamma_1$. A priori, this notion might depend on the enhanced level graph Δ . But the preceding proposition implies that it does in fact not depend on Δ .

The proof of Proposition 5.1 uses dimension estimates and the following lemma. We define

$$d_\Lambda^{[p]} = \dim(B_\Lambda^{[p]}) \text{ for all } \Lambda \in \text{LG}_L(B) ,$$

where $B_\Lambda^{[p]}$ is the projectivised substratum at level $p \in \{0, \dots, -L\}$ of D_Λ defined in Proposition 4.4. Note that the sum $\sum_{p=0}^{-L} (d_\Lambda^{[p]} + 1) = N = 1 + \dim(\overline{B})$ is the unprojectivised dimension of the total stratum.

Lemma 5.3. *The dimensions of the levels of a boundary stratum D_Λ and the boundary divisor $D_{\delta_k(\Lambda)}$ given by its k th undegeneration are related by*

$$d_{\delta_k(\Lambda)}^{[0]} = k - 1 + \sum_{p=0}^{k-1} d_\Lambda^{[-p]} , \quad d_{\delta_k(\Lambda)}^{[-1]} = L - 1 - k + \sum_{p=k}^{L-1} d_\Lambda^{[-p]} .$$

Proof. This follows directly from the description of undegeneration; see [BCGGM3]. □

Proof of Proposition 5.1. Assume that, after reordering, $\cap_{i=1}^L D_i$ is not empty and D_Λ is a component of $D_{1, \dots, L}$. Assume furthermore that there is a permutation $\sigma \in S_j$ such that $D_{\sigma(1), \dots, \sigma(L)}$ also is nonempty, containing a component D_{Λ^σ} . Now, by definition,

$$\delta_1(\Lambda^\sigma) = \delta_{\sigma(1)}(\Lambda) = D_{\Gamma_{\sigma(1)}} .$$

By Lemma 5.3, we can then write the dimension of the top component of D_{Γ_1} and $D_{\Gamma_{\sigma(1)}}$ in two different ways, namely

$$d_{\Gamma_1}^{[0]} = d_{\Lambda}^{[0]} = \sigma^{-1}(1) - 1 + \sum_{p=0}^{\sigma^{-1}(1)-1} d_{\Lambda^{\sigma}}^{[-p]}$$

$$d_{\Gamma_{\sigma(1)}}^{[0]} = d_{\Lambda^{\sigma}}^{[0]} = \sigma(1) - 1 + \sum_{p=0}^{\sigma(1)-1} d_{\Lambda}^{[-p]}.$$

By substituting the first expression into the second one, we obtain

$$d_{\Lambda^{\sigma}}^{[0]} = \sigma(1) - 1 + \sigma^{-1}(1) - 1 + \sum_{p=0}^{\sigma^{-1}(1)-1} d_{\Lambda^{\sigma}}^{[-p]} + \sum_{p=1}^{\sigma(1)-1} d_{\Lambda}^{[-p]},$$

which simplifies to

$$0 = \sigma(1) - 1 + \sigma^{-1}(1) - 1 + \sum_{p=1}^{\sigma^{-1}(1)-1} d_{\Lambda^{\sigma}}^{[-p]} + \sum_{p=1}^{\sigma(1)-1} d_{\Lambda}^{[-p]}.$$

This implies that $\sigma(1) = 1$. By induction, we get that $\sigma = \text{id}$.

To prove the second statement, assume by contradiction that the orbifold D_{i_1, \dots, i_L} is nonempty, with $i_1 = i_k$ for $1 < k \leq L$. Let D_{Λ} be a component of D_{i_1, \dots, i_L} . Then by Lemma 5.3, we get

$$d_{\delta_1(\Lambda)}^{[0]} = d_{\Lambda}^{[0]} = k - 1 + \sum_{p=0}^{k-1} d_{\Lambda}^{[-p]}.$$

This implies that $k = 1$, which is already a contradiction. □

6. Euler sequence for strata of Abelian differentials

The characteristic classes of the tangent bundle to the projective space $\mathbb{P}(V)$ of a vector space V are conveniently computed using the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(V)}^1 \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(-1)^{\oplus \dim(V)} \xrightarrow{\text{ev}} \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0. \tag{36}$$

Our main computational tool uses the affine structure of strata to provide a similar Euler sequence on the compactified strata $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.

Theorem 6.1. *There is a vector bundle \mathcal{K} on \overline{B} that fits into an exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow (\overline{\mathcal{H}}_{rel}^1)^{\vee} \otimes \mathcal{O}_{\overline{B}}(-1) \xrightarrow{\text{ev}} \mathcal{O}_{\overline{B}} \longrightarrow 0, \tag{37}$$

where $\overline{\mathcal{H}}_{rel}^1$ is the Deligne extension of the relative cohomology such that the restriction of \mathcal{K} to the interior B is the cotangent bundle Ω_B^1 .

An explicit description of local generators of \mathcal{K} is part of the proof in this section. We will set up the tools to describe \mathcal{K} intrinsically in Theorem 9.2.

We will define the evaluation map ev in the course of the construction. The construction happens first over the open part and then the finite covering charts that exhibit $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ locally as a quotient stack.

6.1. Over the open stratum

Recall that moduli spaces of Abelian differentials have an affine structure given by period coordinates. Concretely, for a pointed flat surface (X, ω, \mathbf{z}) , we denote by $Z = \{z_1, \dots, z_{r+s}\}$ the zeros and by $P = \{z_{r+s+1}, \dots, z_n\}$ the poles among the marked points, thus including marked ordinary points in Z . By [HM79] or [Vee86] (see also [BCGGM2]), integration of the one-form along relative homology classes is a local biholomorphism and thus provides local charts of $\Omega\mathcal{M}_{g,n}(\mu)$ in the vector space

$$V = V_{(X, \omega, \mathbf{z})} := H^1(X \setminus P, Z; \mathbb{C}).$$

The changes in charts are linear with \mathbb{Z} -coefficients. This makes the projectivisation B into a $(\text{PGL}_N, \mathbb{P}^{N-1})$ -manifold.

We denote by $\mathcal{H}_{\text{rel}}^1$ the local system on B with fibre the relative cohomology $V = H^1(X \setminus P, Z; \mathbb{C})$ and recall that $N = \dim(V) = \dim(B) + 1$. Recall that the fibre of $\mathcal{O}_B(-1)$ at the point (X, ω, \mathbf{z}) is the complex one-dimensional vector space generated by ω . We thus obtain the evaluation map

$$\text{ev}: (\mathcal{H}_{\text{rel}}^1)^\vee \otimes \mathcal{O}_B(-1) \rightarrow \mathcal{O}_B, \quad \gamma \otimes \omega \mapsto \int_\gamma \omega$$

by integrating the one-form.

Proposition 6.2. *There is a short exact sequence of vector bundles on B*

$$0 \rightarrow \Omega_B^1 \rightarrow (\mathcal{H}_{\text{rel}}^1)^\vee \otimes \mathcal{O}_B(-1) \xrightarrow{\text{ev}} \mathcal{O}_B \rightarrow 0$$

that locally on a chart $\mathbb{P}V$ is given by the standard Euler sequence.

Proof. Let $\pi: \tilde{B} \rightarrow B$ be the universal cover of B . Consider the developing map $\text{dev}: \tilde{B} \rightarrow \mathbb{P}(V)$, which is a $\pi_1(B)$ -equivariant local isomorphism. We use the sequence on the standard charts of $\mathbb{P}(V)$, and we claim that its dev-pullback descends to an exact sequence B .

To justify this, consider paths $\{\alpha_i\}_{i=1}^N$ that form a local frame of $(\mathcal{H}_{\text{rel}}^1)^\vee$. Let $\{a_i\}_{i=1}^N$ be the corresponding local coordinates and $\{da_i\}$ the local frame of $\Omega_{\mathbb{P}(V)}^1$. On the open subset $U_k = \{a_k \neq 0\} \subseteq \mathbb{P}(V)$, the monomorphism of the Euler sequence in equation (36) is given by

$$da_i \mapsto \left(\alpha_i - \frac{a_i}{a_k} \alpha_k \right) \otimes \omega, \quad i = 1, \dots, \hat{k}, \dots, N, \tag{38}$$

where ω is the representative of the line bundle with $\int_{\alpha_k} \omega = 1$. The pullback sequence gives rise to an isomorphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{dev}^* \left(\Omega_{\mathbb{P}(V)}^1 \right) & \longrightarrow & \text{dev}^* (V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)) & \xrightarrow{\text{ev}} & \text{dev}^* (\mathcal{O}_{\mathbb{P}(V)}) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \pi^* (\Omega_B^1) & \longrightarrow & \pi^* (\mathcal{H}_{\text{rel}}^1)^\vee \otimes \pi^* (\mathcal{O}_B(-1)) & \xrightarrow{\text{ev}} & \pi^* (\mathcal{O}_B) \longrightarrow 0 \end{array}$$

Each vector bundle appearing is provided with a canonical $\pi_1(B)$ action, and the vertical maps are isomorphisms of $\pi_1(B)$ -vector bundles. The first vertical map is an isomorphism since the developing map is a local isomorphism and $\pi^* (\Omega_B^i) \cong \Omega_B^i$ for every i . Since the evaluation map is $\pi_1(B)$ -equivariant, so is the kernel. Hence the short exact sequence passes to the quotient by the action of $\pi_1(B)$ and yields the claim. □

6.2. Coordinates near the boundary

Coordinates near the boundary of the moduli space $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are *perturbed period coordinates* ([BCGGM3, Section 11] or [CMZ19, Section 3]) that we now illustrate in typical cases that exhibit all the relevant features. The reader is encouraged to read this subsection in parallel with the subsequent one, where the Euler sequence is extended step by step to these boundary strata.

Case 1: only horizontal nodes

Suppose that the level graph Γ consists of $k \geq 1$ horizontal edges only, all of which must necessarily be nonseparating. At a smooth point near D_Γ , the relative homology can be grouped into

- the vanishing cycles α_i for $i = 1, \dots, k$ around the nodes,
- loops β_i symplectically dual to α_i , and
- paths $\gamma_1, \dots, \gamma_{N-2k}$ completing the above to a basis of relative homology.

Coordinates in a chart of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ near D_Γ are given by the periods $c_i = \int_{\gamma_i} \omega$, by $a_i = \int_{\alpha_i} \omega$ and by the exponentiated period ratio $q_i = \exp(2\pi i b_i / a_i)$, where $b_i = \int_{\beta_i} \omega$. To provide charts of the projectivisation \overline{B} , we fix a_1 to be identically one.

Case 2: two levels, only vertical nodes

For concreteness, we suppose that in the 2-level graph $\Gamma \in \text{LG}_1(B)$ there is only one vertex on each level and for concreteness, say, with three edges e_1, e_2, e_3 joining the two vertices. Suppose moreover that there is no marked zero on the lower level. (If there is such a marked point on each level, the loops β_i below have to be replaced by relative periods across the level, leading to similar constructions.) At a point close to D_Γ , the relative homology can be grouped into

- loops β_1 through e_1 and e_3 and β_2 through e_2 and e_3 ,
- loops α_1 and α_2 , the vanishing cycles corresponding to e_1 and e_2 ,
- paths $\gamma_1^{[0]}, \dots, \gamma_{d_0}^{[0]}$ forming a basis of the relative homology on the top level,
- loops $\gamma_1^{[-1]}, \dots, \gamma_{d_1}^{[-1]}$ forming a basis of the homology on the bottom level,

for some $d_0, d_1 \in \mathbb{Z}$; see also Figure 3.

On the other hand, the surfaces on the boundary stratum D_Γ have a basis of relative homology that can be grouped into

- relative periods $\tilde{\beta}_i$ joining the marked points at the upper ends of the edge e_i to the upper end of the edge e_3 for $i = 1, 2$,
- loops $\tilde{\alpha}_i$ around the poles at the lower ends of e_i for $i = 1, 2$,

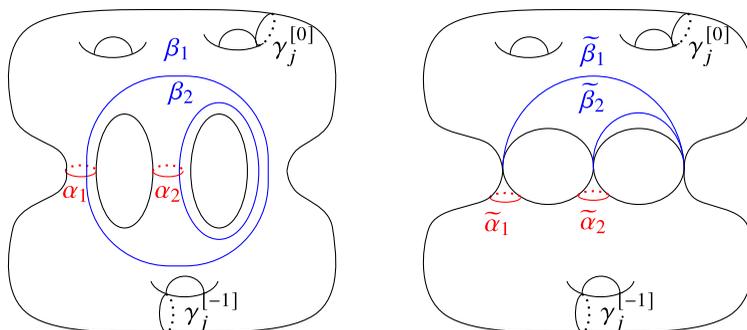


Figure 3. Cycles in Case 2, near the boundary stratum and at the boundary stratum.

- paths $\tilde{\gamma}_1^{[0]}, \dots, \tilde{\gamma}_{d_0}^{[0]}$ forming a basis of the relative homology on the top level,
- loops $\tilde{\gamma}_1^{[-1]}, \dots, \tilde{\gamma}_{d_1}^{[-1]}$ forming a basis of the homology on the bottom level.

From this description it is apparent that d_i is related to the projectivised and unprojectivised dimensions of the level strata previously introduced by $d_i = N_\Gamma^{[i]} - 2 = d_\Gamma^{[i]} - 1$.

The main statement about perturbed period coordinates [BCGGM3, Section 11] is that on the one hand, coordinates near the boundary are given by the periods on the boundary surfaces and on the other hand, periods with and without tildes are nearly the same after appropriate rescaling. To make this statement concrete, let κ_i be the enhancements corresponding to the edges e_i , and let $\ell = \text{lcm}(\kappa_1, \kappa_2, \kappa_3)$. Near our current boundary divisor D_Γ , the universal family of curves has a (universal) family of differentials ω and ℓ is chosen so that rescaling $\eta_{(-1)} = t^{-\ell} \omega_{(-1)}$ is holomorphic and generically nonzero for a coordinate with $D_\Gamma = \{t = 0\}$ locally ([BCGGM3, Section 12]). At each point $p \in D_\Gamma$, we find a nonzero η -period on the lower level, say the period along $\tilde{\gamma}_1^{[-1]}$, and choose t and thus η so that $\int_{\tilde{\gamma}_1^{[-1]}} \eta = 1$.

A chart of $\Xi \overline{\mathcal{M}}_{g,n}(\mu)$ near p is then nearly the product of a neighbourhood of the irreducible components (X_0, ω) and (X_1, η) of the fibre over p in their respective strata of meromorphic differentials. Here, ‘nearly’ refers to the fact that, because of prong-matchings; it is an ℓ -fold cover fully ramified over $t = 0$, and moreover, because of enhanced level graph automorphisms, it is a quotient stack by the subgroup G of S_3 that exchanges edges with the same enhancement.

Coordinates on this chart are then given by t and the periods

$$\begin{aligned} \tilde{b}_i &= \int_{\tilde{\beta}_i} \eta_{(0)} \quad (i = 1, 2), & r_i &= \int_{\tilde{\alpha}_i} \eta_{(-1)} \quad (i = 1, 2), \\ \tilde{c}_i^{[0]} &= \int_{\tilde{\gamma}_i^{[0]}} \eta_{(0)} \quad (i = 1, \dots, d_0), & \tilde{c}_i^{[-1]} &= \int_{\tilde{\gamma}_i^{[-1]}} \eta_{(-1)} \quad (i = 2, \dots, d_1). \end{aligned}$$

To provide charts of the projectivisation \overline{B} , we simply fix one of the periods on the top level, say $\tilde{c}_1^{[0]}$, to be identically one. (If $d_0 = 0$, we take $\tilde{b}_1 \equiv 1$ instead.)

In each sector near the boundary, the perturbed period coordinates are related to the ω -periods by

$$\begin{aligned} b_i &:= \int_{\beta_i} \omega \sim \tilde{b}_i & a_i &:= \int_{\alpha_i} \omega = t^\ell r_i \\ c_i^{[0]} &:= \int_{\gamma_i^{[0]}} \omega \sim \tilde{c}_i^{[0]}, & c_i^{[-1]} &:= \int_{\gamma_i^{[-1]}} \omega = t^\ell \tilde{c}_i^{[-1]}, \end{aligned} \tag{39}$$

where \sim indicates that the difference is $O(t^\ell)$. The difference stems (for $c_i^{[0]}$) from the fact that the ω in the universal family is not just the deformation of the twisted differential $(\eta_{(0)}, \eta_{(-1)})$ in the fibre over p in its product moduli space, but blurred by some modification differentials. For the b_i , there is an additional error term in the same order of magnitude due to a choice of a nearby base point in the plumbing construction.

Case 3: two levels, additional horizontal nodes

This is a mixture of the previous two cases. To see the effects, we assume that we are in the situation of Case 2, with one horizontal node and thus additionally a pair of cycles $\alpha^{[j]}$ and $\beta^{[j]}$ with $j = 0$ or $j = -1$ depending on the level where the horizontal node is attached. We may then uniformly write the periods $a^{[j]} = \int_{\alpha^{[j]}} \eta_{(j)}$ and $b^{[j]} = \int_{\beta^{[j]}} \eta_{(j)}$. The additional coordinates are $a^{[j]}$ and the exponentiated period ratio $q^{[j]} = \exp(2\pi i b^{[j]} / a^{[j]})$.

Case 4: three levels, three nodes

This is the generalisation of the triangle case (Figure 1 left), where each edge can possibly be replaced by multiple edges: say, k_i edges instead of a single edge e_i . We include the case $k_i = 0$ (e.g., if e_3 is the long edge in the triangle case, then Figure 1 right represents a situation with $k_3 = 0$). Let ℓ_1 be the lcm of the enhancements on the edges starting at level 0 and ℓ_2 the lcm of the edges ending at level -2 , as defined in equation (22).

A point $p \in D_\Gamma$ on the corresponding divisor is given by meromorphic differential forms $(X_{(0)}, \eta_{(0)})$, $(X_{(-1)}, \eta_{(-1)})$, $(X_{(-2)}, \eta_{(-2)})$ together with prong-matchings. We denote by $\tilde{\gamma}_i^{[j]}$ for $j = 0, -1, -2$ and $i = 1, \dots, N_j$ paths of the relative homology of the surfaces. (There are no global residue conditions in this example.) We may suppose that $\int_{\tilde{\gamma}_1^{[j]}} \eta_{(j)} = 1$ for $j = 0, -1, -2$ to fix the scale of the $\eta_{(j)}$ on the lower level and for $j = 0$ to fix an open subset of the projectivisation.

A chart of $\Xi \overline{\mathcal{M}}_{g,n}(\mu)$ near p is then nearly the product of a neighbourhood of the irreducible components $(X_{(j)}, \eta_{(j)})$, where $j = 0, -1, -2$ of the fibre over p in their respective strata of meromorphic differentials. Slightly abusing notation, we also call the universal differentials over these neighbourhoods $\eta_{(j)}$. A coordinate system for the neighbourhood of $p \in \overline{B}$ is given by functions t_1 and t_2 that correspond to rescalings of the two levels together with the functions $\tilde{c}_i^{[j]} = \int_{\tilde{\gamma}_i^{[j]}} \eta_{(j)}$ for $j = 0, -1, -2$ and $i = 2, \dots, N_j$. In particular, $N_0 + N_{-1} + N_{-2} = N$.

To give the relation of these coordinates to nearby periods, note that the universal differential ω over $\Xi \overline{\mathcal{M}}_{g,n}(\mu)$ has by construction the property that the periods of ω on the bottom level agree with those of $t_1^{\ell_1} t_2^{\ell_2} \eta_{(-2)}$, the periods on level -1 of ω differ from those of $t_1^{\ell_1} \eta_{(-1)}$ by functions that decay like $t_1^{\ell_1} t_2^{\ell_2}$ and the periods on the top level of ω differ from those of $\eta_{(0)}$ by functions that decay like $t_1^{\ell_1}$. Here, as we have illustrated in Case 2, the loops around the nodes corresponding to the $k_1 + k_2 + k_3$ edges can be treated as residues and thus as periods on the level at the lower end of the edge, while the loops through those edges (denoted previously by β_i) can be treated as relative periods on the highest level that the loop touches.

