

SET RECONSTRUCTION BY VORONOI CELLS

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Abstract

For a Borel set A and a homogeneous Poisson point process η in \mathbb{R}^d of intensity $\lambda > 0$, define the Poisson–Voronoi approximation A_η of A as a union of all Voronoi cells with nuclei from η lying in A . If A has a finite volume and perimeter, we find an exact asymptotic of $E \text{Vol}(A \Delta A_\eta)$ as $\lambda \rightarrow \infty$, where Vol is the Lebesgue measure. Estimates for all moments of $\text{Vol}(A_\eta)$ and $\text{Vol}(A \Delta A_\eta)$ together with their asymptotics for large λ are obtained as well.

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1. Introduction

Let A be a Borel set in \mathbb{R}^d , and let η be a Poisson point process in \mathbb{R}^d . Assume that we observe η and that the only information about A at our disposal is which points of η lie in A , i.e. we have the partition of the process η into $\eta \cap A$ and $\eta \setminus A$. We try to reconstruct the set A just by the information contained in these two point sets. To this end, we approximate A by the set A_η of all points in \mathbb{R}^d which are closer to $\eta \cap A$ than to $\eta \setminus A$.

More formally, let η be a homogeneous Poisson point process of intensity $\lambda > 0$, and denote by $v_\eta(x) = \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\| \text{ for all } y \in \eta\}$ the Voronoi cell generated by η with nucleus $x \in \eta$. Then the set A_η is just the union of the Poisson–Voronoi cells with nuclei lying in A , i.e.

$$A_\eta = \bigcup_{x \in \eta \cap A} v_\eta(x).$$

We call this set the *Poisson–Voronoi approximation* of the set A . It was first introduced by Khmaladze and Toronjadze [8]. They proposed A_η to be an estimator for A when λ is large (potential applications are listed in [7, Section 1]). They conjectured that, for an arbitrary

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bounded Borel set $A \subset \mathbb{R}^d$, $d \geq 1$, it holds that

$$\begin{aligned} \text{Vol}(A_\eta) &\rightarrow \text{Vol}(A) \quad \text{as } \lambda \rightarrow \infty, \\ \text{Vol}(A \Delta A_\eta) &\rightarrow 0 \quad \text{as } \lambda \rightarrow \infty, \end{aligned} \tag{1}$$

almost surely, where $\text{Vol}(\cdot)$ stands for the Lebesgue measure (volume) and Δ is the operation of the symmetric difference of sets. This conjecture was proved in [8] for $d = 1$. The case of general d was treated by Einmahl and Khmaladze [5] with some technical assumption on the boundary of A , and then generalized by Penrose [11] to an arbitrary bounded Borel set A .

It can be easily shown (see Section 3 for details) that, for any Borel set A , it holds that

$$E \text{Vol}(A_\eta) = \text{Vol}(A).$$

Thus, $\text{Vol}(A_\eta)$ is an unbiased estimator for the volume of A . In this paper we also consider the n th moment of $\text{Vol}(A_\eta)$ and approximate it by the n th degree of the volume of the original set $\text{Vol}^n(A)$ asymptotically as $\lambda \rightarrow \infty$ (Theorem 2). For the case when $n = 2$ and A is a convex compact, similar estimates were obtained in [7].

It might be suggested from (1) that

$$E \text{Vol}(A \Delta A_\eta) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty, \tag{2}$$

although it is not a direct corollary. The more interesting problem is to find an exact asymptotic of $E \text{Vol}(A \Delta A_\eta)$. This problem was initially considered by Heveling and Reitzner [7]. They proved that, for any compact convex set A with $(d - 1)$ -dimensional Hausdorff measure $\mathcal{H}_{d-1}(\partial A)$ of the boundary ∂A of A , it holds that

$$E \text{Vol}(A \Delta A_\eta) = c_d \mathcal{H}_{d-1}(\partial A) \lambda^{-1/d} (1 + O(\lambda^{-1/d})) \quad \text{as } \lambda \rightarrow \infty,$$

where the constant c_d is independent of λ and A was calculated by them in an explicit form (see Section 2 for details). Here we obtain a similar asymptotic formula (Theorem 1) for a much wider class of sets. Namely, we consider Borel sets with finite volume $\text{Vol}(A)$ and perimeter $\text{Per}(A)$ (see Section 3 for the precise definition). Our methods are completely different from those of Heveling and Reitzner. The key observations are the connection between the Poisson–Voronoi approximation and the covariogram of A , and the connection between the covariogram and the perimeter of a set recently established by Galerne [6]. As a by-product of our calculations, we prove that (2) holds for any Borel set A with finite volume (Corollary 3).

