ON THE EXISTENCE OF AN EXACT SOLUTION OF THE NAVIER-STOKES EQUATION PERTAINING TO CERTAIN VORTEX MOTION

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1. Introduction

In 1942, Burgers [1] observed that in cylindrical polar coordinates, the steady Navier-Stokes equation governing viscous incompressible fluid motion can be reduced to a set of ordinary differential equations if the velocity components v_r, v_0 and v_z are assumed to have a special form. Specifically, if we write

$$v_r = -\frac{f(x)}{r}; \quad v_0 = \frac{1}{r}\, g(x); \quad v_z = 2zf'(x) \tag{1}$$

where $x = r^2$ and the prime denotes differentiation with respect to x, the equations governing f and g are

$$2\nu x g'' + f g' = 0 \tag{2}$$

$$2\nu x f'''' + (4\nu + f) f''' - f' f'' = 0 \tag{3}$$

Here, the parameter ν is the kinematic viscosity, and is positive. The solution Burgers considered is

$$f = Ax, \quad A > 0$$
$$g = B(1 - e^{-Ax/2\nu})$$

which describes a type of vortex motion. The radial velocity $v_r = -Ar$ does not change sign, and the vortex is usually referred to as one-celled. In 1962, Donaldson and Sullivan [2] considered the numerical solution of equations (2) and (3) with boundary conditions at $r = 0$ and $r = R$, in an attempt to understand the flow pattern in a vortex tube. In general, oscillatory solution corresponding to multi-celled vortices were obtained.

Now the special form assumed for the velocity components places severe restrictions on the functional form of the pressure. It is easily verified from the

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Navier-Stokes equation that $\partial^2 p/\partial r \partial z = 0$. For this reason, solutions of equations (2) and (3), satisfying two point boundary conditions, may not necessarily described any real fluid flow. However, no approximation has been made in the derivation of (2) and (3), so that their solutions are exact solutions of the Navier-Stokes equations. As such they are of some intrinsic interest. In this note, we prove that the system consisting of equations (2) and (3), together with suitable boundary conditions at $r = 1$ and $r = \infty$, admits a solution with two-celled structure, that is, the function $f(x)$ changes sign.

2. The boundary value problem

Writing $2\nu f = \tilde{f}$, we can scale out the parameter ν in equations (2) and (3). If we then omit the tilde in $\tilde{f}$, we have

$$(2') \qquad xg'' + fg' = 0$$

$$(3') \qquad xf^{iv} + (2+f)f''' - f'f'' = 0.$$

Equation (3′) can be integrated once to give

$$(4) \qquad xf''' + (1+f)f'' = f'^2 - a^2$$

where a^2 is the integration constant, assumed positive. We impose the following boundary conditions:

$$(5) \qquad \begin{aligned} f(1) = b < 0; \quad f'(1) = 0; \quad g(1) = c \\ f'(\infty) = a > 0; \quad g(\infty) = k. \end{aligned}$$

We shall prove that there exists at least one solution to the system (2′), (4), (5) by a shooting method. This method of proving existence consists of seeking appropriate initial conditions for $f''(1)$ and $g'(1)$ so that the solution of the resulting initial value problem has the correct limiting behaviour as x tends to infinity. The method has been used by Ho and Wilson [3] and McLeod and Serrin [4].

Since equation (4) is independent of g, we shall first prove that (4) admits a solution satisfying the boundary conditions. A solution to (2′) will be established separately.

3. The sets S^+ and S^-

We consider equation (4) with the initial values

$$f(1) = b; \quad f'(1) = 0; \quad f''(1) = \beta$$

where $-\infty < \beta < \infty$. The solution of this initial value problem will be simply referred to as f. It is evident that the solution can be continued for all $x > 1$ as long as f' remains uniformly bounded. We define two sets S^+ and S^- of values of β as follows:

S^+: *$\beta \in S^+$ if there exists a value $x^+ > 1$ such that $f'(x^+) > a$ and $f' > 0$ for $1 < x < x^+$.*

S^-: *$\beta \in S^-$ if there exists a value $x^- > 1$ such that $f'(x^-) < 0$ and $f' < a$ for $1 < x < x^-$.*

LEMMA 1. *The sets S^+ and S^- are disjoint and open.*

PROOF. That S^+ and S^- are disjoint follows from their definition. Since solutions of (4) depend continuously on their initial values, it is clear that S^+ and S^- are open sets.

LEMMA 2. *β is in S^+ if*

$$\beta \geqq e^a[2a+a^2] \tag{6}$$

and β is in S^- if $\beta \leqq 0$.

PROOF. Clearly $\beta < 0$ is in S^-. Since

$$f'''(1) = -a^2-(1+b)\beta,$$

it is clear that

$$f'''(1) = -a^2 \text{ if } \beta = 0$$

so that $f' < 0$ for some $x > 1$. Hence $\beta = 0$ is also in S^-.