6.3. The Euler sequence on the Deligne extension

Recall that the Deligne extension of a local system on B is a canonical extension to a vector bundle on \overline{B} admitting an extension of the Gauss-Manin connection to a connection with regular singular points ([Del70]). In this section, we want to extend the Euler sequence across the boundary to construct equation (37). For this purpose, we exhibit local generators of the Deligne extension $\overline{\mathcal{H}}_{\text{rel}}^1$ of $\mathcal{H}_{\text{rel}}^1$, extend the map ev and determine its kernel in each of the cases as we discussed perturbed period coordinates in Section 6.2, adopting notation from there.

Case 1: only horizontal nodes

A basis of $(\overline{\mathcal{H}}_{\text{rel}}^1)^\vee$ consists of the cycles $\alpha_1, \dots, \alpha_k$ and $\gamma_1, \dots, \gamma_{N-2k}$ that extend across D_Γ , together with the linear combinations

$$\widehat{\beta}_i = \beta_i - \frac{1}{2\pi i} \log(q_i) \alpha_i$$

designed to be monodromy invariant. Since the family of one-forms ω extends across D_Γ to a family of stable differentials, the definition

$$\text{ev}(\widehat{\beta}_i \otimes \omega) = \int_{\beta_i} \omega - \frac{1}{2\pi i} \log(q_i) \int_{\alpha_i} \omega = b_i - \frac{1}{2\pi i} \log(q_i) a_i = 0$$

extends the definition of ev in the interior and gives a well-defined holomorphic function. To check the surjectivity of ev , we can use any of the periods that extend across D_Γ . We claim that the kernel of ev is on the chart U with $a_1 \equiv 1$

$$\mathcal{K} = \langle dq_1/q_1, da_2, dq_2/q_2, \dots, da_k, dq_k/q_k, dc_1, \dots, dc_{N-2k} \rangle \tag{40}$$

as \mathcal{O}_U -module. In fact, using the definition in equation (38) in the interior, one checks that

$$dq_i/q_i = d \log(q_i) = d \left(2\pi i \frac{b_i}{a_i} \right) \mapsto \frac{2\pi i}{a_i} \left(\beta_i - \frac{b_i}{a_i} \alpha_i \right) \otimes \omega \tag{41}$$

is mapped to a local generator of $(\overline{\mathcal{H}}_{rel}^1)^\vee \otimes \mathcal{O}_{\overline{B}}(-1)$ since the functions a_i do not vanish near such a boundary point. Moreover, dq_i/q_i is mapped to the kernel of ev by the preceding calculation. For the other elements, these claims follow as in the interior.

Case 2: two levels, only vertical nodes

We first work in the special case near a boundary divisor D_Γ , where Γ has three edges as in the case discussed in Section 6.2. A basis of $(\overline{\mathcal{H}}_{rel}^1)^\vee$ consists of the cycles $\alpha_1, \alpha_2, \gamma_1^{[0]}, \dots, \gamma_{d_0}^{[0]}, \gamma_1^{[-1]}, \dots, \gamma_{d_1}^{[-1]}$ that extend across D_Γ , together with the linear combinations

$$\widehat{\beta}_i = \beta_i - \log(t)(\alpha_i + (\alpha_1 + \alpha_2)) \quad (i = 1, 2) \tag{42}$$

that are monodromy invariant since turning once around the divisor acts by simultaneous Dehn-twists around the core curves of the three plumbing cylinders. Sending cycles that extend across D_Γ to their ω -integrals and letting

$$ev(\widehat{\beta}_i \otimes \omega) = \int_{\widehat{\beta}_i} \omega - \log(t) \left(\int_{\alpha_i} \omega + \int_{\alpha_1 + \alpha_2} \omega \right) \quad (i = 1, 2)$$

extends the definition of ev in the interior and is well-defined since the function

$$\log(t) \left(2 \int_{\alpha_i} \omega + \int_{\alpha_1 + \alpha_2} \omega \right) = O(t^\ell \log(t))$$

is bounded near D_Γ and $\int_{\beta_i} \omega \rightarrow \int_{\widehat{\beta}_i} \omega$ is bounded as well.

Obviously, on the chart with $c_1^{[0]} = 1$, the kernel of ev is

$$\begin{aligned} \text{Ker}(ev) = \langle & \gamma_i^{[0]} - c_i^{[0]} \gamma_1^{[0]} \quad (i = 2, \dots, d_0); \quad \alpha_i - a_i \gamma_1^{[0]} \quad (i = 1, 2); \\ & \gamma_i^{[-1]} - c_i^{[-1]} \gamma_1^{[0]} \quad (i = 1, \dots, d_1); \quad \widehat{\beta}_i - \widehat{b}_i \gamma_1^{[0]} \quad (i = 1, 2) \rangle, \end{aligned} \tag{43}$$

where \widehat{b}_i is the integral of $\widehat{\beta}_i$. We claim that via the identification of periods in equation (39), this kernel is precisely the image of

$$\mathcal{K} = \langle d\widehat{c}_2^{[0]}, \dots, d\widehat{c}_{d_0}^{[0]}, d\widehat{b}_1, d\widehat{b}_2, t^\ell dt/t, t^\ell d\widehat{c}_2^{[-1]}, \dots, t^\ell d\widehat{c}_{d_1}^{[-1]}, t^\ell dr_1, t^\ell dr_2, \rangle$$

under the map in equation (38). First, since we used the coordinate $\widehat{c}_1^{[1]}$ to fix the scaling on the bottom level, the differential form $t^\ell dt/t = dc_1^{[-1]}$ is mapped to $\gamma_1^{[-1]} - c_1^{[-1]} \gamma_1^{[0]}$. Then from equation (39), we see that $t^\ell d\widehat{c}_i^{[-1]}$ is mapped to a linear combination of $\gamma_i^{[-1]} - c_i^{[-1]} \gamma_1^{[0]}$ and the previous generator for any $i \geq 2$. Similarly, $t^\ell dr_i$ maps to $\alpha_i - a_i \gamma_1^{[0]}$ and a linear combination of the previous generators. In the second step, we consider the generators that correspond to top level. The form $d\widehat{c}_i^{[0]}$ does not

quite map to $\gamma_i^{[0]} - c_i^{[0]}\gamma_1^{[0]}$ because of the presence of modification differentials, but the difference is a linear combination of the differential of some η -periods that we have shown already in the first step to belong to $\text{Ker}(ev)$. Similarly, the images of $d\widehat{b}_i$ and $\widehat{\beta}_i - \widehat{b}_i\gamma_1^{[0]}$ are differentials of periods supported on the lower level (from equation (39) to compare with db_i and from equation (42)).

We now rename and regroup the generators of \mathcal{K} in a form that generalises to other level graphs. Since the β -periods become relative periods, and since the α -periods for the edges joining the levels are simply residues appearing on a lower level, we may name the set of all periods on the top level by $\widetilde{c}_i^{[0]}$ for $1 \leq i \leq N_0$ and those on the bottom level by $\widetilde{c}_i^{[-1]}$ for $1 \leq i \leq N_1$. Then the above argument gives that

$$\mathcal{K} = \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]}, t^\ell dt/t, t^\ell d\widetilde{c}_2^{[-1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[-1]} \rangle. \tag{44}$$

Case 3: two levels, additional horizontal nodes

We mix the conclusion of the two previous cases. If the horizontal node is at the top level, then

$$\mathcal{K} = \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]}, da, dq/q, t^\ell dt/t, t^\ell d\widetilde{c}_2^{[1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[1]} \rangle, \tag{45}$$

while in the case of a horizontal node at the bottom level

$$\mathcal{K} = \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]}, t^\ell dt/t, t^\ell d\widetilde{c}_2^{[-1]}, \dots, t^\ell d\widetilde{c}_{N_1}^{[-1]}, t^\ell da, t^\ell dq/q \rangle. \tag{46}$$

Case 4: three levels, three nodes

Using the same arguments as in Case 2, we can show that the α_j -periods corresponding to the edges joining different levels become residues and that the monodromy-invariant modifications $\widehat{\beta}_j$ of the dual β_j -periods have ev -images that tend to the β_j -integrals. We claim that thus $\text{Ker}(ev)$ is the image of

$$\mathcal{K} = \langle d\widetilde{c}_2^{[0]}, \dots, d\widetilde{c}_{N_0}^{[0]}, t_1^{\ell_1} dt_1/t_1, t_1^{\ell_1} d\widetilde{c}_2^{[-1]}, \dots, t_1^{\ell_1} d\widetilde{c}_{N_1}^{[-1]}, t_1^{\ell_1} t_2^{\ell_2} dt_2/t_2, t_1^{\ell_1} t_2^{\ell_2} d\widetilde{c}_2^{[-2]}, \dots, t_1^{\ell_1} t_2^{\ell_2} d\widetilde{c}_{N_2}^{[-2]} \rangle \tag{47}$$

under the map in equation (38). We justify this, starting at the bottom level. The differential form

$$d(\widetilde{c}_1^{[-2]}) = d(t_1^{\ell_1} t_2^{\ell_2}) = \ell_2 t_1^{\ell_1} t_2^{\ell_2} dt_2/t_2 + \ell_1 t_1^{\ell_1} t_2^{\ell_2} dt_1/t_1 \in \mathcal{K},$$

since it is mapped to $\gamma_1^{[-2]} - c_1^{[-2]}\gamma_1^{[0]}$, which in analogy with equation (43) belongs to the natural basis of $\text{Ker}(ev)$. Next, the form $dt_1^{\ell_1} t_2^{\ell_2} d\widetilde{c}_i^{[-2]}$ map to a linear combination of the elements $\gamma_i^{[-2]} - c_i^{[-2]}\gamma_1^{[0]}$ in the natural basis of $\text{Ker}(ev)$ and the previous generator.

We next proceed to the middle level. There, the form $\ell_1 t_1^{\ell_1} dt_1/t_1$ is not quite equal to $d(c_1^{[-1]})$ because of the presence of modification differentials. It thus does not quite map to the basis element $\gamma_i^{[-1]} - c_i^{[-1]}\gamma_1^{[0]}$ of $\text{Ker}(ev)$. But the difference is a combination of elements that we have already shown to belong to \mathcal{K} . As a combination of this form and $d(\widetilde{c}_1^{[-2]})$, we now have $\ell_2 t_1^{\ell_1} t_2^{\ell_2} dt_2/t_2 \in \mathcal{K}$. Considering the remaining form $dc_i^{[-1]}$ from periods on the middle level and then all the form $dc_i^{[0]}$ for $i \geq 2$ on the top level identifies the remaining elements listed in \mathcal{K} with elements of $\text{Ker}(ev)$, up to the effect of modification differentials, which produce differentials of periods already shown to belong to \mathcal{K} .

The notation

$$t_{[j]} = \prod_{i=1}^j t_i^{\ell_i}, \quad j \in \mathbb{N} \tag{48}$$

will be convenient here and in the sequel.

Proof of Theorem 6.1. Continuing the argument as in the preceding cases, we see that near a point $p \in D_\Gamma$, the elements

- $t_{[j]} dt_j / t_j$, for every level $-j$,
- the $t_{[j]}$ -multiples of differential forms associated to periods on level $-j$,
- $t_{[j]} dq_k^{[-j]} / q_k^{[-j]}$ for every horizontal node with parameter q_k on level $-j$

freely generate \mathcal{K} . □

7. The normal bundle to boundary strata

In this section, we provide formulas to compute the first Chern class of the normal bundle $\mathcal{N}_\Gamma = \mathcal{N}_{D_\Gamma}$ to a boundary divisor D_Γ . We will frequently encounter here and in the sequel the top level correction line bundle

$$\mathcal{L}_\Gamma^\top = \mathcal{O}_{D_\Gamma} \left(\sum_{\substack{\widehat{\Delta} \in \text{LG}_2(\overline{B}) \\ \delta_2(\widehat{\Delta}) = \Gamma}} \ell_{\widehat{\Delta}, 1} D_{\widehat{\Delta}} \right) \tag{49}$$

on D_Γ that records all the degenerations of the top level of Γ .

Theorem 7.1. *Suppose that D_Γ is a divisor in \overline{B} corresponding to a graph $\Gamma \in \text{LG}_1(\overline{B})$. Then*

$$c_1(\mathcal{N}_\Gamma) = \frac{1}{\ell_\Gamma} \left(-\xi_\Gamma^\top - c_1(\mathcal{L}_\Gamma^\top) + \xi_\Gamma^\perp \right) \text{ in } \text{CH}^1(D_\Gamma). \tag{50}$$

In case the graph Γ contains an edge e (which is automatic if the ambient stratum parametrises connected curves but is often not satisfied in the generalisation to higher codimension strata below), there is an alternative expression for the Chern class of the normal bundle that gives a comparison to the situation in the moduli space of curves. Let e^\pm be the half-edges that form the edge e .

Proposition 7.2. *The first Chern class of the normal bundle \mathcal{N}_Γ of a boundary divisor D_Γ is*

$$c_1(\mathcal{N}_\Gamma) = -\frac{\kappa_e}{\ell_\Gamma} \left(\psi_{e^+} + \psi_{e^-} \right) - \frac{1}{\ell_\Gamma} \sum_{\widehat{\Delta} \in \text{LG}_{2,e}^\Gamma(B)} \ell_{\widehat{\Delta}, a_{\widehat{\Delta}, \Gamma}} [D_{\widehat{\Delta}}] \tag{51}$$

as an element of $\text{CH}^1(D_\Gamma)$, where $\text{LG}_{2,e}^\Gamma(B)$ is the set of 3-level graphs in $\text{LG}_2^\Gamma(B)$, where the edge e goes from level zero to level -2 and $a_{\widehat{\Delta}, \Gamma} \in \{1, 2\}$ is the index such that the $a_{\widehat{\Delta}, \Gamma}$ th undegeneration of $\widehat{\Delta}$ is not equal to Γ .

We say that $\text{LG}_{2,e}^\Gamma(B)$ are the 3-level graphs where the edge e becomes long. We give direct proofs of both expressions for the normal bundle. The equivalence of the statements follows from an application of the relation in Proposition 8.2 below.

Proof of Theorem 7.1. We consider over the boundary stratum D_Γ the line bundles $\mathcal{L}_1 = \mathcal{O}_\Gamma^{[0]}(-1) \otimes \mathcal{L}_\Gamma^\top$ and $\mathcal{L}_2 = \mathcal{O}_\Gamma^{[-1]}(-1)$, where the tautological bundles on the levels have been introduced in Section 4.3. Roughly, the content of the theorem is that the ratio of local sections of these line bundles is the function $t_1^{\ell_\Gamma}$, which is also the ℓ_Γ -th power of a transversal coordinate. For the precise statement, we compare the cocycles defining the line bundles $\mathcal{L}_1^{-1} \otimes \mathcal{L}_2$ and $\mathcal{N}_\Gamma^{\ell_\Gamma}$.

We start by considering the open subset of D_Γ where Γ does not degenerate further. A local section of $\mathcal{L}_1^{-1} \otimes \mathcal{L}_2$ is the ratio of two relative differential forms, thus a function on the base, that we may compute as $u = \int_{\alpha_1} \eta_{(-1)} / \int_{\alpha_0} \eta_{(0)}$ for some paths α_1 at level -1 and α_0 at level 0 . Here α_1 can be taken as a (usual) relative cycle, and for α_0 , we might have to use a path starting and ending at points near the upper ends of connecting nodes, as in the definition of perturbed period coordinates in [BCGGM3,

Section 11]. We consider a nearby coordinate patch where now the ratio is $\tilde{u} = \frac{\int_{\tilde{\alpha}_1} \eta_{(-1)} / \int_{\tilde{\alpha}_0} \eta_{(0)}}$ for some new cycles related to the original ones by a base change $\tilde{\alpha}_1 = \alpha_1 + \gamma_1$ and $\tilde{\alpha}_0 = \alpha_0 + \gamma_0$ in the homology of the upper and lower level subsurfaces, respectively. One computes that

$$\tilde{u} = u \cdot \frac{1+y}{1+x}, \quad \text{where } x = \frac{\int_{\gamma_0} \eta_{(0)}}{\int_{\alpha_0} \eta_{(0)}} \quad \text{and } y = \frac{\int_{\gamma_1} \eta_{(-1)}}{\int_{\alpha_1} \eta_{(-1)}}.$$

In particular, these x, y are local functions on the upper- and lower-level strata.

On the other hand, by construction (of the perturbed period coordinates), the ℓ_Γ th power of a transversal coordinate is given by

$$t_1^{\ell_\Gamma} = s = \int_{\alpha_1} \eta_{(-1)} / \int_{\alpha_0} (\eta_{(0)} + \xi_{(0)}),$$

where $\xi_{(0)}$ is the modification differential at level 0 constructed in [BCGGM3, Section 11] and the α_i are as above. Again, a nearby coordinate patch is given by $\tilde{s} = \frac{\int_{\tilde{\alpha}_1} \eta_{(-1)} / \int_{\tilde{\alpha}_0} (\eta_{(0)} + \xi_{(0)})}$ with cycles as above. The main point now is that ξ_0 is divisible by s by construction: its contribution vanishes after s -derivation and setting $s = 0$, so

$$\left. \frac{\partial \tilde{s}}{\partial s} \right|_{s=0} = \frac{1+y}{1+x} = \frac{\tilde{u}}{u}, \tag{52}$$

showing that the cocycles from $\mathcal{L}_1^{-1} \otimes \mathcal{L}_2$ and $\mathcal{N}_\Gamma^{\ell_\Gamma}$ agree on the subset under consideration.

If the bottom level degenerates, or in the case of horizontal degenerations of Γ , the above claims remain valid without modification if we take α_1 to be a period that does not go to the lower level. If the top level degenerates into two levels (without loss of generality, higher codimension degenerations do not affect the first Chern classes), the above cocycle comparison is valid verbatim if all pairs of level indices are shifted from $(0, -1)$ to $(-1, -2)$: that is, if we compare the periods of a form on the middle level with the periods of a form at the bottom level. Since the multi-scale differential on the middle level is $t_1^{\ell_{\hat{\Delta},1}}$ times a multi-scale differential on the top level at the intersection with $D_{\hat{\Delta}}$, and since $t_1^{\ell_{\hat{\Delta},1}}$ is a local generating section of \mathcal{L}_Γ^\top , the functions whose ratios we take to get the same cocycle as $\mathcal{N}_\Gamma^{\ell_\Gamma}$ are indeed sections of $\mathcal{L}_1 = \mathcal{O}_\Gamma^{[0]}(-1) \otimes \mathcal{L}_\Gamma^\top$ and \mathcal{L}_2 as we claimed. \square

Sketch of proof of Proposition 7.2. We let $m_e = \ell(\Gamma)/\kappa_e$. In $\overline{\mathcal{M}}_{g,n}$, consider the divisor D_e corresponding to the single edge e and denote by \mathcal{N}_e its normal bundle. With the same symbol, we denote also the pullback of this normal bundle under the forgetful map $D_\Gamma \rightarrow D_e$. We claim that (at least outside a subvariety of codimension two) there is a short exact sequence of quasi-coherent \mathcal{O}_{D_Γ} -modules

$$0 \longrightarrow \mathcal{N}_\Gamma^{m_e} \longrightarrow \mathcal{N}_e \longrightarrow \mathcal{Q}_\Gamma \longrightarrow 0, \tag{53}$$

where the coherent sheaf \mathcal{Q}_Γ is supported on the set $\text{LG}_{2,e}^\Gamma(B)$ and this sheaf is given by

$$\mathcal{Q}_\Gamma = \bigoplus_{\Delta \in \text{LG}_{2,e}^\Gamma(B)} \mathcal{O}_{D_\Gamma} / I_{D_{\hat{\Delta}}}^{\ell_{\hat{\Delta},a}/\kappa_e}, \tag{54}$$

where $a = a_{\hat{\Delta},\Gamma}$ as above and where $I_{D_{\hat{\Delta}}}$ is the ideal sheaf of the divisor $D_{\hat{\Delta}} \subseteq D_\Gamma$. This claim obviously implies the proposition.