We also consider higher moments of $\text{Vol}(A \Delta A_\eta)$. For an arbitrary Borel set A , we approximate $E \text{Vol}^n(A \Delta A_\eta)$ by the n th degree of $E \text{Vol}(A \Delta A_\eta)$ asymptotically as $\lambda \rightarrow \infty$ (Theorem 3). Thus, assuming that $\text{Vol}(A), \text{Per}(A) < \infty$ and using the asymptotic for $E \text{Vol}(A \Delta A_\eta)$ from Theorem 1, we obtain the asymptotic for $E \text{Vol}^n(A \Delta A_\eta)$ (Corollary 1).

The paper is organized as follows. Our main results are stated in Section 2. In Section 3 we introduce the necessary background and notation, in particular the perimeter and the covariogram of a set A . Proofs are given in Section 4.

2. Main results

Our first result yields the asymptotic of the average volume of $A \Delta A_\eta$ with increasing intensity λ . To formulate it, we need to define the notion of a perimeter of a Borel set. The definition is somewhat technical, so we postpone it till Section 3. If A is a compact

set with Lipschitz boundary (e.g. a convex body), then $\text{Per}(A)$ equals the $(d - 1)$ -dimensional Hausdorff measure $\mathcal{H}_{d-1}(\partial A)$ of the boundary ∂A of A . In the general case it holds that $\text{Per}(A) \leq \mathcal{H}_{d-1}(\partial A)$ (see, e.g. [2, Proposition 3.62]). Therefore, $\text{Per}(A)$ could be replaced by $\mathcal{H}_{d-1}(\partial A)$ in the assumptions of the theorem.

Theorem 1. *If $A \subset \mathbb{R}^d$ is a Borel set with $\text{Vol}(A) < \infty$ and $\text{Per}(A) < \infty$, then*

$$E \text{Vol}(A\Delta A_\eta) = c_d \text{Per}(A) \lambda^{-1/d} (1 + o(1)) \quad \text{as } \lambda \rightarrow \infty, \tag{3}$$

where $c_d = 2d^{-2} \Gamma(1/d) \kappa_{d-1} \kappa_d^{-1-1/d}$ and κ_n is the volume of the unit n -dimensional ball.

The probabilistic intuition behind this asymptotic is the following. The set difference $A\Delta A_\eta$ is formed out of the Poisson–Voronoi cells with nuclei lying almost on ∂A . Thus, its volume behaves asymptotically like the volume of a small tube neighbourhood $B(\partial A)$ of the boundary ∂A . The size of the boundary is given by the perimeter of A . Since the volume of a typical Poisson–Voronoi cell is λ^{-1} , the width of this tube neighbourhood should have order $\lambda^{-1/d}$, and so the volume of the tube neighbourhood has the order $\text{Per}(A) \lambda^{-1/d}$.

In the following, when we say that some inequality holds asymptotically as $\lambda \rightarrow \infty$, we mean that it holds for sufficiently large $\lambda \geq \lambda_0$. The choice of λ_0 might depend on A . Thus, all estimates are not uniform with respect to A (including those of Theorem 1).

Theorem 2. *If $A \subset \mathbb{R}^d$ is a Borel set with $\text{Vol}(A) < \infty$ then*

$$|E \text{Vol}^n(A_\eta) - \text{Vol}^n(A)| \leq C_{n,d} \text{Vol}^{n-1}(A) \lambda^{-1} \quad \text{as } \lambda \rightarrow \infty,$$

where $C_{n,d}$ is some constant independent of λ and A .