Let $J = (1, x^*)$, where $(x^*-1) \leqq 1$, be the maximal open interval in which $0 < f' < a$. Clearly, for $x \in J$, we have

$$b \log x < \int_1^x \frac{f}{t}\, dt < a(x-1)-(a-b) \log x. \tag{7}$$

Let $E(x) = f'' \exp [\int_1^x f/t\, dt]$, and $F = f''E(x)$. Equation (4) can be written as

$$(xF)' = (f'^2-a^2)E(x). \tag{8}$$

Using (7), and that $a-b > 0$, by assumption, we have

$$(xF)' > -a^2e^a$$

and

$$xF > \beta-a^2e^a(x-1) > \beta-a^2e^a$$

from which we obtain after some simplication

$$f'' > a\ x \in J$$

and hence

$$f' > a(x-1).$$

From the definition of J, it is clear that there exists an $x^+ > x^*$ at which $f'(x^+) > a$. Hence $\beta \in S^+$.

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4. The complement of S^+ and S^-

Since S^+ and S^- are disjoint non-empty open sets, their complement is also non-empty. Clearly the complement is the union of three sets A, B and C defined as follows:

A: *$\beta \in A$ if there exists x_A such that $f'(x_A) = a$ and $0 < f'$ for $1 < x < x_A$.*

B: *$\beta \in B$ if there exists x_B such that $f'(x_B) = 0$ and $f' < a$ for $1 < x < x_B$.*

C: *$\beta \in C$ if the solution of the initial value problem can be continued for all $x > 1$ with $0 < f' < a$.*

We want to show that $B \equiv S^-$ and $A = S^+ \cap D$ where D is defined as

D: *$\beta \in D$ if $\beta \in A$ and $f''(x_A) = 0$.*

It is clear that $B \subset S^-$. Conversely, suppose $\beta \in B$. If $f''(x_B) < 0$, then $\beta \in S^-$. If $f''(x_B) > 0$, again $\beta \in S^-$ for we must have $x^- < x_B$. If $f''(x_B) = 0$, then it follows from (4) that $f'''(x_B) = -a^2/x_B$ and so $\beta \in S^-$. Hence we have $B \equiv S^-$ as claimed.

In a similar manner, we note that if $f''(x_A) \gtrless 0$, then $\beta \in S^+$. If however $f''(x_A) = 0$, that is, $\beta \in D$, then it follows from (3) and (4) that $f^{(n)}(x_A) = 0$ for $n \geqq 2$, so that if D is non-empty, a solution of the boundary value problem is obtained. If D is empty, then clearly C is non-empty. We proceed as follows:

LEMMA 3. *If $\beta \in C$, then* $\lim f' = a$.

PROOF. Let $G = xf''E$, then equation (3′) can be written as

$$xG'' - fG' - 2f'G = 0.$$

Since $f' > 0$ if $\beta \in C$, it is evident that G cannot have a positive maximum. Hence G is ultimately zero or of one sign, and so f' is ultimately zero or of one sign. That f' is bounded implies that $\lim f'' = 0$ and $\lim f'$ exists.

Suppose $\lim f' = 0$, then since $\beta > 0$, f' must have a local maximum at which $f''' < 0$. Since f' cannot have a local minimum for $0 < f' < a$, f''' must change from negative to positive through a zero at which $f'' < 0$. This is however impossible since it follows from (3′) that at the zero of f'''

$$xf^{iv} = f'f'' < 0.$$

Hence, $\lim f' \neq 0$.

From the above, we readily establish $\lim f/x = f'(\infty)$, and that $\lim (f'^2 - a^2)$ exists. Suppose $\lim f' \neq a$, then from (8), we have

$$\lim ff'' = \lim \frac{\dfrac{f}{x}\displaystyle\int_1^x (f'^2 - a^2)E(t)dt}{E(x)}.$$

Using L'Hopital's rule, we readily obtain

$$\lim ff'' = \lim (f'^2 - a^2).$$

If $\lim (f'^2 - a^2) \neq 0$, then since $|f| \leqq \text{const.}\, x$, we have $|f''| \geqq \text{const.}/x$ for all sufficiently large x. Thus f'' is not integrable, contradicting the boundedness of f'. Hence, we must have $\lim f' = a$.

Since we have already shown that C is non-empty, it follows that equation (4) with the prescribed boundary conditions has at least one solution.

5. The g equation

Equation (2) can be integrated to give

$$g = c + g'(1) \int_1^x \frac{dt}{E(t)}. \tag{9}$$

Clearly, with the solution for f just determined, the integral in (9) converges for x tending to infinity. Writing $I = \lim \int_1^x dt/E(t)$, we have

$$k = c + g'(1) I.$$

Hence if we choose $g'(1) = k - c/I$, equation (3) with the prescribed boundary condition will have a solution.

We have proved:

THEOREM. *The boundary value problem* (2′), (4) *and* (5) *has a solution.*

REMARK. We have obtained a bound for $f''(1)$, $0 < f''(1) < e^a(2a+a^2)$ in the course of the proof.

References

[1] J. M. Burgers, 'Application of a model system to illustrate some points of the statistical theory of free turbulence', *Proc. Acad. Sci. Amsterdam* 43 (1940), 2–12.

[2] C. Du P. Donaldson and R. D. Sullivan, 'Behaviour of solutions of the Navier-Stokes equations for a complete class of three-dimensional viscous vortices', *Proc. Heat Transfer and Fluid Mech.* (1960), 16–30.

[3] D. Ho and H. K. Wilson, 'On the existence of a similarity solution for a compressible boundary layer'. *Arch. Rational Mech. Anal.* 27 (1967), 165–174.

[4] J. B. McLeod and J. Serrin, 'The existence of similar solutions for some laminar boundary layer problem'. *Arch. Rational Mech. Anal.* 31 (1968), 288–303.

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