To prove it, we use the local description of the universal family over $\Xi \overline{\mathcal{M}}_{g,n}(\mu)$ given by the plumbing construction described in Section 12 of [BCGGM3]. At a boundary point that is precisely in

the intersection of divisors D_{Γ_i} , we let $m_{e,i} = \ell_{\Gamma_i}/\kappa_e$. Then the construction states in particular that the universal family is constructed using the plumbing fixture

$$\mathbb{V}_e = \left\{ (u, v) \in \Delta^2 : uv = \prod_{i=L(e^-)}^{L(e^+)} t_i^{m_{e,i}} \right\}$$

at the node corresponding to the edge e , where u and v are coordinates on the surfaces at the upper and lower ends of the edge and $L(e^\pm)$ denotes the levels at the ends of the edge. A local generator of \mathcal{N}_e is $\partial/\partial f$ if $uv = f$ is a local equation of the node. On the other hand, a local generator of $\mathcal{N}_\Gamma^{m_e}$ is $\partial/\partial(t_i^{m_{e,i}})$ if Γ is the undegeneration of the i th level at the point under consideration. (In particular, $m_e = m_{e,i}$ in this situation.) This follows from the form of perturbed period coordinates. This implies that at a generic point of D_Γ (and more generally whenever the edge does not become long), the natural map $\mathcal{N}_\Gamma^{m_e} \rightarrow \mathcal{N}_e$ is an isomorphism. At the remaining points,

$$\frac{\partial}{\partial f} = \prod_{\substack{j=L(e^-) \\ j \neq i}}^{L(e^+)} t_j^{m_{e,j}} \frac{\partial}{\partial(t_i^{m_{e,i}})} + \dots,$$

where the suppressed tangent vectors vanish when restricted to D_Γ . Since t_j are the defining equations of divisors D_Δ , where the edge becomes long, this implies equation (54). □

Example 7.3. Consider the stratum $\mathbb{P}\Omega\mathcal{M}_{0,5}(a_1, a_2, a_3, a_4, -b)$ with $a_i \geq 0$ and $b = 2 + \sum a_i \geq 0$. We study the ‘cherry’ divisor Γ (see also [BCGM3, Section 14.4]) with one vertex on the top level carrying the unique pole and two vertices on the lower level, each carrying two legs and assuming that the first two legs are on the same vertex. The vertices on the lower level are each connected to the top level by a single edge, denoted by e_1 and e_2 respectively, where e_1 is attached to the vertex carrying the first two legs. The enhancements are given by $\kappa_1 = a_1 + a_2 + 1$ and $\kappa_2 = a_3 + a_4 + 1$. Hence $\ell_\Gamma = \text{lcm}(\kappa_1, \kappa_2)$.

We compute the degree of the normal bundle using either of the two edges. Note that the boundary divisor D_Γ has two intersection points with other boundary strata, where e_1 and e_2 become long edges. Neighbourhoods of these points are quotient stacks by a cyclic group of order $m_i = \ell_\Gamma/\kappa_i$. To see this, say where e_1 becomes long, we check that $\text{Tw}_\Lambda^s = \ell\mathbb{Z} \oplus \kappa_1\mathbb{Z}$ and $\text{Tw}_\Lambda = \langle (0, \kappa_1), (\kappa_2, -\kappa_2) \rangle$, hence the index is m_1 , as claimed.

In this example, the bundle \mathcal{N}_{e_1} has degree zero when pulled back to D_Γ , since D_Γ is contracted when mapped to $\overline{\mathcal{M}}_{0,5}$. Applying Proposition 7.2, we get

$$\text{deg}(\mathcal{N}_\Gamma) = -\frac{1}{m_1 m_2},$$

and using e_2 , we arrive at the same conclusion.

Our next task is to identify the normal bundle as a sum of two contributions from the top and bottom levels via push-pull through the level projections and clutching maps. For this purpose, we define

$$\mathcal{L}_{B_\Gamma^\top} = \mathcal{O}_{B_\Gamma^\top} \left(\sum_{\Delta \in \text{LG}_1(B_\Gamma^\top)} \ell_\Delta D_\Delta \right) \tag{55}$$

and we let p_Γ^\top and c_Γ be the maps defined in Section 4.2.

Lemma 7.4. *We have $p_\Gamma^{\top,*} \mathcal{L}_{B_\Gamma^\top} = c_\Gamma^* \mathcal{L}_\Gamma^\top$.*

Proof. We sum the first equation in Proposition 4.7 over all $\Delta \in \text{LG}_1(B_\Gamma^\top)$. Each $\widehat{\Delta}$ will appear for all graphs in $J(\Gamma^\dagger, \widehat{\Delta})$, as discussed at the beginning of Section 4.3. However, thanks to Lemma 4.6, this factor cancels with all the automorphism factors in that proposition to give the statement we claim. □

The lemma obviously implies

$$c_1(\mathcal{L}_\Gamma^\top) = \frac{1}{\text{deg}(c_\Gamma)} \cdot c_{\Gamma,*} p_\Gamma^\top{}^* c_1(\mathcal{L}_{B_\Gamma^\top}).$$

Since the tautological bundles on the top and bottom levels have a pullback description by Proposition 4.9, we have shown that there exist $\nu^\top \in \text{CH}^1(B_\Gamma^\top)$ and $\nu^\perp \in \text{CH}^1(B_\Gamma^\perp)$ such that

$$\nu_\Gamma := c_1(\mathcal{N}_\Gamma) = c_{\Gamma,*} (p^\top)^\top{}^* \nu_\Gamma^\top + c_{\Gamma,*} (p^\perp)^\perp{}^* \nu_\Gamma^\perp. \tag{56}$$

The normal bundle computation has a generalisation to an inclusion $\mathfrak{j}_{\Gamma,\Pi}: D_\Gamma \hookrightarrow D_\Pi$ between non-horizontal boundary strata of relative codimension one, say defined by the L -level graph Π and one of its $(L + 1)$ -level graph degenerations Γ . This generalisation is needed in Section 8 for recursive evaluations. Such an inclusion is obtained by splitting one of the levels of Π , say the level $i \in \{0, -1, \dots, -L\}$. We define

$$\mathcal{L}_\Gamma^{[i]} = \mathcal{O}_{D_\Gamma} \left(\sum_{\Gamma \xrightarrow{[i]} \widehat{\Delta}} \ell_{\widehat{\Delta}, -i+1} D_{\widehat{\Delta}} \right) \quad \text{for any } i \in \{0, -1, \dots, -L\}, \tag{57}$$

where the sum is over all graphs $\widehat{\Delta} \in \text{LG}_{L+2}(\overline{B})$ that yield divisors in D_Γ by splitting the i th level, which in terms of undegenerations means $\delta_{-i+1}^{\mathbb{C}}(\widehat{\Delta}) = \Gamma$. With the same proof as above, simply shifting attention to level i of Π , we obtain:

Proposition 7.5. *For $\Pi \xrightarrow{[i]} \Gamma$ (or equivalently $\delta_{-i+1}^{\mathbb{C}}(\Gamma) = \Pi$), the Chern class of the normal bundle $\mathcal{N}_{\Gamma,\Pi} = \mathcal{N}_{D_\Gamma/D_\Pi}$ is given by*

$$c_1(\mathcal{N}_{\Gamma,\Pi}) = \frac{1}{\ell_{\Gamma,(-i+1)}} (-\xi_\Gamma^{[i]} - c_1(\mathcal{L}_\Gamma^{[i]}) + \xi_\Gamma^{[i-1]}) \quad \text{in } \text{CH}^1(D_\Gamma). \tag{58}$$

With the same proof as in Lemma 7.4, we obtain

$$p_\Gamma^{[i]*} \mathcal{L}_{B_\Gamma^{[i]}} = c_\Gamma^* \mathcal{L}_\Gamma^{[i]} \quad \text{where } \mathcal{L}_{B_\Gamma^{[i]}} = \mathcal{O}_{B_\Gamma^{[i]}} \left(\sum_{\Delta \in \text{LG}_1(B_\Gamma^{[i]})} \ell_\Delta D_\Delta \right). \tag{59}$$

We can thus write the normal bundle as a sum of bundles that are c_Γ -pushforwards of pullbacks from $B_\Gamma^{[i]}$ and $B_\Gamma^{[i-1]}$. We express this by saying that the normal bundle is *supported on the levels i and $i - 1$* (for $i \in \mathbb{Z}_{\leq 0}$).

We need some compatibility statements for pullbacks of normal bundles to more degenerate graphs. We start with auxiliary bundles, whose pullback we need, too.

Lemma 7.6. *Let $\Gamma \in \text{LG}_L(B)$, and let $\Gamma \xrightarrow{[i]} \widehat{\Delta}$ be a codimension one degeneration of Γ obtained by splitting the level $i \in \{0, \dots, -L\}$. Then for every $j \in \{0, \dots, -L\}$,*

$$\mathfrak{i}_{\widehat{\Delta},\Gamma}^* (\xi_\Gamma^{[j]}) = \begin{cases} \xi_{\widehat{\Delta}}^{[j]}, & \text{if } j \geq i \\ \xi_{\widehat{\Delta}}^{[j-1]} & \text{if } j < i \end{cases}$$

and

$$\mathfrak{i}_{\widehat{\Delta},\Gamma}^* \left(c_1 \left(\mathcal{L}_\Gamma^{[j]} \right) \right) = \begin{cases} c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[j]} \right), & \text{if } j > i \\ c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]} \right) & \text{if } j < i \\ c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[j-1]} \right) + \xi_{\widehat{\Delta}}^{[j-1]} - \xi_{\widehat{\Delta}}^{[j]} & \text{if } j = i. \end{cases}$$

Proof. For the cases $j \neq i$, the claims are obvious since level j is untouched in the degeneration from Γ to $\widehat{\Delta}$. If $i = j$, then the second claim follows from

$$\begin{aligned} i_{\widehat{\Delta}, \Gamma}^* \left(c_1 \left(\mathcal{L}_{\Gamma}^{[i]} \right) \right) &= i_{\widehat{\Delta}, \Gamma}^* \left(\sum_{\Gamma \rightsquigarrow \Lambda, \Lambda \neq \widehat{\Delta}} \ell_{\Lambda, -i+1} [D_{\Lambda}] + \ell_{\widehat{\Delta}, -i+1} [D_{\widehat{\Delta}}] \right) \\ &= c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[i]} \right) + c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[i-1]} \right) + \ell_{\widehat{\Delta}, -i+1} c_1 \left(\mathcal{N}_{D_{\widehat{\Delta}}/D_{\Gamma}} \right) \\ &= c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[i]} \right) + c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[i-1]} \right) + \left(-\xi_{\widehat{\Delta}}^{[i]} + \xi_{\widehat{\Delta}}^{[i+1]} - c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[i]} \right) \right) \\ &= c_1 \left(\mathcal{L}_{\widehat{\Delta}}^{[i-1]} \right) + \xi_{\widehat{\Delta}}^{[i-1]} - \xi_{\widehat{\Delta}}^{[i]}. \end{aligned}$$

The case $j = i$ for the first claim about pulling back $\xi_{\Gamma}^{[i]}$ follows directly from the definition of $\mathcal{O}_{\Gamma}^{[i]}(-1)$ by local generators. Alternatively, one can compute it by applying the relation in equation (62) shown in the next section. More specifically, if the chosen marked point is supported on the i th level of $\widehat{\Delta}$, the calculation is straightforward. If the marked point h is supported on the $(i - 1)$ st level of $\widehat{\Delta}$, then $\widehat{\Delta}$ appears among the boundary terms of equation (62). Pulling back makes the normal bundle appear, and thus $\xi_{\widehat{\Delta}}^{[i]}$ in the formula from Theorem 7.1. The remaining boundary terms of equation (62) can be grouped into those where h ends up at level $i - 1$ or $i - 2$ after pulling back to $\widehat{\Delta}$. These groups cancel with the remaining two terms of the normal bundle. \square

As a consequence of the preceding lemma and Theorem 7.1, we obtain:

Corollary 7.7. *Let $\Gamma \in \text{LG}_L(B)$, and let $\widehat{\Delta}$ be a codimension one degeneration of the $(-i + 1)$ -level of Γ : that is, such that $\Gamma = \delta_i^{\mathbb{C}}(\widehat{\Delta})$, for some $i \in \{1, \dots, L + 1\}$. Then*

$$i_{\widehat{\Delta}, \Gamma}^* \left(\ell_{\Gamma, j} c_1 \left(\mathcal{N}_{\Gamma/\delta_j^{\mathbb{C}}(\Gamma)} \right) \right) = \begin{cases} \ell_{\widehat{\Delta}, j} c_1 \left(\mathcal{N}_{\widehat{\Delta}/\delta_j^{\mathbb{C}}(\widehat{\Delta})} \right), & \text{for } j < i \\ \ell_{\widehat{\Delta}, j+1} c_1 \left(\mathcal{N}_{\widehat{\Delta}/\delta_{(j+1)}^{\mathbb{C}}(\widehat{\Delta})} \right) & \text{otherwise.} \end{cases}$$

8. The tautological ring

In this section, we give the precise definition of the tautological ring and prove Theorem 1.5. We define the *tautological rings of strata* as the smallest set of \mathbb{Q} -subalgebras $R^{\bullet}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)) \subset \text{CH}^{\bullet}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu))$ that

- contains the ψ -classes attached to the marked points,
- is closed under the pushforward of the map forgetting a regular marked point (a zero of order zero), and
- is closed under the maps $\zeta_{\Gamma, * } p^{[i], *}$ defined in Proposition 4.4 for all level graphs Γ .

Our goal is to provide additive generators of this ring and show that the main players, normal bundles and logarithmic cotangent bundle have Chern classes in this ring. The main tool is the excess intersection formula that allows computing the intersection product of boundary strata, possibly decorated with ψ -classes.

There are two definitions of other (refined) tautological rings. One option is the refined ring $R_{\text{ref}}^{\bullet}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu))$ that is closed under all the clutching morphisms $\zeta_*^{\text{ref}} p^{[i], *}$ that distinguish the components of boundary strata that are reducible due to inequivalent prong-matchings. Obviously, $R^{\bullet}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)) \subseteq R_{\text{ref}}^{\bullet}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)) \subset \text{CH}^{\bullet}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu))$. There is an analogue of Theorem 1.5, replacing in the additive generators the inclusion maps i_{Γ} of reducible boundary strata with the inclusion maps of irreducible components. The proofs below can be adapted to that setting.

The second option is to include D_h or equivalently clutching morphism for horizontal nodes into the definition of the tautological ring $R_h^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ (and not distinguish inequivalent prong-matchings, although one could obviously do both). Obviously $R^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subseteq R_h^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$. For our purposes, working in the smaller ring is enough, since the main result Theorem 1.2 shows that the Chern character of the logarithmic cotangent bundle belongs to the smaller ring. Moreover, working in the smaller tautological ring is computationally advantageous as the number of boundary strata is smaller, and they can be efficiently encoded using the notion of profiles (see [CMZ20]).

To express $c_1(\Omega_{\overline{B}})$, we need D_h , so we need to work in $R_h^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$. However, one of the main points of this section is that the Chern polynomial of the logarithmic cotangent bundle belongs to the smallest of the natural candidates for a tautological ring. It seems interesting to decide which of the two inclusions of tautological rings defined above are strict.

8.1. Excess intersection formula

Suppose we are given two level graphs Λ_1 and Λ_2 without horizontal nodes and the corresponding inclusion maps $i_{\Lambda_j} : D_{\Lambda_j} \rightarrow \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ into a compactified stratum. For a class $\alpha \in \text{CH}^\bullet(D_{\Lambda_2})$, we want to compute $i_{\Lambda_1}^* i_{\Lambda_2,*} \alpha$ as the pushforward from the maximal-dimensional boundary strata in the support of $D_{\Lambda_1} \cap D_{\Lambda_2}$ in terms of an α -pullback and normal bundle classes encoding the excess intersection of D_{Λ_1} and D_{Λ_2} . We say that a level graph Π is a (Λ_1, Λ_2) -graph if there are undegeneration morphisms $\rho_i : \Pi \rightarrow \Lambda_i$: that is, edge contraction morphisms with the property that there are subsets I_{Λ_1} and I_{Λ_2} of level passages of Π such that $\delta_{I_{\Lambda_1}}(\Pi) = \Lambda_1$ and $\delta_{I_{\Lambda_2}}(\Pi) = \Lambda_2$. (Automorphisms of Λ_i – that is, the stack structure of D_{Λ_i} – stemming from permuting the edges require the distinction between δ 's and the ρ_i 's.) We call Π a generic (Λ_1, Λ_2) -graph if $I_{\Lambda_1}^c \cap I_{\Lambda_2}^c = \emptyset$. The intersection formula will use the inclusion maps as indicated in the diagram:

$$\begin{array}{ccc} D_\Pi & \xrightarrow{i_{\Pi,\Lambda_2}} & D_{\Lambda_2} \\ \downarrow i_{\Pi,\Lambda_1} & & \downarrow i_{\Lambda_2} \\ D_{\Lambda_1} & \xrightarrow{i_{\Lambda_1}} & \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu). \end{array}$$

Proposition 8.1. *For any $\alpha \in \text{CH}^\bullet(D_{\Lambda_2})$, we can express its pushforward pulled back to Λ_1 as*

$$i_{\Lambda_1}^* i_{\Lambda_2,*} \alpha = \sum_{\Pi} i_{\Pi,\Lambda_1,*} \left(v_{\Lambda_1 \cap \Lambda_2}^\Pi \cdot i_{\Pi,\Lambda_2}^* \alpha \right), \tag{60}$$

where the sum is over all generic (Λ_1, Λ_2) -graphs Π . In this expression,

$$v_{\Lambda_1 \cap \Lambda_2}^\Pi = \prod_{k \in I_{\Lambda_1} \cap I_{\Lambda_2}} i_{\Pi,\delta_k(\Pi)}^* (v_{\delta_k(\Pi)})$$

is the product of the pullback to D_Π of the first Chern classes of the normal bundles of the divisors containing both D_{Λ_1} and D_{Λ_2} .

Proof. By the excess intersection formula ([Ful98, Proposition 17.4.1]), we have to show that the fibre product $\mathcal{F}_{\Lambda_1,\Lambda_2} = D_{\Lambda_1} \times_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)} D_{\Lambda_2}$ is the coproduct $\mathcal{D} = \coprod D_\Pi$ over all generic (Λ_1, Λ_2) -graphs Π and identify the excess normal bundle.

First we define a map $\varphi : \mathcal{D} \rightarrow \mathcal{F}_{\Lambda_1,\Lambda_2}$ via the universal properties of the coproduct and the fibre product. It is the map induced by the inclusions $i_{\Pi,\Lambda_i} : D_\Pi \rightarrow D_{\Lambda_i}$ for each generic (Λ_1, Λ_2) -graph Π .

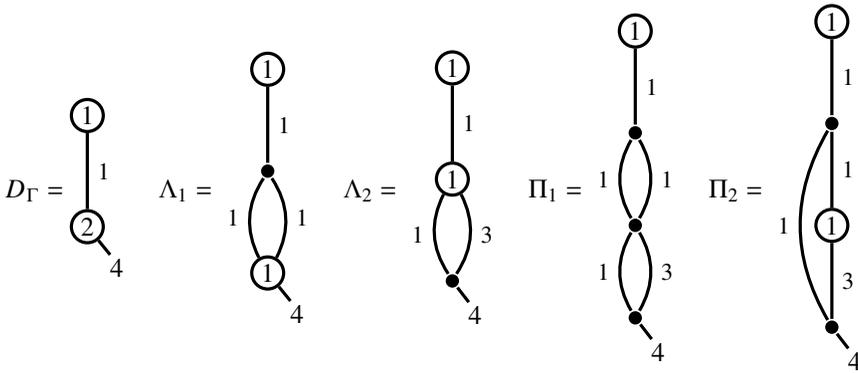


Figure 4. An example for excess intersection computations.