Remark 1. In fact, we show that the following nonasymptotic inequality holds: for any $\lambda > 0$,

$$|E \text{Vol}^n(A_\eta) - \text{Vol}^n(A)| \leq C_{n,d} \sum_{k=1}^{n-1} \text{Vol}^{n-k}(A) \lambda^{-k}.$$

Theorem 3. *If $A \subset \mathbb{R}^d$ is a Borel set with $\text{Vol}(A) < \infty$ and $\text{Per}(A) < \infty$, then*

$$|E \text{Vol}^n(A\Delta A_\eta) - (E \text{Vol}(A\Delta A_\eta))^n| \leq C'_{n,d} \text{Per}(A)^{n-1} \lambda^{-1-(n-1)/d} \quad \text{as } \lambda \rightarrow \infty,$$

where $C'_{n,d}$ is some constant independent of λ and A .

Remark 2. We conjecture that the following limit theorems can be proven by the method of moments (see, e.g. [4, Theorems 30.1, 30.2]):

$$\begin{aligned} \lambda^{1/2(1+1/d)} (\text{Vol}(A_\eta) - \text{Vol}(A)) &\rightarrow N(0, \sigma_1 \text{Per}(A)), \\ \lambda^{1/2(1+1/d)} (\text{Vol}(A\Delta A_\eta) - c_d \text{Per}(A) \lambda^{-1/d}) &\rightarrow N(0, \sigma_2 \text{Per}(A)) \end{aligned} \tag{4}$$

in distribution as $\lambda \rightarrow \infty$, $\sigma_1, \sigma_2 > 0$.

Recently, (4) was proved by Schulte [13] for *convex* sets A using a central limit theorem for Wiener–Itô chaos expansions. In his Remark 4 Schulte points out that the result can be extended to all sets where the volume of a small tube neighbourhood $B(\partial A)$ of ∂A can be bounded in a nice way. Yet the general conjecture seems to be open.

Corollary 1. *If $A \subset \mathbb{R}^d$ is a Borel set with $\text{Vol}(A) < \infty$ and $\text{Per}(A) < \infty$, then*

$$E \text{Vol}^n(A \Delta A_\eta) = (E \text{Vol}(A \Delta A_\eta))^n (1 + O(\lambda^{-1+1/d})) \quad \text{as } \lambda \rightarrow \infty,$$

and, for $d \geq 2$,

$$E \text{Vol}^n(A \Delta A_\eta) = (c_d \text{Per}(A))^n \lambda^{-n/d} (1 + o(1)) \quad \text{as } \lambda \rightarrow \infty.$$

The asymptotic order of the variance of A_η and $A \Delta A_\eta$ as $\lambda \rightarrow \infty$ was first studied in [7] for convex sets A . We extend these results to arbitrary Borel sets.

Corollary 2. *If $A \subset \mathbb{R}^d$ is a Borel set with $\text{Vol}(A) < \infty$ and $\text{Per}(A) < \infty$, then*

$$\text{varVol}(A_\eta) \leq C_d \text{Per}(A) \lambda^{-1-1/d} \quad \text{as } \lambda \rightarrow \infty$$

and

$$\text{varVol}(A \Delta A_\eta) \leq C_d \text{Per}(A) \lambda^{-1-1/d} \quad \text{as } \lambda \rightarrow \infty,$$

where C_d is some constant independent of λ and A .

The second inequality follows immediately from Theorem 3. The first inequality will be proved in Section 4.2.

The probabilistic heuristic explaining the asymptotic behaviour of the variances is the following. The symmetric difference $A \Delta A_\eta$ consists of parts $\tilde{v}_\eta(x)$ of almost independent Poisson–Voronoi cells $v_\eta(x)$. Their nuclei $x \in B(\partial A)$ are asymptotically contained in a small tube neighbourhood $B(\partial A)$ of ∂A already mentioned in the heuristic intuition after Theorem 1. We use the formula for the variance of the compound Poisson distribution,

$$\text{varVol}(A \Delta A_\eta) = \text{var} \left(\sum_{x \in \eta \cap B(\partial A)} \text{Vol}(\tilde{v}_\eta(x)) \right) \approx \text{var} \left(\sum_{i=1}^N Y_i \right),$$

where the random variables $Y_i \stackrel{D}{=} \text{Vol}(\tilde{v}_\eta(x))$ are independent and identically distributed and

$$N \stackrel{D}{=} \text{card}(\eta \cap B(\partial A)) \sim \text{Pois}(\lambda \text{Vol}(B(\partial A)))$$

is independent of Y_i . Here ‘ $\stackrel{D}{=}$ ’ denotes equality in distribution and $\text{card}(B)$ is the cardinality of a set B . Then

$$\begin{aligned} \text{var} \left(\sum_{i=1}^N Y_i \right) &= E N \text{var} Y_1 + \text{var} N (E Y_1)^2 \\ &= \lambda \text{Vol}(B(\partial A)) E Y_1^2 \\ &\leq \lambda \text{Vol}(B(\partial A)) (E \text{Vol}(v_\eta(x)))^2 \\ &= O(\lambda \text{Per}(A) \lambda^{-1/d} \lambda^{-2}) \\ &= \text{Per}(A) O(\lambda^{-1-1/d}) \end{aligned}$$

since $\tilde{v}_\eta(x) \subset v_\eta(x)$ for any x , the second moment of the volume of a typical Poisson–Voronoi cell is of order λ^{-2} , and the volume of $B(\partial A)$ is of order $\text{Per}(A) \lambda^{-1/d}$.

The results of Corollary 2 can also be obtained by using the Poincaré inequality which gives an upper bound on the variance of a functional of a Poisson point process. Let \mathcal{N} be the set of all locally finite configurations on \mathbb{R}^d . Consider a nonnegative measurable function $F: \mathcal{N} \rightarrow \mathbb{R}$. If $E F^2(\eta) < \infty$ then

$$\text{var } F(\eta) \leq \lambda E \int_{\mathbb{R}^d} (F(\eta \cup \{y\}) - F(\eta))^2 dy, \tag{5}$$

where we have added a point y to the Poisson point process η . Putting $F(\eta) = \text{Vol}(A_\eta)$ in (5), we obtain

$$\text{var Vol}(A_\eta) \leq \lambda \int_{\mathbb{R}^d} E(\text{Vol}(A_{\eta \cup \{y\}}) - \text{Vol}(A_\eta))^2 dy,$$

where the right-hand side can be estimated from above to obtain the upper bound in Corollary 2. The reasoning for the symmetric difference $A \Delta A_\eta$ is similar.

In full generality, inequality (5) was proved by Wu [15]. As was shown by Last and Penrose [9, Theorem 1.2], it is a consequence of an even more general inequality following from the Fock space representation of Poisson point processes.

3. Preliminaries

For basic facts from integral geometry, stochastic geometry, and Voronoi tessellations which are not explained in the following, we refer the reader to [10], [12], and [14].

Define the perimeter of a Borel set A as

$$\text{Per}(A) = \sup \left\{ \int_A \text{div} \varphi(x) dx : \varphi \in \mathcal{C}_c^1(\mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\},$$

cf. [2], where

$$\text{div} \varphi(x) = \sum_{i=1}^d \frac{\partial \varphi_i}{\partial x_i} \quad \text{and} \quad \|\varphi\|_\infty = \max_{i=1, \dots, d} \sup_{x \in \mathbb{R}^d} |\varphi_i(x)|$$

for $\varphi = (\varphi_1, \dots, \varphi_d)$. The class $\mathcal{C}_c^1(\mathbb{R}^d)$ consists of all continuously differentiable vector-valued functions from \mathbb{R}^d to \mathbb{R}^d with compact support. It is well known that if A is a compact set with Lipschitz boundary (e.g. a convex body), then $\text{Per}(A)$ equals the $(d - 1)$ -dimensional Hausdorff measure $\mathcal{H}_{d-1}(\partial A)$ of the boundary ∂A of A . Therefore, in most applications, $\text{Per}(A)$ could be replaced by $\mathcal{H}_{d-1}(\partial A)$. In the general case,

$$\text{Per}(A) \leq \mathcal{H}_{d-1}(\partial A)$$

(see, e.g. [2, Proposition 3.62]). More detailed information about the perimeter of a set and its connection to the Hausdorff measure of the boundary is given in [1] where, e.g. it is proved [1, Theorem 3] that the perimeter equals the Hausdorff measure of the essential boundary of the set if the perimeter is finite.

Let A be a Borel set with finite volume. Then

$$g_A(x) = \text{Vol}((A + x) \cap A), \quad x \in \mathbb{R}^d,$$

is a covariogram of A . For the history on the covariogram problem, see the references in [6] and also the recent breakthrough by Averbkov and Bianchi [3].