To give a converse natural transformation on objects, we take a family parametrised by $\mathcal{F}_{\Lambda_1, \Lambda_2}$: that is, a pair of a family (\mathcal{X}_1, η_1) of multi-scale differentials compatible with an undegeneration of Λ_1 and a family (\mathcal{X}_2, η_2) compatible with an undegeneration of Λ_2 . If we forget the differentials, we can construct a family of pointed stable curves $(\mathcal{X}, \mathbf{z})$ over some stable graph Π , which is generic as a (Λ_1, Λ_2) -stable graph (see [GP03] or [SZ20]). We make Π into a level graph by declaring a vertex v_1 to be on top of v_2 if this holds for either of their images in Λ_1 or in Λ_2 . Compatibility of the fibre product ensures that this definition is consistent. Moreover, this definition ensures that Π is (Λ_1, Λ_2) -generic in our sense of enhanced level graphs. The construction of \mathcal{X} exhibits a bijection of its f -relative components (relative to the structure morphism f to the base) with the f -relative components of \mathcal{X}_1 (and also those of \mathcal{X}_2). We can thus pull back the differential η_1 on each of those components of \mathcal{X} (or we could pull back η_2) to a collection of differentials $\boldsymbol{\eta}$ on \mathcal{X} . To see that this indeed defines a twisted differential compatible with Π , only the global residue condition requires a nontrivial verification. By definition of (Λ_1, Λ_2) -genericity and because of the unique ordering of profiles shown in Proposition 5.1, for each level $-i$ of Π there is an index $j \in \{1, 2\}$ and a level $-i'$ of Λ_j such that the connected components of the subgraph of Π above level $-i$ are in natural bijection with the connected components of the subgraph of Λ_j above level $-i'$. This implies the global residue condition. The enhancements of the edges Π are given by the identification of the edges with those of Λ_1 and Λ_2 in the first step of the converse construction. In the same way, we provide $(\mathcal{X}, \mathbf{z}, \boldsymbol{\eta})$ with a collection of prong-matchings and pull back the rescaling ensembles as in [BCGGM3, Section 7] to complete the construction of a family of multi-scale differentials compatible with an undegeneration of Π . The converse natural transformation on morphisms is simply the map constructed for families of pointed stable curves.

The excess normal bundle is in general given by $E = \mathbf{i}_{\Pi, \Lambda_1}^* \mathcal{N}_{\Lambda_1} / \mathcal{N}_{\Pi, \Lambda_2}$, where the normal sheaves appearing are the normal sheaves of the morphisms \mathbf{i}_{Λ_1} and $\mathbf{i}_{\Pi, \Lambda_2}$. Since by Proposition 5.1 the nonhorizontal boundary strata are smooth and simple normal crossing, the previous normal sheaves are vector bundles, and they are given as the direct sum of the pullback of the normal bundles of appropriate divisors. More specifically, $\mathcal{N}_{\Lambda_1} = \bigoplus_{i=1}^{L(\Lambda_1)} \mathcal{N}_{\delta_i(\Lambda_1)}$ and $\mathcal{N}_{\Pi, \Lambda_2} = \bigoplus_{i \in I_{\Lambda_2}^c} \mathcal{N}_{\delta_i(\Pi)}$. This implies that E is the direct sum of the normal bundles of the levels common to both Λ_1 and Λ_2 (pulled back to D_Π), and thus its top Chern class is as claimed in the proposition. \square

An example of the excess intersection formula that illustrates both the appearance of a nontrivial normal bundle and of a sum over more than one graph Π is given in Figure 4. In this example, Π_1 and Π_2 are the only generic (Λ_1, Λ_2) -graphs. Since $\Gamma = \delta_1(\Lambda_1) = \delta_1(\Lambda_2)$ and D_Γ is the only boundary divisor containing both D_{Λ_1} and D_{Λ_2} , we can compute $\mathbf{i}_{\Lambda_1}^* \mathbf{i}_{\Lambda_2, *} [1_{D_{\Lambda_2}}] = \sum_{i=1}^2 \mathbf{i}_{\Pi_i, \Lambda_1, *} \left(\mathbf{i}_{\Pi_i, D_\Gamma}^* (\mathcal{N}_{D_\Gamma}) \right)$.

At the expense of introducing more notation, the excess intersection formula can be generalised in two ways. First, the ambient space might be a boundary stratum associated to a codimension L -level

graph Γ , as summarised in the diagram

$$\begin{array}{ccc}
 D_{\Pi} & \xrightarrow{i_{\Pi, \Lambda_2}} & D_{\Lambda_2} \\
 \downarrow i_{\Pi, \Lambda_1} & & \downarrow i_{\Lambda_2, \Gamma} \\
 D_{\Lambda_1} & \xrightarrow{i_{\Lambda_1, \Gamma}} & D_{\Gamma}
 \end{array}$$

of inclusions. In this situation, we define $v_{(\Lambda_1 \cap \Lambda_2)/\Gamma}^{\Pi}$ to be the product of the pullback to Π of the Chern classes of the normal bundles $\mathcal{N}_{\Gamma'/\Gamma}$, where Γ' ranges over all codimension 1 nonhorizontal degenerations Γ' of Γ that are common to Λ_1 and Λ_2 . As above, we denote appropriate pullbacks of this product by the same letter. The excess intersection formula then reads

$$i_{\Lambda_1, \Gamma}^* i_{\Lambda_2, \Gamma}^* \alpha = \sum_{\Pi} i_{\Pi, \Lambda_1, *}\left(v_{(\Lambda_1 \cap \Lambda_2)/\Gamma}^{\Pi} \cdot i_{\Pi, \Lambda_2}^* \alpha\right), \tag{61}$$

where the sum ranges over all (Λ_1, Λ_2) -graphs Π .

In the more general case that the level graphs Λ_i also have horizontal nodes, there is an obvious generalisation of this proposition. A general undegeneration of boundary graphs is given by a pair $\delta = (\delta_{\text{ver}}, \delta_{\text{hor}})$ consisting of a level undegeneration δ_{ver} as in Section 3.3 and an undegeneration of horizontal nodes δ_{hor} . One defines Π to be a (Λ_1, Λ_2) -graph if there are undegenerations δ_i such that $\delta_i(\Pi) = \Lambda_i$, for $i = 1, 2$. Such a graph is generic if the vertical undegenerations are generic as above and, moreover, if the horizontal contractions are generic in the usual sense of $\overline{\mathcal{M}}_g$ (see [GP03] or [ACG11, Chapter XVII]). We leave it to the reader to adapt the previous proposition and the subsequent argument to the general context.

8.2. Relations in the tautological ring and the proof of Theorem 1.5

Before concluding the proof of Theorem 1.5, we need some relations in the tautological ring. These relations are essentially known, but we restate them here for convenience and to justify a version for the spaces $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}^{\mathfrak{R}}(\boldsymbol{\mu})$: that is, possibly disconnected, with residue conditions, and for multi-scale differentials rather than on the incidence variety compactification. Recall the notation of Section 4.1 for generalised strata, where the (i, j) th marked point is the j th marked point of the i th surface and has order $m_{i,j} \in \mathbb{Z}$.

Proposition 8.2 ([Sau19, Theorem 6(1)]). *The class ξ on $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}^{\mathfrak{R}}(\boldsymbol{\mu})$ can be expressed using the ψ -class at the (i, j) th marked point as*

$$\xi = (m_{i,j} + 1)\psi_{(i,j)} - \sum_{\Gamma \in_{(i,j)} \text{LG}_1(\overline{B})} \ell_{\Gamma}[D_{\Gamma}], \tag{62}$$

where ${}_{(i,j)}\text{LG}_1(\overline{B})$ are two-level graphs with the leg (i, j) on the lower level.

The fact that our D_{Γ} records prong-matching equivalence classes makes up for the difference between our formula and the one appearing in [Sau19]. Indeed, the original formula of [Sau19] can be retrieved using the substitution $\ell_{\Gamma}[D_{\Gamma}] = \frac{K_{\Gamma}}{|\text{Aut}(\Gamma)|} c_{\Gamma, *} p_{\Gamma}^* [B_{\Gamma}^{\top} \times B_{\Gamma}^{\perp}]$, which follows from Lemma 4.5.

Proof. We expand the argument given in [Che19, Proposition 2.1] including the boundary terms. Let $\pi : \mathcal{X} \rightarrow \overline{B}$ be the universal family and S_i be the image of the section given by the i th marked point. The evaluation map gives an isomorphism of $\pi^* \mathcal{O}(-1)$ and $\omega_{\mathcal{X}/\overline{B}}$ outside the locus S_i and the lower level components of the boundary divisors. Consider the construction of the universal differential over $\Xi\overline{\mathcal{M}}_{g,n}(\boldsymbol{\mu})$ in [BCGGM3, Section 12], in particular in the plumbing fixture (12.6) of [BCGGM3, Section 12]. The

difference of t -powers at the two branches is just ℓ_Γ in our notation, and all this is unchanged in the presence of a GRC \mathfrak{R} . We deduce that

$$\pi^* \xi = c_1(\omega_{\mathcal{X}/\overline{B}}) - \sum_{i=1}^n m_i S_i - \sum_{\Gamma \in \text{LG}_1(\overline{B})} \ell_\Gamma [\mathcal{X}_\Gamma^\perp], \tag{63}$$

where \mathcal{X}_Γ^\perp is the lower-level component in the universal family over the divisor D_Γ . We intersect both sides with S_i and apply π_* . Using $\pi_*(S_i^2) = -\psi_i$ and $\pi_*(\omega_{\mathcal{X}/\overline{B}} \cdot S_i) = \psi_i$, this gives the claim. \square

We need a similar generalisation of another relation of Sauvaget to our framework that will be needed for the final evaluation of top degree classes (see the end of Section 9 and [CMZ20]). Consider a generalised stratum defined by a residue condition \mathfrak{R} as defined in Section 4.1. Suppose we remove one element from the set $\lambda_{\mathfrak{R}}$ constraining the residues in the definition of \mathfrak{R} . We denote this new set by $\lambda_{\mathfrak{R}_0}$ and \mathfrak{R}_0 the new set of residue conditions. Two cases might occur. Either $\mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}}(\mu) = \mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}_0}(\mu)$ or $\mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}}(\mu) \subsetneq \mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}_0}(\mu)$ is a divisor. We consider the second case here and note that this condition is equivalent to $S := R \cap \mathfrak{R} \subset S_0 := R \cap \mathfrak{R}_0$ is codimension one (rather than the two being equal), where R is the space of residues defined in equation (25). Consider now a boundary stratum D_Γ in $\mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}_0}(\mu)$. For each level i of D_Γ and any GRC \mathfrak{R} containing \mathfrak{R}_0 , we define the *residue condition* $\mathfrak{R}^{[i]}$ induced by \mathfrak{R} to be the residue condition given at level i by the auxiliary level graph $\tilde{\Gamma}_{\mathfrak{R}}$ as defined in Section 4.1, created with the help of the auxiliary vertices of \mathfrak{R} . For the top level, we write \mathfrak{R}^\top for the induced residue condition on the top level. It can be simply computed by discarding from the parts $\lambda_{\mathfrak{R}}$ all indices of edges that go to lower level in D_Γ .

Proposition 8.3 ([Sau19, Proposition 7.6]). *The class of the stratum $\mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}}(\mu)$ with residue condition \mathfrak{R} compares inside Chow ring of the generalised stratum $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}_0}(\mu)$ to the class ξ by the formula*

$$[\mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}}(\mu)] = -\xi - \sum_{\Gamma \in \text{LG}_1^{\mathfrak{R}}(\overline{B})} \ell_\Gamma [D_\Gamma] - \sum_{\Gamma \in \text{LG}_{1,\mathfrak{R}}(\overline{B})} \ell_\Gamma [D_\Gamma], \tag{64}$$

where $\text{LG}_1^{\mathfrak{R}}(\overline{B})$ are two-level graphs with $R_\Gamma \cap \mathfrak{R}^\top = R_\Gamma \cap \mathfrak{R}_0^\top$ – that is, the GRC on the top level induced by \mathfrak{R} no longer introduces an extra condition – and $\text{LG}_{1,\mathfrak{R}}(\overline{B})$ are two-level graphs where all the legs involved in the condition forming $\mathfrak{R} \setminus \mathfrak{R}_0$ go to lower level.

Proof. Consider that map $s : \mathcal{O}_{\overline{B}}(-1) \rightarrow S_0/S$ to the constant rank one vector bundle, mapping a point (X, ω) to the (equivalence class mod S of the) tuple of residues of ω , which defines a point in S_0 . The vanishing locus in the interior of \overline{B} is by definition $\mathbb{P}\Xi\overline{\mathcal{M}}_{\mathfrak{g},n}^{\mathfrak{R}}(\mu)$, and in usual period coordinates, we see that the vanishing order is one there. To understand the boundary contribution, consider first boundary divisors neither in $\text{LG}_1^{\mathfrak{R}}(\overline{B})$ nor in $\text{LG}_{1,\mathfrak{R}}(\overline{B})$. For those, being in the vanishing locus of s is a nontrivial (divisorial) condition, and thus this locus is of codimension two and irrelevant for the equation. It remains to justify the vanishing statement and the vanishing order for the other divisors. Any section of $\mathcal{O}_{\overline{B}}(-1)$ decays like $t_1^{\ell_\Gamma}$ near lower-level components by construction of the compactification in [BCGGM3, Section 12], where t_1 is a transversal coordinate. Consequently, any $D_\Gamma \in \text{LG}_{1,\mathfrak{R}}(\overline{B})$ is in the support of the cokernel of the map s , with multiplicity ℓ_Γ . For D_Γ with $\Gamma \in \text{LG}_1^{\mathfrak{R}}(\overline{B})$, the residues at the poles going to level zero are zero (mod S) all along D_Γ by definition. Transversally, they become nonzero with the growth of the modification differential (see the construction in [BCGGM3, Section 11]), since the modification differential must be generically nonzero on D_Γ if \mathfrak{R} imposes a nontrivial condition generically on the stratum, but none along D_Γ . Since the modification differential scales with $t_1^{\ell_\Gamma}$, this proves the claim on the multiplicity of D_Γ in this case, too. \square

We are now ready to prove that the tautological ring is finitely generated by the additive generators displayed in Theorem 1.5.

We let $\overline{B}(0) = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n+1}(\mu, 0)$, in comparison with the usual $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. There is a forgetful map $\pi' : \overline{B}(0) \rightarrow \overline{B}$ given by forgetting the marked point and contracting edges to vertices that have become unstable while preserving the level structure on the remaining vertices. Note that in contrast to $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$, the map π' is not the universal family $\pi : \mathcal{X} \rightarrow \overline{B}$. The difference is apparent at points where the natural map $\overline{B}(0) \rightarrow \overline{\mathcal{M}}_{g,n+1}$ has positive dimensional fibres: for example, at the cherry divisors (see [BCGGM3, Section 14.4]) with one regular marked point. Indeed, there is a $\mathbb{P}^1 \subset \overline{B}(0)$ that parametrises multi-scale differentials on the same underlying curve with cherry-type level graphs that differ only by the relative scale at lower level, and the image of this \mathbb{P}^1 in the universal curve $\overline{\mathcal{M}}_{g,n+1}$ is just a point.

Proof of Theorem 1.5. We let $R_{f,g}^\bullet(\overline{B})$ be the vector space spanned by the classes $\zeta_{\Gamma,*}(\prod_{i=0}^{-L(\Gamma)} p_{\Gamma}^{[i],*} \alpha_i)$, where α_i is a monomial in the ψ -classes supported on level i of the graph Γ , where $\Gamma \in \text{LG}(\overline{B})$ ranges among all level graphs without horizontal nodes. Obviously this is a finite-dimensional vector space since for any stratum μ there are only finitely many level graphs, and for each of them there is a finite number of monomials that give a nonzero class.

By our definition of the tautological ring of the moduli space of multi-scaled differentials, clearly $R_{f,g}^\bullet(\overline{B}) \subseteq R^\bullet(\overline{B})$.

We show now that $R_{f,g}^\bullet(\overline{B})$ is actually a subring of the tautological ring: that is, it is closed under the intersection product. We prove this by iteratively applying the projection formula and the excess intersection formula (61). In the first step, for any two classes $\alpha_j \in \text{CH}^*(D_{\Lambda_j})$, Proposition 8.1 and the projection formula (together with the relation $i_{\Pi} = i_{\Lambda_1} \circ i_{\Pi, \Lambda_1}$) imply

$$\begin{aligned} i_{\Lambda_1,*}(\alpha_1) \cdot i_{\Lambda_2,*}(\alpha_2) &= \sum_{\Pi} i_{\Lambda_1,*} \left(\alpha_1 \cdot i_{\Pi, \Lambda_1,*} \left(v_{\Lambda_1 \cap \Lambda_2}^{\Pi} \cdot i_{\Pi, \Lambda_2}^*(\alpha_2) \right) \right) \\ &= \sum_{\Pi} i_{\Pi,*} \left(v_{\Lambda_1 \cap \Lambda_2}^{\Pi} \cdot i_{\Pi, \Lambda_1}^*(\alpha_1) \cdot i_{\Pi, \Lambda_2}^*(\alpha_2) \right), \end{aligned} \tag{65}$$

where the sums are over all generic (Λ_1, Λ_2) -graphs Π . The excess intersection class $v_{\Lambda_1 \cap \Lambda_2}^{\Pi}$ is given by pullbacks of normal bundles of divisors. By repeatedly applying Corollary 7.7, we see that the pullback of the class of the normal bundle of a divisor is given by the class of the normal bundle of D_{Π} in a codimension one undegeneration. The shape of such a class was computed in equation (58). By using the compatibility expressed in equation (4.9) between level-wise tautological line classes and the tautological line classes on the level strata, together with Proposition 8.2, we see that the classes of these normal bundles are given by ψ -class contributions and boundary contributions given by codimension one degenerations of Π . If there are no boundary contributions, then we are done since we obtained an expression in terms of elements of $R_{f,g}^\bullet(\overline{B})$ supported on Π . If this is not the case, we can apply the same projection formula and excess intersection formula argument as before to these boundary contributions. (Now we have to use the more general excess intersection formula in equation (61) with ambient Π .) This process has to terminate since the dimension of the boundary strata appearing in the excess intersection factor is decreasing, so at some point the excess class contribution will be trivial. Hence we have shown that $R_{f,g}^\bullet(\overline{B})$ is a subring of the tautological ring.

To show that $R_{f,g}^\bullet(\overline{B})$ is equal to $R^\bullet(\overline{B})$, we need to show that $R_{f,g}^\bullet(\overline{B})$ is closed under pushforward of clutching morphism and under pushforward of the map that forgets a regular point. The first statement is clear. We now argue inductively on the dimension of \overline{B} , starting with the obvious case $\dim(\overline{B}) = 0$. We may assume by induction hypothesis that the π' -pushforwards of elements in $R_{f,g}^\bullet(\overline{B}'(0))$ are in $R_{f,g}^\bullet(\overline{B}')$ for any stratum \overline{B}' of dimension less than the dimension of \overline{B} .

We first show that $\pi'_*(\zeta_{\Gamma,*}\psi_{n+1}^{a+1}) \in R_{f_g}^\bullet(\overline{B})$ for any graph Γ with at least two levels. Let i be the level of Γ that contains the $(n+1)$ st marked point. For ψ_{n+1}^{a+1} to be nonzero, we need the component containing the $(n+1)$ st marked point to be positive-dimensional (taking GRC into account). Let Γ' be the level graph obtained from Γ by forgetting this point and contracting edges to unstable vertices, preserving the level structure otherwise. There is thus a well-defined projection map $(\pi')^{[i]} : B(0)_\Gamma^{[i]} \rightarrow B_{\Gamma'}^{[i]}$ of generalised strata. Recalling that $\psi_{n+1} = p_\Gamma^*\psi_{n+1}$, by our general abuse of notation, we find $\pi'_*(\zeta_{\Gamma,*}\psi_{n+1}^{a+1}) = \zeta_{\Gamma',*}p_{\Gamma'}^*((\pi')^{[i]}\psi_{n+1}^{a+1})$. By induction, we know that $(\pi')_*^{[i]}\psi_{n+1}^{a+1} \in R_{f_g}^\bullet(B_{\Gamma'}^{[i]})$; and since the collection of rings $R_{f_g}^\bullet(\cdot)$ is already known to be stable under $\zeta_{\Gamma',*}p_{\Gamma'}^*$, we conclude that $\pi'_*(\zeta_{\Gamma,*}\psi_{n+1}^{a+1}) \in R_{f_g}^\bullet(\overline{B})$.

Second, to treat the case when Γ is the trivial graph, we consider the commutative diagram

$$\begin{CD} \mathbb{P}\overline{\mathcal{M}}_{g,n+1}(\mu, 0) @>f_{n+1}>> \overline{\mathcal{M}}_{g,n+1} \\ @VV\pi'V @VV\pi_{n+1}V \\ \mathbb{P}\overline{\mathcal{M}}_{g,n}(\mu) @>f_n>> \overline{\mathcal{M}}_{g,n} \end{CD}$$

where π' and π_{n+1} are the maps forgetting the last point and f_{n+1} and f_n are the maps forgetting the twisted differential. Recall that the Arbarello-Cornalba class κ_a^{AC} , which is defined to be the π_{n+1} -pushforward of the $(a+1)$ th power of the first Chern class of the relative log-cotangent bundle of π_{n+1} , satisfies the dilation equation $\kappa_a^{AC} = (\pi_{n+1})_*(\psi_{n+1}^{a+1})$. From the above diagram, we then get

$$f_n^*\kappa_a^{AC} = f_n^*(\pi_{n+1})_*(\psi_{n+1}^{a+1}) = \pi'_*(f_{n+1}^*(\psi_{n+1}^{a+1})).$$

Recall that we abuse notation and identify ψ and κ -classes in $CH^*(\overline{B})$ with their pullback from $\overline{\mathcal{M}}_{g,n}$. We have thus shown that the special version of the dilation equation $\pi'_*(\psi_{n+1}^{a+1}) = \kappa_a^{AC}$ also holds in $CH^*(\overline{B})$. We thus only need to show that $\kappa_a^{AC} \in R_{f_g}^\bullet(\overline{B})$. Let now $\pi : \mathcal{X} \rightarrow \overline{B}$ be the universal family and recall that the (Mumford-Morita-Miller) κ -classes are defined by $\kappa_a = \pi_*(c_1(\omega_\pi)^{a+1})$. Since the difference between κ_a and κ_a^{AC} is given by a sum of powers of ψ -classes, if we can show that κ_a is in $R_{f_g}^\bullet(\overline{B})$, then it would follow that κ_a^{AC} is in the same ring, which is our goal. Recall that $[\mathcal{X}_\Gamma^\perp]$ is the lower-level component in the universal family over the divisor D_Γ . From equation (63), we deduce that

$$\kappa_a = \pi_* \left(\left(\pi^*\xi + \sum_{i=1}^n m_i S_i + \sum_{\Gamma \in \text{LG}_1(B)} \ell_\Gamma[\mathcal{X}_\Gamma^\perp] \right)^{a+1} \right)$$

is a linear combination of terms of the form $\xi^p \pi_* (S_i^{b_i} \prod [\mathcal{X}_\Gamma^\perp]^{c_\Gamma})$ with $p + b_i + \sum_\Gamma c_\Gamma = a + 1$, since the sections S_i are disjoint. The ξ -powers are tautological by Proposition 8.2, so we only need to study the π_* -term. Let $\mathfrak{i} : D_0 := \bigcap_{\Gamma: c_\Gamma > 0, i \in \Gamma^\perp} D_\Gamma \rightarrow \overline{B}$ be the inclusion of the intersection of boundary divisors where the i th marked point is on the bottom level, which is the image under π of the support of $S_i^{b_i} \prod [\mathcal{X}_\Gamma^\perp]^{c_\Gamma}$. Let $\tilde{\mathfrak{i}} : \mathcal{X}_0 := \bigcap_{\Gamma: c_\Gamma > 0, i \in \Gamma^\perp} \mathcal{X}_\Gamma^\perp \rightarrow \mathcal{X}$ be the corresponding inclusion in the total space of the family. Let $\mathfrak{j}_{0,\Gamma} : D_0 \rightarrow D_\Gamma$ and $\tilde{\mathfrak{j}}_{0,\Gamma} : \mathcal{X}_0 \rightarrow \mathcal{X}_\Gamma$ be the inclusions into codimension one divisors. Finally, let σ_i be the section of the i th marked point and abusively also its restriction to D_Γ and to D_0 .

Suppose that $b_i > 0$. Then using $\sigma_i^* S_i^k = (-\psi_i)\sigma_i^*(S_i^{k-1})$, we find

$$\begin{aligned} \pi_* (S_i^{b_i} \prod_\Gamma [\mathcal{X}_\Gamma^\perp]^{c_\Gamma}) &= \pi_* \sigma_{i,*} \sigma_i^* (S_i^{b_i-1} \cdot \tilde{\mathfrak{i}}_* (\prod_\Gamma \tilde{\mathfrak{i}}_{0,\Gamma}^* \mathcal{N}_{\mathcal{X}_\Gamma^\perp}^{c_\Gamma-1})) \\ &= (-\psi_i)^{b_i-1} \cdot \sigma_i^* (\tilde{\mathfrak{i}}_* (\prod_\Gamma \tilde{\mathfrak{i}}_{0,\Gamma}^* \mathcal{N}_{\mathcal{X}_\Gamma^\perp}^{c_\Gamma-1})) \end{aligned}$$

$$\begin{aligned}
 &= (-\psi_i)^{b_i-1} \cdot i_* \left(\prod_{\Gamma} \sigma_i^* (\tilde{i}_{0,\Gamma}^* \mathcal{N}_{\mathcal{X}_{\Gamma}^{\pm}}^{c_{\Gamma}-1}) \right) \\
 &= (-\psi_i)^{b_i-1} \cdot i_* \left(\prod_{\Gamma} \tilde{i}_{0,\Gamma}^* \mathcal{N}_{\Gamma}^{c_{\Gamma}-1} \right),
 \end{aligned}$$

which is in $R_{f_g}^{\bullet}(\bar{B})$ by Theorem 7.1. If $b_i = 0$, the expression $\pi_* (\prod_{\Gamma} [\mathcal{X}_{\Gamma}^{\pm}]^{c_{\Gamma}})$ is the π_* -pushforward of a sum of tautological generators supported on nontrivial boundary strata, and we have already shown that they belong to $R_{f_g}^{\bullet}(\bar{B})$.

Since we have shown that $R_{f_g}^{\bullet}(\bar{B})$ is a subring of the tautological ring closed under clutching and π -pushforward, it has to be the same as the tautological ring by minimality.

We finally show the last statement of the theorem, namely that the i_{Γ^*} of the level-wise tautological classes $\xi_{\Gamma}^{[i]}$ and the κ -classes are tautological. For the ξ -classes, it is enough to notice that by Proposition 8.2, the class $\xi_{B_{\Gamma}^{[i]}}$ can be expressed as a linear combination of a ψ -class and boundary classes, so it is tautological on $B_{\Gamma}^{[i]}$ by the main statement of the theorem that we just proved. Since the tautological rings are closed under clutching morphisms, the class $\zeta_{\Gamma} p_{\Gamma}^{[i],*} \xi_{B_{\Gamma}^{[i]}}$ is also tautological. Notice that this is, up to constant, the same as $i_{\Gamma^*}(\xi_{\Gamma}^{[i]})$. Finally, the κ -classes are tautological since we have previously shown that they belong to $R_{f_g}^{\bullet}(\bar{B})$, which we have proven to be the same as the tautological ring. \square

9. The Chern classes of the logarithmic cotangent bundle

In this section, we relate the logarithmic cotangent bundle to bundles whose Chern classes can be expressed in standard generators. We will first prove in Theorem 9.2, a restatement of Theorem 1.4. We will then complete the proofs of the remaining main theorems of the introduction, Theorem 1.2 and Theorem 1.3.

The first step is a direct consequence of the Euler sequence in equation (37).

Corollary 9.1. *The Chern character and the Chern polynomial of the kernel \mathcal{K} of the Euler sequence are given by*

$$\text{ch}(\mathcal{K}) = Ne^{\xi} - 1 \quad \text{and} \quad c(\mathcal{K}) = \sum_{i=0}^{N-1} \binom{N}{i} \xi^i.$$

Proof. The result follows from the properties of the Chern character and the Chern polynomial, together with the fact that all higher Chern classes of the Deligne extension $\overline{\mathcal{H}}_{\text{rel}}^1$ vanish. Indeed, the Chern classes of a logarithmic sheaf are given in terms of symmetric polynomials of residues of the logarithmic connection (see [EV86, B3]), and the Deligne extension is defined such that all these terms are zero, since the residues are given by nilpotent matrices. (See also the discussion around [ACG11, Theorem 17.5.21].) \square

The second step relates the kernel of the Euler sequence to the vector bundle we are actually interested in. We will use the abbreviations

$$\mathcal{E}_B = \Omega_B^1(\log D) \quad \text{and} \quad \mathcal{L}_B = \mathcal{O}_{\bar{B}} \left(\sum_{\Gamma \in \text{LG}_1(B)} \ell_{\Gamma} D_{\Gamma} \right) \tag{66}$$

throughout in the sequel.

Theorem 9.2. *There is a short exact sequence of quasi-coherent $\mathcal{O}_{\bar{B}}$ -modules*

$$0 \longrightarrow \mathcal{E}_B \otimes \mathcal{L}_B^{-1} \longrightarrow \mathcal{K} \longrightarrow \mathcal{C} \longrightarrow 0, \tag{67}$$

where $\mathcal{C} = \bigoplus_{\Gamma \in \text{LG}_1(\mathbb{B})} \mathcal{C}_\Gamma$ is a coherent sheaf supported on the nonhorizontal boundary divisors, whose precise form is given in Lemma 9.4 below.

Proof. We start analysing the injection claimed in equation (67). As in Section 6, all local calculations happen on the finite covering charts of $\overline{\mathbb{P}\Xi\mathcal{M}}_{g,n}(\mu)$. At a generic point of a divisor D_Γ , the vector bundle $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$ is generated (using the notation of Case 2 of Section 6.2) by $\langle t^\ell d\tilde{c}_2^{[0]}, \dots, t^\ell d\tilde{c}_{N_0}^{[0]}, t^\ell dt/t, t^\ell d\tilde{c}_2^{[-1]}, \dots, t^\ell d\tilde{c}_{N_1}^{[-1]} \rangle$. It is hence obviously a subbundle of the kernel \mathcal{K} as given in equation (44). Similarly, at the intersection point of L divisors different from D_h , the vector bundle $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$ is generated by the elements $t_{[L]} d\tilde{c}_j^{[-i]}$ and $t_{[L]} dt_i/t_i$ for $j = 2, \dots, N_i$ and for $i = 0, \dots, L$, where we recall that $t_{[L]} = \prod_{i=1}^L t_i^{\ell_i}$ was introduced in equation (48). This is obviously a subbundle of \mathcal{K} as given in proof of Theorem 6.1. In the presence of a horizontal edge, this argument still works; see the form of the cokernel in Case 1 and Case 3 above. The precise form of \mathcal{C} is isolated in several lemmas below. \square

To start with the computation of \mathcal{C} , we will also need an infinitesimal thickening of the boundary divisor D_Γ , namely we define $D_{\Gamma,\bullet}$ to be its ℓ_Γ th thickening, the nonreduced substack of $\overline{\mathbb{P}\Xi\mathcal{M}}_{g,n}(\mu)$ defined by the ideal $\mathcal{I}_{D_\Gamma}^{\ell_\Gamma}$. We will factor the above inclusion using the notation

$$i_\Gamma = i_{\Gamma,\bullet} \circ j_{\Gamma,\bullet} : D_\Gamma \xrightarrow{j_{\Gamma,\bullet}} D_{\Gamma,\bullet} \xrightarrow{i_{\Gamma,\bullet}} \overline{B}.$$

We need three more bundles. First, we recall from equation (49) the definition of the line bundle \mathcal{L}_Γ^\top , and we define $\mathcal{L}_{\Gamma,\bullet}^\top = (j_{\Gamma,\bullet})_* \mathcal{L}_\Gamma^\top$. Second, we need the analogue of \mathcal{E}_B , but as a bundle on D_Γ . Since the projections are defined only on D_Γ^s rather than on D_Γ , we cannot define this bundle as a p^\top -pullback, but we need to define it by local generators. That is, we define \mathcal{E}_Γ^\top to be the vector bundle of rank $N_\Gamma^\top - 1$ on D_Γ with generators $dc_j^{[0]}$ as \mathcal{O}_{D_Γ} -module at a generic point of Γ with the usual coordinates from equation (39). At a point where the top level degenerates, into say k levels, it is generated as \mathcal{O}_{D_Γ} -module by the differentials $dc_j^{[-i]}$ of level-wise periods and by dt_i/t_i for $i = 0, \dots, k - 1$. Third, we define $\mathcal{E}_{\Gamma,\bullet}^\top = (j_{\Gamma,\bullet})_*(\mathcal{E}_\Gamma^\top)$.

Lemma 9.3. *There is an equality of Chern characters*

$$\text{ch}\left((i_{\Gamma,\bullet})_*(\mathcal{E}_{\Gamma,\bullet}^\top \otimes (\mathcal{L}_{\Gamma,\bullet}^\top)^{-1})\right) = \text{ch}\left((i_\Gamma)_*\left(\bigoplus_{j=0}^{\ell_\Gamma-1} \mathcal{N}_\Gamma^{\otimes -j} \otimes \mathcal{E}_\Gamma^\top \otimes (\mathcal{L}_\Gamma^\top)^{-1}\right)\right).$$

Proof. If \mathcal{F}_Γ is a vector bundle on D_Γ and $\mathcal{F}_{\Gamma,\bullet} = (i_{\Gamma,\bullet})_*(\mathcal{F}_\Gamma)$ is its pushforward to the ℓ_Γ -thickening, we consider the exact sequences

$$0 \rightarrow \mathcal{I}_{D_\Gamma}^{k+1} \mathcal{F}_{\Gamma,\bullet} \rightarrow \mathcal{I}_{D_\Gamma}^k \mathcal{F}_{\Gamma,\bullet} \rightarrow (j_{\Gamma,\bullet})_* \left(\frac{\mathcal{I}_{D_\Gamma}^k}{\mathcal{I}_{D_\Gamma}^{k+1}} \otimes_{\mathcal{O}_D} \mathcal{F}_\Gamma \right) \rightarrow 0, \quad k = 0, \dots, \ell_\Gamma - 1.$$

Notice that $\mathcal{I}_{D_\Gamma}^{\ell_\Gamma} \mathcal{F}_{\Gamma,\bullet} = 0$.

We specialise to $\mathcal{F}_\Gamma = \mathcal{E}_\Gamma^\top \otimes (\mathcal{L}_\Gamma^\top)^{-1}$ and compute the Chern character of its pushforward to the thickening via the previous sequences. The statement then follows from the identification $\mathcal{I}_{D_\Gamma}^k / \mathcal{I}_{D_\Gamma}^{k+1} = \mathcal{N}_\Gamma^{\otimes -k}$ and from the fact that $(i_{\Gamma,\bullet})_*$ is exact, since $i_{\Gamma,\bullet}$ is a closed embedding. \square

The cokernel of equation (67) can be described using the bundles we just introduced.

Lemma 9.4. *The cokernel of equation (67) is given by*

$$\mathcal{C} = \bigoplus_{\Gamma \in \text{LG}_1(\mathbb{B})} \mathcal{C}_\Gamma \quad \text{where} \quad \mathcal{C}_\Gamma = (i_{\Gamma,\bullet})_*(\mathcal{E}_{\Gamma,\bullet}^\top \otimes (\mathcal{L}_{\Gamma,\bullet}^\top)^{-1}). \tag{68}$$

Proof. Recall that local generators of \mathcal{K} had been given in the proof of Theorem 6.1. At a generic point of the boundary divisor D_Γ , there is a map of coherent sheaves $\mathcal{K} \rightarrow (i_{\Gamma, \bullet})_*(\mathcal{E}_\Gamma^\top \otimes (\mathcal{L}_\Gamma^\top, \bullet)^{-1})$ given in terms of the generators in equation (44) by $t^\ell dt/t \mapsto 0$, by $t^\ell d\tilde{c}_j^{[-1]} \mapsto 0$ and by $d\tilde{c}_j^{[0]} \mapsto d\tilde{c}_j^{[0]} \bmod t^\ell$ for all j . The kernel of this map is obviously $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$.

In a neighbourhood U of the intersection of L boundary divisors D_{Γ_i} , labeled so that Γ_i is the i th undegeneration, we recall the shorthand notation $t_{[s]} = \prod_{i=1}^s t_i^{\ell_i}$, and we assign for every level $-i \in \{0, \dots, L\}$

$$\begin{aligned} t_{[i]} dt_s/t_s &\mapsto t_{[i]} dt_s/t_s && \bmod t_{i+1}^{\ell_{i+1}}, && \in \mathcal{C}_{\Gamma_{i+1}} \\ t_{[i]} d\tilde{c}_j^{[-s]} &\mapsto t_{[i]} d\tilde{c}_j^{[-s]} && \bmod t_{i+1}^{\ell_{i+1}}, && \in \mathcal{C}_{\Gamma_{i+1}} \\ t_{[i]} dq_k^{[-s]}/q_k^{[-s]} &\mapsto t_{[i]} dq_k^{[-s]}/q_k^{[-s]} && \bmod t_{i+1}^{\ell_{i+1}}, && \in \mathcal{C}_{\Gamma_{i+1}} \end{aligned} \tag{69}$$

for all $s = 0, \dots, i$ and all j and k . Again, this map is designed so that the kernel is $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}|_U$. A local computation of transition functions shows that these maps glue together. \square

The proof of Theorem 9.2. is completed by the two preceding lemmas. \square

Proposition 9.5. *The Chern character of the twisted logarithmic cotangent bundle $\mathcal{E}_B \otimes \mathcal{L}_B^{-1}$ can be expressed in terms of the twisted logarithmic cotangent bundles of the top levels of nonhorizontal divisors as*

$$\text{ch}(\mathcal{E}_B \otimes \mathcal{L}_B^{-1}) = Ne^\xi - 1 - \sum_{\Gamma \in \text{LG}_1(\mathbb{B})} i_{\Gamma*} \left(\text{ch}(\mathcal{E}_\Gamma^\top) \cdot \text{ch}(\mathcal{L}_\Gamma^\top)^{-1} \cdot \frac{(1 - e^{-\ell_\Gamma c_1(\mathcal{N}_\Gamma)})}{c_1(\mathcal{N}_\Gamma)} \right).$$

Proof. First, by Corollary 9.1, we have $\text{ch}(\mathcal{K}) = Ne^\xi - 1$. Second, from the sequence in equation (67), we get

$$\text{ch}(\mathcal{E}_B \otimes \mathcal{L}_B^{-1}) = \text{ch}(\mathcal{K}) - \text{ch}(\mathcal{C}). \tag{70}$$

From the additivity of the Chern character, we get $\text{ch}(\mathcal{C}) = \oplus_{\Gamma \in \text{LG}_1(\mathbb{B})} \text{ch}(\mathcal{C}_\Gamma)$. We now aim to apply Lemma 9.3 and the Grothendieck-Riemann-Roch Theorem in equation (10) to the map $f = i_\Gamma$, a smooth embedding. The contribution of the Todd classes simplifies, since the normal bundle exact sequence

$$0 \rightarrow \mathcal{T}_{D_\Gamma} \rightarrow i_\Gamma^* \mathcal{T}_B \rightarrow \mathcal{N}_\Gamma \rightarrow 0$$

implies $\text{td}(\mathcal{T}_{D_\Gamma}) \cdot \text{td}(\mathcal{N}_\Gamma) = \text{td}(i_\Gamma^* \mathcal{T}_B) = i_\Gamma^* \text{td}(\mathcal{T}_B)$. If \mathcal{F}_Γ is a vector bundle on D_Γ , we can thus simplify (10) and get

$$\begin{aligned} \text{ch}(i_{\Gamma,*} \mathcal{F}_\Gamma) &= i_{\Gamma,*} (\text{ch}(\mathcal{F}_\Gamma) \cdot \text{td}(\mathcal{T}_{D_\Gamma})) \cdot \text{td}(\mathcal{T}_B)^{-1} = i_{\Gamma,*} (\text{ch}(\mathcal{C}) \cdot \text{td}(\mathcal{T}_{D_\Gamma}) \cdot i_\Gamma^* \text{td}(\mathcal{T}_B)^{-1}) \\ &= i_{\Gamma,*} (\text{ch}(\mathcal{F}_\Gamma) \cdot \text{td}(\mathcal{N}_\Gamma)^{-1}). \end{aligned}$$

Using the previous remark and Lemma 9.3, we get

$$\begin{aligned} \text{ch}(\mathcal{C}_\Gamma) &= (i_\Gamma)_* \left(\sum_{j=0}^{\ell_\Gamma-1} \text{ch}(\mathcal{E}_\Gamma^\top) \cdot \text{ch}(\mathcal{L}_\Gamma^\top)^{-1} \cdot \text{ch}(\mathcal{N}_\Gamma)^{-j} \text{td}([\mathcal{N}_\Gamma]^{-1}) \right) \\ &= \sum_{j=0}^{\ell_\Gamma-1} i_{\Gamma*} \left(\text{ch}(\mathcal{E}_\Gamma^\top) \cdot \text{ch}(\mathcal{L}_\Gamma^\top)^{-1} \cdot \frac{e^{-j c_1(\mathcal{N}_\Gamma)} (1 - e^{-c_1(\mathcal{N}_\Gamma)})}{c_1(\mathcal{N}_\Gamma)} \right). \end{aligned}$$

Cancelling terms in the telescoping sum and substituting back the previous expression in equation (70) gives the proposition. \square

From this proposition, we get some concrete expansions.

Proof of Theorem 1.1. Since the first Chern character is the same as the first Chern class, by extracting the first-degree parts from the expression given in Proposition 9.5, we compute the left-hand side to be

$$\text{ch}_1(\mathcal{E}_B \otimes \mathcal{L}_B^{-1}) = c_1(\mathcal{E}_B) + (N - 1) \sum_{\Gamma \in \text{LG}_1(B)} \ell_\Gamma [D_\Gamma]$$

and the right-hand side to be

$$N\xi - \sum_{\Gamma \in \text{LG}_1(B)} \ell_\Gamma \mathbf{i}_{\Gamma,*}((N_\Gamma^\top - 1)[1_{D_\Gamma}]) = N\xi - \sum_{\Gamma \in \text{LG}_1(B)} \ell_\Gamma (N_\Gamma^\top - 1)[D_\Gamma].$$

By comparing the two expressions, we get the claim. □

To translate Proposition 9.5 into a formula that can be recursively evaluated, we compare the bundle \mathcal{E}_Γ^\top to the analogous object

$$\mathcal{E}_{B_\Gamma^\top} = \Omega_{B_\Gamma^\top}^1(\log D_{B_\Gamma^\top})$$

on the top level of the divisor D_Γ for $\Gamma \in \text{LG}_1(B)$, where $D_{B_\Gamma^\top}$ is the total boundary of the generalised stratum B_Γ^\top , including the horizontal divisor.

Lemma 9.6. *We have*

$$p_\Gamma^{\top,*} \mathcal{E}_{B_\Gamma^\top} = c_\Gamma^* \mathcal{E}_\Gamma^\top. \tag{71}$$

Proof. The statement can be checked on the local generators. Indeed, recall that the generators of \mathcal{E}_Γ^\top as introduced before Lemma 9.3 are $dc_j^{[0]}$ at a generic point of D_Γ and $dc_j^{[-i]}$ and dt_i/t_i for $i = 0, \dots, k - 1$. Note that even though the map c_Γ is branched at the preimage of $\{t_i = 0\}$, say given by $\{\tilde{t}_i = 0\}$, the pullback of the standard generators dt_i/t_i of the log cotangent bundle are proportional to the standard generators $d\tilde{t}_i/\tilde{t}_i$. We can apply the same argument for the finite-degree map $p^\top \times p^\perp$ and check that the pullback of the local generators of $\mathcal{E}_{B_\Gamma^\top}$ coincide with the previous ones. □

For the inductive proof, we introduce the following shorthand notation. Let

$$P_B = \text{ch}(\mathcal{E}_B) \prod_{\Gamma \in \text{LG}_1(B)} e^{-\ell_\Gamma [D_\Gamma]} \quad \text{and} \quad P_{B_\Gamma^\top} = \text{ch}(\mathcal{E}_{B_\Gamma^\top}) \prod_{\Delta \in \text{LG}_1(B_\Gamma^\top)} e^{-\ell_\Delta [D_\Delta]}$$

be the Chern characters of the logarithmic cotangent bundles twisted by a boundary contribution, and let

$$P_\Gamma^\top = \text{ch}(\mathcal{E}_\Gamma^\top) \cdot \text{ch}(\mathcal{L}_\Gamma^\top)^{-1} = \text{ch}(\mathcal{E}_\Gamma^\top) \prod_{\Gamma \rightsquigarrow \tilde{\Delta}^{[0]}} e^{-\ell_{\tilde{\Delta},1} [D_\Delta]}. \tag{72}$$

In these terms, Proposition 9.5 reads

$$P_B = (Ne^\xi - 1) - \sum_{\Gamma \in \text{LG}_1(B)} \mathbf{i}_{\Gamma,*} \left(\ell_\Gamma P_\Gamma^\top \text{td}(\mathcal{N}_\Gamma^{\otimes \ell_\Gamma})^{-1} \right). \tag{73}$$

We set $\delta_{L+1}(\Gamma) = \{\cdot\}$, the only graph with one level corresponding to the open stratum B , for $\Gamma \in \text{LG}_L(B)$, to make boundary terms well-defined in the sequel. In particular, $N_{\delta_{L+1}}^\top(\Gamma) = N$.

Proposition 9.7. *The twisted Chern character P_B is given by*

$$P_B = \sum_{L=0}^{N-1} \sum_{\Gamma \in \text{LG}_L(B)} (N_{\delta_1(\Gamma)}^\top e^{\xi_B} - 1) \mathbf{i}_{\Gamma^*} \left(\prod_{i=1}^L -\ell_{\Gamma,i} \text{td} \left(\mathcal{N}_{\Gamma/\delta_i^C(\Gamma)}^{\otimes \ell_{\Gamma,i}} \right)^{-1} \right). \tag{74}$$

Proof. We prove the formula by induction on N . For one-dimensional strata ($N = 2$), the formula is equation (73) since P_Γ^\top is trivial then. We claim that by induction hypothesis

$$P_\Gamma^\top = \sum_{L=0}^{N-2} \sum_{\substack{\widehat{\Delta} \in \text{LG}_{L+1}(B) \\ \delta_{L+1}(\widehat{\Delta}) = \Gamma}} (N_{\delta_1(\Gamma)}^\top e^{\xi_B|_{D_\Gamma}} - 1) \mathbf{i}_{\widehat{\Delta}, \Gamma^*} \left(\prod_{i=1}^L -\ell_{\widehat{\Delta},i} \text{td} \left(\mathcal{N}_{\widehat{\Delta}/\delta_i^C(\widehat{\Delta})}^{\otimes \ell_{\widehat{\Delta},i}} \right)^{-1} \right) \tag{75}$$

holds in $\text{CH}^*(D_\Gamma)$. We insert this formula into equation (73). Note that for the degeneration of arbitrary codimension appearing in equation (75), we have

$$j_{\widehat{\Delta}, \Gamma}^* c_1(\mathcal{N}_\Gamma^{\otimes \ell_\Gamma}) = c_1 \left(\mathcal{N}_{\widehat{\Delta}/\delta_{L+1}^C(\widehat{\Delta})}^{\otimes \ell_{\widehat{\Delta}, L+1}} \right) \tag{76}$$

by splitting the degeneration into codimension one degenerations and applying successively Corollary 7.7 in the case $\delta_{L+1}(\widehat{\Delta}) = \Gamma$. An application of the push-pull formula now gives the expression in the proposition.

To prove the claim, note that the induction hypothesis directly implies that

$$P_{B_\Gamma^\top} = \sum_{L=0}^{N-2} \sum_{\Delta \in \text{LG}_L(B_\Gamma^\top)} (N_{\delta_1(\Delta)}^\top e^{\xi_{B_\Gamma^\top}} - 1) \mathbf{i}_{\Delta^*} \left(\prod_{i=1}^L -\ell_{\Delta,i} \text{td} \left(\mathcal{N}_{\Delta/\delta_i^C(\Delta)}^{\otimes \ell_{\Delta,i}} \right)^{-1} \right) \tag{77}$$

in $\text{CH}^*(B_\Gamma^\top)$. We now pull back this equation and our claimed equation to D_Γ^s and compare. Agreement in D_Γ^s implies the claim, since we are working with rational Chow groups throughout. The agreement follows from the comparison of the normal bundles in the argument of the Todd classes, which in turn is a consequence of the comparison results in Proposition 4.9 and equation (59). \square

Corollary 9.8. *The Chern character of the logarithmic cotangent bundle is*

$$\text{ch}(\mathcal{E}_B) = \sum_{L=0}^{N-1} \sum_{\Gamma \in \text{LG}_L(B)} (N_{\delta_1(\Gamma)}^\top e^{\xi_B} - 1) \mathbf{i}_{\Gamma^*} \left(e^{\mathcal{L}_\Gamma} \prod_{i=1}^L -\ell_{\Gamma,i} \text{td} \left(\mathcal{N}_{\Gamma/\delta_i^C(\Gamma)}^{\otimes -\ell_{\Gamma,i}} \right)^{-1} \right),$$

where $\mathcal{L}_\Gamma = \sum_{i=0}^{-L} \mathcal{L}_\Gamma^{[i]}$.

The subsequent simplifications of this formula are based on the following observation. Suppose that $\Gamma \mapsto a_\Gamma$ is an assignment of a rational number to every level graph $\Gamma \in \text{LG}_L(B)$ for every L with the property that if $L > 1$. Then

$$a_\Gamma = \prod_{i=1}^L a_{\delta_i(\Gamma)} \tag{78}$$

is the product of those numbers over all undegenerations to two-level graphs. We use the abbreviation $a_{\Gamma,i} = a_{\delta_i(\Gamma)}$.

Lemma 9.9. For a collection of a_Γ satisfying equation (78), the identity

$$\exp\left(\sum_{\Gamma \in \text{LG}_1(\overline{B})} a_\Gamma [D_\Gamma]\right) = 1 + \sum_{L=1}^{N-1} \sum_{\Gamma \in \text{LG}_L(\overline{B})} a_\Gamma \mathbf{i}_{\Gamma,*} \left(\prod_{i=1}^L \text{td}(\mathcal{N}_{\Delta/\delta_i^c(\Delta)}^{\otimes -a_{\Gamma,i}})\right)^{-1}$$

holds in $\text{CH}^*(\overline{B})$.

Proof. The proof shows that this equality holds if we restrict to any subset $S \subset \text{LG}_1(\overline{B})$ on the left-hand side and if we restrict on the right-hand side to the sum of those $\Gamma \in \text{LG}_L(\overline{B})$ such that all their two-level undegenerations belong to S . The proof now proceeds by induction over $|S|$.

For $|S| = 1$, this is the identity $\exp(a_\Gamma [D_\Gamma]) = 1 + a_\Gamma \mathbf{i}_{\Gamma,*}(\text{td}(\mathcal{N}_\Gamma^{\otimes -a_\Gamma})^{-1})$ that follows from the adjunction formula $\mathbf{i}_\Gamma^* \mathbf{i}_{\Gamma,*} \alpha = c_1(\mathcal{N}_\Gamma) \cdot \alpha$ and the relation between the generating series of the exponential and the Todd class.

For $|S| \geq 2$, this follows from the uniqueness of the intersection orders shown in Proposition 5.1 and induction. We give details for $|S| = 2$, leaving it to the reader to set up the notation for the general case. Let $\Gamma_k \in \text{LG}_1(\overline{B})$ for $k = 1, 2$, and abbreviate $D_k = D_{\Gamma_k}$, $\mathcal{N}_k = c_1(\mathcal{N}_{\Gamma_k})$, $\mathbf{i}_k = \mathbf{i}_{\Gamma_k}$ and $\mathbf{j}_k = \mathbf{j}_{\Delta, \Gamma_k}$ for any graph Δ with $\delta_k(\Delta) = \Gamma_k$ for $k = 1, 2$. We denote by $[1, 2]$ the set of such 3-level graphs Δ . Then by equation (76),

$$\begin{aligned} \sum_{\Delta \in [1,2]} \mathbf{i}_{\Delta,*} \left(c_1(\mathcal{N}_{\Delta/\delta_1^c(\Delta)})^{x-1} c_1(\mathcal{N}_{\Delta/\delta_2^c(\Delta)})^{y-1}\right) &= \sum_{\Delta \in [1,2]} \mathbf{i}_{\Delta,*} (\mathbf{i}_1^* \mathcal{N}_1^{x-1} \mathbf{i}_2^* \mathcal{N}_2^{y-1}) \\ &= \sum_{\Delta \in [1,2]} \mathbf{i}_{1,*} (\mathbf{i}_1^* ([D_1]^{x-1} \mathbf{i}_{1,*} \mathbf{i}_2^* \mathcal{N}_2^{y-1})) = [D_1]^{x-1} \cdot \sum_{\Delta \in [1,2]} \mathbf{i}_{2,*} \mathbf{i}_{2,*} \mathbf{i}_2^* \mathcal{N}_2^{y-1} \\ &= [D_1]^x \cdot \mathbf{i}_{2,*} \mathcal{N}_2^{y-1} = [D_1]^x \cdot [D_2]^y. \end{aligned}$$

Taking the generating series over this expression proves the claim. □

Proof of Theorem 1.2. To deduce this theorem from Corollary 9.8, we introduce shorthand notations for the products of inverse Todd classes: namely for any $\Gamma \in \text{LG}_L(\overline{B})$, we let

$$X_{\Gamma,i} = \text{td}\left(\mathcal{N}_{\Gamma/\delta_i^c(\Gamma)}^{\otimes -\ell_{\Gamma,i}}\right)^{-1} \quad \text{and} \quad X_\Gamma = \prod_{i=1}^L X_{\Gamma,i}, \tag{79}$$

and

$$X_{\Delta \setminus \Gamma} = \prod_{i \in I^c} \text{td}\left(\mathcal{N}_{\Gamma/\delta_i^c(\Gamma)}^{\otimes -\ell_{\Gamma,i}}\right)^{-1}$$

if $\Gamma = \delta_I(\Delta)$ is the undegeneration keeping only the level passages in I of Δ . Now the argument of Lemma 9.9 with ℓ_Γ playing the role of a_Γ and with both sides restricted to degenerations of a fixed $\Gamma \in \text{LG}_L(\overline{B})$ gives

$$\exp(\mathcal{L}_\Gamma) = \exp\left(\sum_{\Gamma \in \text{LG}_{L+1}^+(\overline{B})} \ell_\Gamma [D_\Gamma]\right) = 1 + \sum_{L'=L+1}^{N-1} \sum_{\Delta \in \text{LG}_{L'}^+(\overline{B})} \ell_\Delta \mathbf{i}_{\Delta, \Gamma,*} (X_{\Delta \setminus \Gamma}),$$

where $\text{LG}_{L'}^+(\overline{B})$ are the graphs with L' levels below zero that are degenerations of Γ . We inject this formula into the right-hand side of Corollary 9.8. Since

$$\mathbf{i}_{\Gamma,*} (\mathbf{i}_{\Delta, \Gamma,*} (X_{\Delta \setminus \Gamma}) \cdot X_\Gamma) = \mathbf{i}_{\Delta,*} (X_\Delta),$$

by the projection formula and equation (76), we obtain

$$\text{ch}(\mathcal{E}_B) = \sum_{L=0}^{N-1} (-1)^L \sum_{\Gamma \in \text{LG}_L(B)} \left(N_{\delta_I(\Gamma)}^\top e^{\xi_B} - 1 \right) \sum_{L'=L}^{N-1} \sum_{\Delta \in \text{LG}_{L'}(\bar{B})} \ell_\Delta \mathbf{i}_{\Delta,*}(X_\Delta).$$

It remains to sort this expression as a sum over $\ell_\Delta \mathbf{i}_{\Delta,*}(X_\Delta)$. Since each $\Delta \in \text{LG}_{L'}(B)$ appears in the expression of each Γ with $\delta_I(\Delta) = \Gamma$, its coefficient in the final expression of $\text{ch}(\mathcal{E}_B)$ is (defining $\min(\{\emptyset\}) = L' + 1$)

$$\begin{aligned} \sum_{I \subseteq \{1, \dots, L'\}} (-1)^{|I|} \cdot \left(N_{\delta_{\min(I)}(\Delta)}^\top e^{\xi_B} - 1 \right) &= e^{\xi_B} \cdot \sum_{I \subseteq \{1, \dots, L'\}} (-1)^{|I|} N_{\delta_{\min(I)}(\Delta)} \\ &= e^{\xi_B} \cdot \left(N - N_{\delta_{L'}(\Gamma)}^\top \right), \end{aligned}$$

where the disappearance of $(-1)^{|I|+1}$ in the first equality and the cancellation in the second equality stem from canceling the contributions of pairs under the involution $I \mapsto I \cup \{L'\}$ if $L' \notin I$ and $I \mapsto I \setminus \{L'\}$, if $L' \in I$. □

In preparation for the next theorem, we switch to the language of profiles introduced in Section 5 and recall that the notation depends on the choice of the numbering of $\text{LG}_1(B) = \{\Gamma_1, \dots, \Gamma_M\}$. We claim that Theorem 1.2 can equivalently be restated as

$$\text{ch}(\mathcal{E}_B) = e^{\xi_B} \cdot \sum_{L=0}^{N-1} \sum_{[j_1, \dots, j_L] \in \mathcal{P}_L} (N - N_{j_L}^\top) \prod_{i=1}^L \left(e^{\ell_{j_i} [D_{j_i}]} - 1 \right), \tag{80}$$

where \mathcal{P}_L is the set of profiles of length L . To see the equivalence, it suffices to expand the product in equation (80) and to use Proposition 5.1 about the uniqueness of the order of letters in a profile. Note that we cannot replace \mathcal{P}_L with $\text{LG}_L(B)$ in equation (80), as this would give wrong multiplicities.

We abbreviate the difference of dimensions $r_{\Gamma,i} = N - N_{\delta_i(\Gamma)}^\top$ and write $r_\Gamma = \prod_{i=1}^L r_{\Gamma,i}$. It is useful to remember that $r_{\Gamma,i} = \sum_{j=i+1}^L N^{[-j]} = \sum_{j=i+1}^L (d^{[-j]} + 1)$ is the sum of the unprojectivised dimensions of the lower levels. If we work with profiles and the elements of $\text{LG}_1(B)$ are numbered, we write $r_j = r_{\Gamma_j}$ and $\ell_i = \ell_{\Gamma_i}$. We can now state an additive and a multiplicative decomposition of the Chern polynomial.

Theorem 9.10. *The Chern polynomial of the logarithmic cotangent bundle is*

$$\begin{aligned} c(\mathcal{E}_B) &= \prod_{L=0}^{N-1} \prod_{[j_1, \dots, j_L] \in \mathcal{P}_L} \prod_{I \subseteq \{1, \dots, L\}} \left(1 + \xi + \sum_{i \in I} \ell_{j_i} [D_{j_i}] \right)^{(-1)^{|I|} c_{1,r_{j_L}}} \\ &= \sum_{L=0}^{N-1} \sum_{\Gamma \in \text{LG}_L(B)} \ell_\Gamma \mathbf{i}_{\Gamma,*} \left(\sum_{\mathbf{k}} \binom{N - \sum_{i=1}^L k_i}{k_0} \xi^{k_0} \cdot \prod_{i=1}^L \binom{r_{\Gamma,i} - \sum_{j>i} k_j}{k_i} (\ell_i v_{\Gamma,i})^{k_i-1} \right), \end{aligned} \tag{81}$$

where $\mathbf{k} = (k_0, k_1, \dots, k_L)$ is a tuple with $k_0 \geq 0$ and $k_i \geq 1$ for $i = 1, \dots, L$ and where $v_{\Gamma,i} = c_1(\mathcal{N}_{\Gamma/\delta_i^c(\Gamma)})$. For $L = 0$, the exponent r_{j_L} is to be interpreted as N .

Proof. We deduce the first line from equation (80). We compute the degree- d -part of its interior product to be

$$\left[e^{\xi_B} \prod_{i=1}^L \left(e^{\ell_{j_i} [D_{j_i}]} - 1 \right) \right]_d = \frac{1}{(d-1)! \cdot d} \sum_{I \subseteq \{1, \dots, L\}} (-1)^{|I|} c_I \left(\xi + \sum_{i \in I} [D_{j_i}] \right)^d.$$

On the other hand, recall from [ACG11, p. 586] that the Chern polynomial is given in terms of the graded pieces of the Chern character by

$$c(\mathcal{E}_B) = \exp\left(\sum_{d \geq 1} (-1)^{d-1} (d-1)! \text{ch}_d(\mathcal{E}_B)\right).$$

Using the generating series of the logarithmic function, we then obtain the first line of the statement by combining the previous two expressions.

To pass to the second line, we show that the first line formally fits with Lemma 9.11 below. We want to replace the two exterior products over all L and profiles \mathcal{P}_L by all subsets of the integer interval $[[1, \dots, M]]$ without altering the value of the product. For this purpose, we claim that for each element of \mathcal{P}_L the interior product

$$P = \prod_{I \subseteq \{1, \dots, L\}} \left(1 + \xi + \sum_{i \in I} \ell_{j_i} [D_{j_i}]\right)^{(-1)^{|I|} c_{1, r_{j_L}}}$$

considered as an element in the polynomial ring is in $1 + D_1 \cdots D_L \cdot \mathbb{Q}[\xi, D_1, \dots, D_L]$. This claim implies that the additional products give zero in the Chow ring and considering the profiles as subsets of $[[1, \dots, M]]$ rather than as ordered tuples is no loss of information thanks to Proposition 5.1. To justify the claim, we may assume $r_{j_L} = 1$, since the claim persists when raising to an integral power. For $L = 1$, the claim is obvious; and for the inductive step, one replaces ξ successively with $\xi + \ell_{j_k} D_k$ to see that $P - 1$ is divisible by D_i for all $i \neq k$.

Now we are in the situation to apply the image of the formula of Lemma 9.11 in the Chow ring. To match the second line of the lemma and the theorem, we define, for a tuple $\mathbf{k} = (k_0, k_1, \dots, k_M)$ as in the lemma, the integer L to be the number of entries k_i that are positive. Consider a summand $\mathbf{k} = (k_0, k_1, \dots, k_M)$ in the second line of the statement of Lemma 9.11, and say that i_1, \dots, i_L are those indices where the entries k_{i_j} are positive. Then the contribution of this summand to the second line of equation (82) equals the contributions of the (possibly empty) set of level graphs in $D_{i_1} \cap \dots \cap D_{i_L}$ to the second line of equation (81). \square

Lemma 9.11. *In the polynomial ring $\mathbb{Q}[\xi, D_1, \dots, D_M]$, the identity*

$$\begin{aligned} & \prod_{[j_1, \dots, j_L] \subseteq \{1, \dots, M\}} \prod_{I \subseteq \{1, \dots, L\}} \left(1 + \xi + \sum_{i \in I} \ell_{j_i} D_{j_i}\right)^{(-1)^{|I|} c_{1, N_{j_L}}} \\ &= \sum_{\mathbf{k}} \binom{N - \sum_{i=1}^M k_i}{k_0} \xi^{k_0} \cdot \prod_{i=1}^M \binom{\sum_{j \geq i} N^{[-j]} - \sum_{j > i} k_j}{k_i} (\ell_i D_i)^{k_i} \end{aligned} \tag{82}$$

holds, where $\mathbf{k} = (k_0, k_1, \dots, k_M)$ is a tuple of nonnegative integers and $N_s := \sum_{j=s+1}^M N^{[-j]}$ and $N = N_0 = \sum_{j=0}^M N^{[-j]}$.

Proof. We proceed by induction on M . The case $M = 0$ is the binomial expansion. The effect of the passage from $M - 1$ to M is given on the left-hand side by replacing $N^{[-(M-1)]}$ with $N^{[-(M-1)]} + N^{[-M]}$ in all those factors where $j_L < M$ and by multiplying by the factors where $j_L = M$: that is, by multiplication with

$$\begin{aligned} & \prod_{\substack{[j_1, \dots, j_{L-1}] \subseteq \{1, \dots, M-1\} \\ I \subseteq \{1, \dots, L-1\}}} \left(1 + \xi + D_M + \sum_{i \in I} \ell_{j_i} D_{j_i}\right)^{(-1)^{|I|} c_{1, N^{[M]}}} \\ &= \sum_{k_M \geq 0} \binom{r_M}{k_M} D^{k_M} \cdot \prod_{\substack{[j_1, \dots, j_{L-1}] \subseteq \{1, \dots, M-1\} \\ I \subseteq \{1, \dots, L-1\}}} \left(1 + \xi + \sum_{i \in I} \ell_{j_i} D_{j_i}\right)^{(-1)^{|I|} c_{1, (N^{[M]} - k_M)}}. \end{aligned}$$

Applying the induction hypothesis with $N^{[-(M-1)]}$ replaced by $N^{[-(M-1)]} + N^{[-M]} - k_M$ gives the claim. □

The following step concludes the proof of all main theorems.

Lemma 9.12. *Suppose that $\alpha_\Gamma \in \text{CH}_0(D_\Gamma)$ is a top-degree class and $c_\Gamma^* \alpha_\Gamma = \prod_{i=0}^{-L(\Gamma)} p_\Gamma^{[i],*} \alpha_i$ for some α_i . Then*

$$\int_{D_\Gamma} \alpha_\Gamma = \frac{K_\Gamma}{|\text{Aut}(\Gamma)| \ell_\Gamma} \prod_{i=0}^{-L(\Gamma)} \int_{B_\Gamma^{[i]}} \alpha_i.$$

Proof. We have

$$\int_{D_\Gamma} \alpha_\Gamma = \frac{1}{\text{deg}(c_\Gamma)} \int_{D_\Gamma^s} c_\Gamma^*(\alpha_\Gamma) = \frac{\text{deg}(p_\Gamma)}{\text{deg}(c_\Gamma)} \prod_{i=0}^{-L(\Gamma)} \int_{B_\Gamma^{[i]}} \alpha_i,$$

and the claim follows from Lemma 4.5. □

Proof of Theorem 1.3. By Proposition 2.1, it is enough to compute the top Chern class $c_d(\mathcal{E}_B)$, where $d = \dim(B) = N - 1$. We investigate for each L and each $\Gamma \in \text{LG}_L(B)$ the contribution of the second line of equation (81) in Theorem 9.10 to $c_d(\mathcal{E}_B)$. It suffices then to show that the expression inside the $\mathfrak{i}_{\Gamma,*}$ is equal to $N_\Gamma^\top \prod_{i=0}^{L-1} (\xi_\Gamma^{[i]})^{d^{[i]}}$. Note that by Proposition 7.5 the first Chern class of the normal bundle $\mathcal{N}_{\Gamma/\delta_i^c(\Gamma)}$ is supported on the levels $-i + 1$ and $-i$ of Γ . Considering the bottom level, we deduce that if the summand \mathbf{k} contributes nontrivially to the top Chern class c_d , then we must have $k_L \geq d^{[L]} + 1$ so that the $v_{\Gamma,L}$ -power is large enough for its binomial expansion to contain a top ξ -power for the bottom level. On the other hand, for the binomial coefficient in front of it to be nonzero, we need $r_{\Gamma,L} \geq k_L$, which is equivalent to $k_L \leq d^{[L]} + 1$. So $k_L = d^{[L]} + 1$, the binomial coefficient is one and we have to select from the expansion of $v_{\Gamma,L}^{d^{[L]}}$ the term that does not contribute to level $-i - 1$. Since the top entry of the binomial coefficient is $r_{\Gamma,i} - \sum_{j>i}^L k_j = 1 + N_i + \sum_{j>i}^L (d^{[j]} + 1 - k_j)$, we can inductively repeat this argument for all levels and deduce $k_j = d^{[j]} + 1$ for all $j \geq 1$ and $k_0 = d^{[0]}$. The only nontrivial factor is now N_Γ^\top , which stems from the first binomial coefficient in the second line of equation (81). The final shape of the statement follows then directly from Proposition 4.9 and Lemma 9.12, after noticing that the ℓ_Γ coefficients cancel. □

10. Examples: Geometry and values

In this section, we explain how to evaluate top-degree classes. We provide examples illustrating the geometry at the boundary of the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and examples of our formulas for the normal bundles, the Chern polynomials and the Euler characteristic.

10.1. Evaluation of top ξ -powers

First we explain how to evaluate the expression in Theorem 1.3; see [CMZ20] for many algorithmic details. We only need to explain how to evaluate $\int_B \xi^d$: that is, top powers of ξ on generalised strata.

Suppose that $B = \mathbb{P}\Omega\mathcal{M}_{g,1}(2g - 2)$ is a stratum parametrising connected surfaces with a single zero. Then the generating series of top ξ -powers is given by a simple power series inversion that arises in the computation of Masur-Veech volumes; see [Sau18] and [CMSZ20, Theorem 3.1] and Table 2 for some values.

Table 2. Integrals of top ξ -powers for some connected strata.

| μ | (0) | (2) | (4) | (6) | (0, 0, -2) | (2, -2) |
|-------------------------------------|----------------|--------------------|-----------------------|----------------------------|------------------|--------------------|
| $\int_{\overline{B}} \xi^{\dim(B)}$ | $\frac{1}{24}$ | $-\frac{1}{640}$ | $-\frac{305}{580608}$ | $-\frac{87983}{199065600}$ | 1 | $-\frac{1}{8}$ |
| μ | (1, 1, -2) | (4, -2) | (3, 1, -2) | (2, 1, -3) | (5, -3) | (8, -2, -2, -2) |
| $\int_{\overline{B}} \xi^{\dim(B)}$ | 0 | $-\frac{23}{1152}$ | 0 | $\frac{5}{8}$ | $-\frac{21}{20}$ | $-\frac{4527}{32}$ |

Suppose that $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is a stratum parametrising connected surfaces of holomorphic type: that is, with all $m_i \geq 0$ and with $n \geq 2$. Then $\int_B \xi^d = 0$ by [Sau18, Proposition 3.3].

It remains to explain how to evaluate $\int_{\overline{B}} \xi^d$, top powers of ξ on meromorphic generalised strata. First we write ξ with the help of Proposition 8.2 as a ψ -class and boundary strata. The product of such objects, which are standard generators of the tautological ring as defined in 6, can be rewritten as a sum of standard additive generators via the algorithm explained in the proof of Theorem 1.5, more specifically in the part in which we show that $R_{f_g}^*(\overline{B}) = R^*(\overline{B})$ is a ring. Now that we have rewritten ξ^d in terms of standard additive generators, by Lemma 9.12, it only remains to explain how to evaluate a top-dimensional standard generator – that is, the top power of a ψ -class – on a generalised stratum \overline{B} . Since ψ -classes are pulled back from $\overline{\mathcal{M}}_{g,n}$, we can use a push-pull argument and express

$$\int_{\overline{B}} \psi_i^d = \int_{\overline{\mathcal{M}}_{g,n}} \pi_*([\overline{B}])\psi_i^d,$$

where $\pi : \overline{B} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism and we use as always the abuse of notation $\psi_i = \pi^*(\psi_i)$.

If we can express the class $\pi_*([\overline{B}])$ in terms of the standard generators of $\overline{\mathcal{M}}_{g,n}$, we can use the sage package `admcycles` to obtain a number.

If \overline{B} is a stratum parametrising meromorphic differentials on connected surfaces without residue conditions, the class $\pi_*([\overline{B}])$ was computed in [Sau19] and [BHPSS20], and the algorithmic task can be performed again by the sage package `admcycles`, which implements the algorithm based on the formula in [Sch18] and [BHPSS20].

If the stratum \overline{B} more generally parametrises differentials on disconnected surfaces and with residue conditions, we first use Proposition 8.3 repetitively to write the class of \overline{B} into the associated stratum without residue conditions in terms of additive generators of the stratum with no conditions. We then reduce to the computation of the class $\pi_*([\overline{B}])$ in the case that \overline{B} has no more residue conditions but is potentially disconnected. If \overline{B} is disconnected, then $\pi_*([\overline{B}])$ is zero. Since we can scale the differentials on the components independently, the fibre dimension to a product of $\overline{\mathcal{M}}_{g_i,n_i}$ is positive, and by definition of pushforward, we get the zero class.

For the subsequent examples, we present some cases where we can directly evaluate the top ξ -power for meromorphic strata in genus 0 and genus 1.

Proposition 10.1. *The integrals of the top ξ -power are given ($a_i, k \geq 0$)*

$$\text{for } B = \mathbb{P}\Omega_{0,n+1} \left(-2 - \sum_{i=1}^n a_i, a_1, \dots, a_n \right) \text{ by } \int_{\overline{B}} \xi_B^{n-2} = \left(-1 - \sum_{i=1}^n a_i \right)^{n-2},$$

$$\text{for } B = \mathbb{P}\Omega_{1,2}(-k, k) \text{ by } \int_{\overline{B}} \xi_B = -\frac{(k-1)(k^2-1)}{24},$$

$$\text{for } B = \mathbb{P}\Omega_{1,3}(-k-1, 1, k) \text{ by } \int_{\overline{B}} \xi_B^2 = \frac{(k^4-1)}{24}.$$

Proof. The first statement follows easily from Proposition 8.2, which in this case implies $\xi_{\bar{B}} = (-1 - \sum_{i=1}^n a_i)\psi_1$. Indeed, there cannot be any boundary divisors that have the pole on the lower level. Hence

$$\int_{\bar{B}} \xi_{\bar{B}}^{n-2} = \left(-1 - \sum_{i=1}^n a_i\right)^{n-2} \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_1^{n-2} = \left(-1 - \sum_{i=1}^n a_i\right)^{n-2}.$$

The second statement follows immediately as the previous one, since again there cannot be boundary divisors where the pole is on the lower level. Hence

$$\int_{\bar{B}} \xi = -(k-1) \int_{\overline{\mathcal{M}}_{1,2}} \pi_*([\bar{B}])\psi_1 = -\frac{(k-1)(k^2-1)}{24},$$

where we used [CC14, Proposition 3.1] for the computation of $\pi_*([\bar{B}])$.

For the proof of the last statement, notice that there can be only one nonhorizontal boundary divisor D_3 that has the pole on the lower level (see Section 10.3 for the full boundary description). Using Proposition 8.2, we find then $\xi_{\bar{B}} = -k\psi_1 - D_3$, which yields

$$\begin{aligned} \int_{\bar{B}} \xi^2 &= - \int_{\bar{B}} \xi_{\bar{B}}(D_3 + k\psi_1) = \left(-1/24 + \int_{\bar{B}} k\psi_1(D_3 + k\psi_1)\right) \\ &= \left(-1/24 + k^2 \int_{\overline{\mathcal{M}}_{1,3}} \pi_*([\bar{B}]) \cdot \psi_1^2\right) = (k^4 - 1)/24 \end{aligned}$$

again by the computation in [CC14, Proposition 3.1] of the class $\pi_*([\bar{B}])$ of the stratum in $\overline{\mathcal{M}}_{1,3}$. □

10.2. The minimal stratum $\mathbb{P}\Omega\mathcal{M}_{2,1}(2)$

This stratum is small enough that we can show all the level graphs, including those with horizontal nodes and their adjacency, in Figure 5. The picture shows the dual graphs of stable curves in the boundary of this stratum, where the top level is on top of each graph. The number in a vertex denotes the genus, and a black dot corresponds to genus zero. The numbers associated to the legs are the orders of zero. In this stratum, all interior edges have enhancement $\kappa_e = 1$, so the discussion of prong-matchings is void here. There are only three graphs without horizontal nodes, in fact $|\text{LG}_1(B)| = 2$ and $|\text{LG}_2(B)| = 1$, where $B = \mathbb{P}\Omega\mathcal{M}_{2,1}(2)$ as usual. Taking into also account the entire stratum and the stack structure of the banana graphs and using the values of top ξ -powers from Section 10.1, we get

$$(-1)^3 \cdot \chi(B) = 4 \cdot \frac{-1}{640} + 0 + 2 \cdot \frac{1}{24} \cdot \frac{-1}{8} + 2 \cdot \frac{1}{2} \cdot \frac{1}{24} \cdot 1 \cdot 1 = \frac{1}{40}$$

as in the table in the introduction, in accordance with the fact that this stratum is a 6-fold unramified cover of \mathcal{M}_2 and $\chi(\mathcal{M}_2) = -\frac{1}{240}$.

10.3. The stratum $\mathbb{P}\Omega\mathcal{M}_{1,3}(-k-1, 1, k)$

This example illustrates the quotient stack structure at the boundary of the smooth compactification that results from prong-matchings: that is, from points with $\text{Tw}_\Gamma \neq \text{Tw}_\Gamma^s$. We have chosen a genus-one stratum with a simple zero since the projection to $\mathcal{M}_{1,2}$ forgetting the simple zero provides an alternative way to compute all invariants in this case. We label the points z_1 (pole), z_2 (simple zero) and z_3 . The

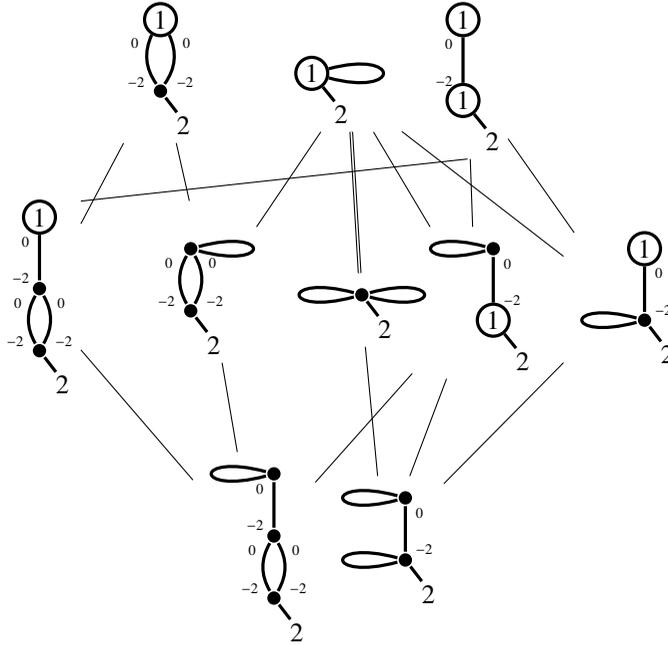


Figure 5. Level graphs appearing in the boundary of $\Omega\mathcal{M}_{2,1}(2)$. Graphs corresponding to components of the same dimension are in the same row (divisors in the first row, points in the bottom row). The lines connecting the graphs symbolise degeneration. The double line indicates a self-intersection.

boundary divisors here are D_h and five more types of divisors, namely there are the divisors

$$D_{1,a} = \left[\begin{array}{c} -k-1 \\ \diagdown \bullet \diagup \\ a-1 \quad b-1 \\ \diagup \bullet \diagdown \\ -a-1 \quad -1-b \\ \diagdown \bullet \diagup \\ 1 \quad k \end{array} \right], \quad D_2 = \left[\begin{array}{c} -k-1 \\ \diagdown \bullet \diagup \\ \textcircled{1} \\ \diagup \bullet \diagdown \\ k+1 \\ \diagdown \bullet \diagup \\ -k-3 \\ \diagdown \bullet \diagup \\ k \quad 1 \end{array} \right], \quad D_3 = \left[\begin{array}{c} \textcircled{1} \\ \diagdown \bullet \diagup \\ 0 \\ \diagup \bullet \diagdown \\ -2 \\ \diagdown \bullet \diagup \\ 1 \quad k \quad -k-1 \end{array} \right],$$

where $a, b \geq 1$ and $a + b = k + 1$. Here $D_{1,a} = D_{1,k+1-a}$, and if k is odd, the middle divisor $D_{1,(k+1)/2}$ has an $\mathbb{Z}/2$ -involution. Moreover, there are the divisors

$$D_4 = \left[\begin{array}{c} -k-1 \quad 1 \\ \diagdown \bullet \diagup \\ k-2 \\ \diagup \bullet \diagdown \\ -k \\ \diagdown \bullet \diagup \\ \textcircled{1} \\ \diagdown \bullet \diagup \\ k \end{array} \right], \quad D_{5,a'} = \left[\begin{array}{c} -k-1 \quad 1 \\ \diagdown \bullet \diagup \\ a'-1 \quad b'-1 \\ \diagup \bullet \diagdown \\ -a'-1 \quad -1-b' \\ \diagdown \bullet \diagup \\ k \end{array} \right]$$

where $a' \in \{1, \dots, k-1\}$ and $b' = k - a'$. Again, $D_{5,a'} = D_{5,k-a'}$ with an involution on $D_{5,k/2}$ if k is even. The local exponents are

$$\ell_{1,a} = \text{lcm}(a, k + 1 - a), \quad \ell_2 = k + 2, \quad \ell_3 = 1, \quad \ell_4 = k - 1, \quad \ell_{5,a'} = \text{lcm}(a', k - a')$$

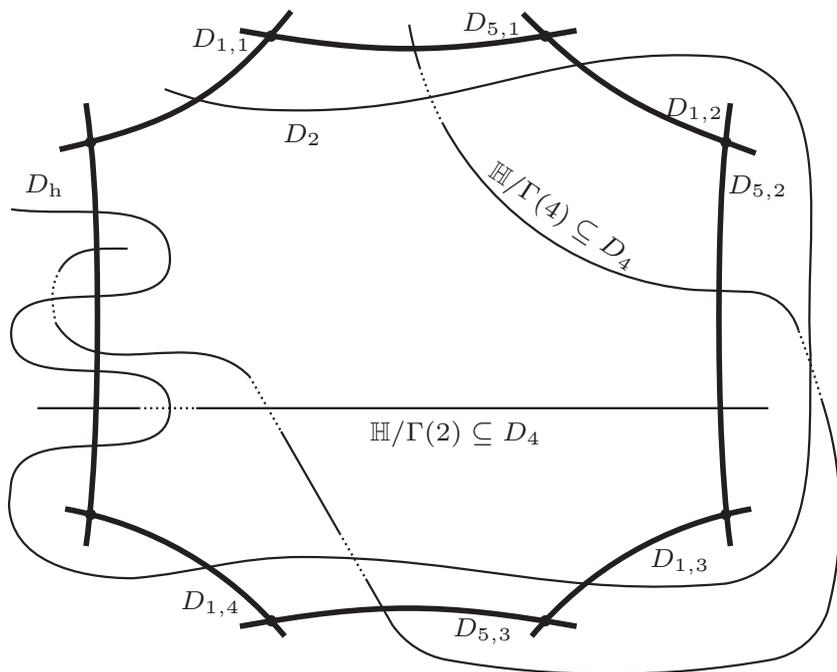


Figure 6. The intersection behavior of the boundary in the stratum $\mathbb{P}\Omega\mathcal{M}_{1,3}(-5, 1, 4)$. The figure has to be considered as quotient by the elliptic involution that interchanges $D_{1,1}$ with $D_{1,4}$ and $D_{5,1}$ with $D_{5,3}$ and so on.

and the dimensions of the top-level components are

$$N_1^T = 1, \quad N_2^T = 2, \quad N_3^T = 2, \quad N_4^T = 1, \quad N_5^T = 2.$$

We abbreviate $D_1 = \frac{1}{2} \sum_{a=1}^k D_{1,a}$ and $D_5 = \frac{1}{2} \sum_{a'=1}^{k-1} D_{5,a'}$.

The local geometry of the boundary divisors

We give a summary of the boundary points and intersection behaviour of the boundary divisors listed above. We start with the boundary divisors that map to the interior of $\mathcal{M}_{1,2}$. These are represented by the thin lines in Figure 6, while thick lines are mapped to the point at infinity of $\mathcal{M}_{1,1}$. The divisor D_3 is simply a copy of the modular curve, intersecting D_h once.

The divisor D_2 minus its intersection with other boundary divisors is the union of the modular curves $X_1(d) = \mathbb{H}/\Gamma_1(d)$ for all divisors $d > 1$ of $k + 1$. The only intersections with other boundary divisors are $\lceil (k + 1)/2 \rceil - 1$ intersection points with D_h and $\gcd(a, b)$ -points with $D_{1,a}$.

The divisor D_4 minus its intersection with other boundary divisors is the union of the modular curves $X_1(d) = \mathbb{H}/\Gamma_1(d)$ for all divisors $d > 1$ of k . The only intersections with other boundary divisors are $\lceil k/2 \rceil - 1$ intersection points with D_h and $\gcd(a', b')$ -points with $D_{5,a'}$.

The curves $D_{1,a}$ and $D_{5,a'}$ form the exceptional divisor when realising the level compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{1,3}(-k-1, 1, k)$ as a blowup of $\overline{\mathcal{M}}_{1,2}$ in the node of the universal family over $\overline{\mathcal{M}}_{1,1}$. Without prong-matchings, the curves $D_{1,a}$ were just an $\overline{M}_{0,4}$ (with a stack structure of an involution if $a = (k + 1)/2$). The three boundary points correspond to the intersection with $D_{5,a-1}$ and $D_{5,a}$ (respectively, with D_h and $D_{5,a}$ if $a = 1$) and with D_2 . By the formulas in Section 3.4, at the generic point of $D_{1,a}$ (and also near the intersection with D_2), there are $\gcd(a, b)$ prong-matching equivalence classes. At the

intersections with D_5 , there is just one prong-matching equivalence class. This implies that each $D_{1,a}$ is a $\gcd(a, b)$ -fold cover of $\overline{\mathcal{M}}_{0,4}$, totally ramified over the two points of intersection with D_5 .

Similarly, the divisor D_5 is a $\gcd(a', b')$ -fold cover of $\overline{\mathcal{M}}_{0,4}$, totally ramified over the two points that correspond to the intersections with D_1 . We compute the normal bundles of the boundary divisor using the special geometry of this example, independently of Theorem 7.1.

Proposition 10.2. *The self-intersection number of $D_{1,a}$ is*

$$D_{1,a}^2 = -\delta_a^{k+1} \cdot k g_{1,a} / \ell_{1,a} \quad \text{where} \quad g_{1,a} = \gcd(a, b)$$

and where $\delta_a^{k+1} = 1/2$ if $a = (k + 1)/2$ and $\delta_a^{k+1} = 1$ otherwise. The self-intersection number of $D_{5,a'}$ is

$$D_{5,a'}^2 = -\delta_{a'}^k \cdot (k + 1) g_{5,a'} / \ell_{5,a'} \quad \text{where} \quad g_{5,a'} = \gcd(a', b').$$

Proof. We consider the fibration $\pi : \mathbb{P}\Xi\overline{\mathcal{M}}_{1,3}(-k - 1, 1, k) \rightarrow \overline{\mathcal{M}}_{1,1}$ obtained from forgetting the last two marked points and take a smooth chart of the quotient stack near the image of the curves $D_{1,a}$ and $D_{5,a'}$. From the intersection discussion above, we deduce that the fibre over ∞ in $\overline{\mathcal{M}}_{1,1}$ consists of a ring of rational curves intersecting in the order

$$D_h - D_{1,1} - D_{5,1} - D_{1,2} - D_{5,2} - \dots - D_{1,k-1} - D_{5,k} - D_{1,k+1} - D_h;$$

see again Figure 6. We claim that the multiplicity of $D_{1,a}$ in the fibre $F = \pi^{-1}(\infty)$ is equal to $(k + 1)/\gcd(a, k + 1)$. This can be deduced from the fact that $\pi|_{D_2}$ is a cover of degree $(k + 1)^2 - 1$ and from the order of the cusp stabilisers (see [DS05, Section 3.8], in particular the explanation around Figure 3.2) since D_2 and $D_{1,a}$ intersect transversally in $\mathbb{P}\Xi\overline{\mathcal{M}}_{1,3}(-k - 1, 1, k)$. Similarly, the multiplicity of $D_{5,a'}$ in this fibre is $k/\gcd(a', k)$. Using the orbifold degree of the intersection points given in equation (23) and $D_{1,a} \cdot F = 0$, we find with $a' = a - 1$ and $b' = b - 1$ that

$$\begin{aligned} D_{1,a}^2 &= -\frac{\gcd(a, b)}{k + 1} \cdot \frac{abk}{\ell_{1,a}} \cdot \left(\frac{a'}{\ell_{5,a'} \gcd(a', k - a')} + \frac{b'}{\ell_{5,b'} \gcd(b', k - b')} \right) \\ &= -\frac{\gcd(a, b)}{k + 1} \cdot \frac{abk}{\ell_{1,a}} \cdot \frac{k + 1}{ab} = -k \cdot g_{1,a} / \ell_{1,a}. \end{aligned}$$

The proof of $D_{5,a'}$ is similar. □

Proposition 10.2 agrees with Theorem 1.6. Indeed, since the dimension of the top (respectively, bottom) level stratum of $D_{1,a}$ (respectively, $D_{5,a'}$) is zero (so codimension one classes vanish), we compute

$$\begin{aligned} D_{1,a}^2 &= c_1(\mathcal{N}_{D_{1,a}}) = \frac{1}{\ell_{1,a}} \left(-\xi_{D_{1,a}}^\top - \mathcal{L}_{D_{1,a}}^\top + \xi_{D_{1,a}}^\perp \right) \\ &= \frac{K_{1,a}}{\ell_{1,a}^2 \text{Aut}(D_{1,a})} \xi_{B_{1,a}^+} = \frac{g_{1,a}}{\ell_{1,a} \text{Aut}(D_{1,a})} \cdot (-k), \end{aligned}$$

where in the last two equalities we used Lemma 9.12 about evaluating top classes and the computation of top powers of ξ that can be done analogously as in the first case of Proposition 10.1. Similarly, we also get

$$\begin{aligned}
 D_{5,a'}^2 &= c_1(\mathcal{N}_{D_{5,a'}}) = \frac{1}{\ell_{5,a'}} \left(-\xi_{D_{5,a'}}^\top + \xi_{D_{5,a'}}^\perp - \mathcal{L}_{D_{5,a'}}^\top \right) \\
 &= -\frac{K_{5,a'}}{\text{Aut}(D_{5,a'})\ell_{5,a'}^2} \xi_{B_{5,a'}^\top} - \frac{1}{\ell_{5,a'}} \mathcal{L}_{D_{5,a'}}^\top \\
 &= \frac{g_{5,a'}}{\text{Aut}(D_{5,a'})\ell_{5,a'}} \cdot (-k) - \frac{1}{\ell_{5,a'}} ([D_{5,a'}] \cdot ([D_4] + [D_{1,a'}] + [D_{1,a'+1}])) \\
 &= \frac{g_{5,a'}}{\text{Aut}(D_{5,a'})\ell_{5,a'}} \cdot (-k - 1).
 \end{aligned}$$

The Euler characteristic

We give two ways to prove the following fact.

Proposition 10.3. *The moduli space $\mathbb{P}\Omega\mathcal{M}_{1,3}(-k-1, 1, k)$ has Euler characteristic equal to $k(k+1)/6$.*

Proof. The first proof uses the description of $B = \mathbb{P}\Omega\mathcal{M}_{1,3}(-k-1, 1, k)$ as the complement of D_2 and D_4 in $\mathcal{M}_{1,2}$. By the above description of D_2 and D_4 , we need to compute

$$\sum_{\substack{d|k \\ d \neq k}} \chi(X_1(k/d)) = \chi(\mathcal{M}_{1,1}) \sum_{\substack{d|k \\ d \neq k}} [\text{SL}_2(\mathbb{Z}) : \Gamma_1(k/d)] = -\frac{k^2 - 1}{12},$$

which holds, since the rightmost sum counts the number of nonzero k -torsion points in an elliptic curve. Together with $\chi(\mathcal{M}_{1,2}) = -1/12$ implies the claim.

The second proof evaluates Theorem 1.3, given in the surface case concretely by

$$\chi(B) = 3\xi_B^2 + \sum_{\Gamma \in \text{LG}_1(B)} \frac{K_\Gamma \cdot N_\Gamma^\top}{|\text{Aut}(\Gamma)|} \left(\int_{B_\Gamma^\top} \xi_{B_\Gamma^\top} + \int_{B_\Gamma^\perp} \xi_{B_\Gamma^\perp} \right) + \sum_{\Delta \in \text{LG}_2(B)} \ell_{\delta_0(\Delta)} \ell_{\delta_1(\Delta)} [D_\Delta].$$

Using the third statement of Proposition 10.1, we find

$$3 \int_B \xi^2 = 3(k^4 - 1)/24.$$

For the divisors $D_{1,a}$ and D_4 , the contribution from $\xi_{B_\Gamma^\top}$ is zero and that of $\xi_{B_\Gamma^\perp}$ is nonzero, while for D_2, D_3 and $D_{5,a'}$ the converse holds. We evaluate in detail the contribution of that last divisor type. Its top levels are $D_{5,a'}^\top = \mathbb{P}\Omega_{0,4}(1, a' - 1, b' - 1, -k - 1)$. Using again Proposition 10.1, we get

$$\sum_{\substack{\Gamma = D_{5,a} \\ a=1, \dots, k/2}} \frac{K_\Gamma \cdot N_\Gamma^\top}{|\text{Aut}(\Gamma)|} \int_{B_\Gamma^\top} \xi_{B_\Gamma^\top} = \frac{1}{2} \sum_{a'=1}^{k-1} 2a'(k - a') \cdot (-k) = \frac{-k^2(k^2 - 1)}{6}.$$

Similar computations again using Proposition 10.1 yield

$$\begin{aligned}
 \sum_{\substack{\Gamma = D_{1,a} \\ a=1, \dots, (k+1)/2}} \frac{K_\Gamma N_\Gamma^\top}{|\text{Aut}(\Gamma)|} \int \xi_{B_\Gamma^\top} &= -k^2 \frac{k^2 + 3k + 2}{12}, & 2\ell_{D_3} \int \xi_{D_3}^\top &= \frac{1}{12} \\
 2\ell_{D_2} \int \xi_{D_2}^\top &= -2k(k + 2) \frac{(k + 1)^2 - 1}{24}, & \ell_{D_4} \int \xi_{D_4}^\top &= -(k - 1)^2 \frac{k^2 - 1}{24}.
 \end{aligned}$$

Table 3. Euler characteristics of some meromorphic strata.

| | | | | | |
|-----------|------------------|-----------------|-----------------|------------------------------|---------------------------|
| μ | (4, -2) | (3, 1, -2) | (2, 2, -2) | (2, 1, 1, -2) | (1 ⁴ , -2) |
| $\chi(B)$ | $-\frac{19}{24}$ | $\frac{28}{15}$ | $\frac{17}{10}$ | -6 | 26 |
| μ | (4, -1, -1) | (3, 1, -1, -1) | (2, 2, -1, -1) | (2, 1 ² , -1, -1) | (1 ⁴ , -1, -1) |
| $\chi(B)$ | $-\frac{8}{5}$ | -4 | -4 | 14 | -63 |

Using the evaluation result of Lemma 9.12, we finally get

$$\sum_{\Delta \in \text{LG}_2(B)} \ell_{\delta_0(\Delta)} \ell_{\delta_1(\Delta)} [D_\Delta] = k(k+1) \frac{k^2 + k + 1}{4}.$$

Adding these contributions gives the claim. □

10.4. Hyperelliptic components

We recall from [KZ03] that strata of holomorphic Abelian differentials have up to three connected components, distinguished by the parity of the spin structure and hyperelliptic components. The strata $\Omega\mathcal{M}_{g,1}(2g-2)$ and $\Omega\mathcal{M}_{g,2}(g-1, g-1)$ have hyperelliptic components. Their Euler characteristics are easy to compute.

Proposition 10.4. *The Euler characteristics of the hyperelliptic components are*

$$\begin{aligned} \chi(\mathbb{P}\Omega\mathcal{M}_{g,1}(2g-2)^{\text{hyp}}) &= \frac{-1}{4g(2g+1)} \quad \text{and} \\ \chi(\mathbb{P}\Omega\mathcal{M}_{g,2}(g-1, g-1)^{\text{hyp}}) &= \frac{1}{(2g+1)(2g+2)}. \end{aligned}$$

Proof. In the first case, the surfaces are double covers of surfaces in the stratum $\mathbb{P}\mathcal{Q}_0(-1^{2g+1}, 2g-3)$ of quadratic differentials with unnumbered poles, which is isomorphic to $\mathcal{M}_{0,2g+2}/S_{2g+1}$. The claim follows from $\chi(\mathcal{M}_{0,n+3}) = (-1)^n \cdot n!$, taking into account the global hyperelliptic involution on the stratum.

Double covers of surfaces in the stratum $\mathbb{P}\mathcal{Q}_0(-1^{2g+2}, 2g-2)$ of quadratic differentials with unnumbered poles produce Abelian differentials of the second case, and this stratum is isomorphic to $\mathcal{M}_{0,2g+3}/S_{2g+2}$. The extra factors 2 from labelling the zeros of order $g-1$ and $1/2$ from the global hyperelliptic involution cancel each other. □

10.5. Meromorphic strata and cross-checks

In this section, we provide in Table 3 some Euler characteristics for meromorphic strata. We abbreviate $\chi(\mu) = \chi(\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu))$. Moreover, we provide several cross-checks for our values. First, note that the union of the strata of types (4), (3, 1), (2, 2), (2, 1, 1) and (1⁴) glue together to the projectivised Hodge bundle over $\overline{\mathcal{M}}_3$, if all of them are taken with unmarked zeros. We read off from Table 1 that

$$\chi(4) + \chi(3, 1) + \frac{1}{2}\chi(2, 2) + \frac{1}{2}\chi(2, 1, 1) + \frac{1}{4!}\chi(1^4) = \frac{3}{1008} = \chi(\mathbb{P}^2) \cdot \chi(\mathcal{M}_3).$$

The value $\chi(4) = -55/504$ can also be retrieved from Proposition 10.4 and the computations of Bergvall [Ber19, Table 4], which gives the cohomology of the stratum with odd spin structure $\Omega\mathcal{M}_{g,1}(4)^{\text{odd}}$ with $\mathbb{Z}/2$ -level structure. Computing the alternating sum weighted by dimension gives -141120 . Since

$|\mathrm{Sp}(6, \mathbb{Z})| = 1451520$, this checks with

$$\frac{-55}{504} = \chi(4) = \chi(4^{\mathrm{hyp}}) + \chi(4^{\mathrm{odd}}) = \frac{-1}{84} + \frac{-141120}{1451520}.$$

(A few other strata in $g = 3$ might be cross-checked with the table in [Ber19], but one has to take into account that Bergvall glosses over the existence of hyperelliptic curves in nonhyperelliptic strata.)

Another cross-check is the Hodge bundle twisted by twice the universal section over $\mathcal{M}_{2,1}$. It decomposes into the unordered strata $(4, -2)$, $(3, 1, -2)$, $(2, 2, -2)$, $(2, 1, 1, -2)$, $(1^4, -2)$, $(2, 0)$, $(1, 1, 0)$, (2) and the ordered stratum $(1, 1)$, since the simple zero at the unique marked point is distinguished. Note that $\chi(2, 0) = 3\chi(2)$ and $\chi(1, 1, 0) = 3\chi(1, 1)$. We can now add up the contributions listed in Table 1 and Table 3 and find that the sum equals $\frac{1}{40} = \chi(\mathbb{P}^2) \cdot \chi(\mathcal{M}_{2,1})$. A similar cross-check can be made for the Hodge bundle over $\mathcal{M}_{2,2}$ twisted by every section once, using the second row of Table 3.

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Conflicts of Interest. none.

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