In the proof of Theorem 1 we use the result obtained by Galerne [6, Theorem 14]. The following assertions are equivalent:

- (a) $\text{Per}(A) < \infty$;
- (b) there exists a finite limit

$$\lim_{r \rightarrow +0} \frac{g_A(ru) - g_A(0)}{r} = \frac{\partial g_A}{\partial u}(0) \tag{6}$$

for all $u \in \mathbb{S}^{d-1}$;

- (c) g_A is Lipschitz.

In addition, the Lipschitz constant of g_A satisfies

$$\text{Lip}(g_A) \leq \frac{1}{2} \text{Per}(A) \tag{7}$$

and it holds that

$$\int_{\mathbb{S}^{d-1}} \frac{\partial g_A}{\partial u}(0) \mathcal{H}_{d-1}(du) = -\kappa_{d-1} \text{Per}(A). \tag{8}$$

Another tool we need is the refined Campbell–Mecke formula for stationary point processes (cf., e.g. [14]). Using Slivnyak’s theorem, we give its particular case for the Poisson point process.

As above, let η be a homogeneous Poisson point process of intensity $\lambda > 0$, and let \mathcal{N} be the set of all locally finite point configurations on \mathbb{R}^d . Consider a nonnegative measurable function $f : \mathcal{N} \times (\mathbb{R}^d)^m \rightarrow \mathbb{R}$. Then

$$\mathbb{E} \sum_{(y_1, \dots, y_m) \in \eta_{\neq}^m} f(\eta, y_1, \dots, y_m) = \lambda^m \int_{(\mathbb{R}^d)^m} \mathbb{E} f(\eta \cup \bar{y}_m, y_1, \dots, y_m) dy_1 \cdots dy_m, \tag{9}$$

where η_{\neq}^m denotes the set of all m -tuples of pairwise distinct points from η , and $\eta \cup \bar{y}_m$ is the process η with added point set $\bar{y}_m = \{y_1, \dots, y_m\}$.

As a simple corollary we get two identities which are crucial for us in the sequel.

Proposition 1. *If $A \subset \mathbb{R}^d$ is a Borel set with $\text{Vol}(A) < \infty$ then*

$$\mathbb{E} \text{Vol}(A_\eta) = \lambda \int_{\mathbb{R}^d} \int_A e^{-\lambda \kappa_d \|y-x\|^d} dy dx = \text{Vol}(A) \tag{10}$$

and

$$\mathbb{E} \text{Vol}(A \Delta A_\eta) = 2\lambda \int_{\mathbb{R}^d \setminus A} \int_A e^{-\lambda \kappa_d \|y-x\|^d} dy dx. \tag{11}$$

Proof. By Fubini’s theorem and the Slivnyak–Mecke formula (9), we have

$$\begin{aligned} \mathbb{E} \text{Vol}(A_\eta) &= \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}(x \in A_\eta) dx \\ &= \int_{\mathbb{R}^d} \mathbb{E} \sum_{y \in \eta \cap A} \mathbf{1}(x \in v_\eta(y)) dx \\ &= \lambda \int_{\mathbb{R}^d} \int_A \mathbb{P}(x \in v_{\eta \cup \{y\}}(y)) dy dx \\ &= \lambda \int_{\mathbb{R}^d} \int_A e^{-\lambda \kappa_d \|x-y\|^d} dy dx. \end{aligned}$$

Similarly, we obtain

$$E \text{Vol}(A \setminus A_\eta) = \lambda \int_A \int_{\mathbb{R}^d \setminus A} e^{-\lambda \kappa_d \|x-y\|^d} dy dx$$

and

$$E \text{Vol}(A_\eta \setminus A) = \lambda \int_{\mathbb{R}^d \setminus A} \int_A e^{-\lambda \kappa_d \|x-y\|^d} dy dx.$$

By definition, $\text{Vol}(A \Delta A_\eta) = \text{Vol}(A \setminus A_\eta) + \text{Vol}(A_\eta \setminus A)$, which completes the proof of (11). To prove the second part of (10), we have to apply Fubini’s theorem and then use the formula

$$\int_{\mathbb{R}^d} e^{-c\|x-y\|^d} dx = \frac{\kappa_d}{c}, \quad c > 0, \tag{12}$$

which could be easily proved by introducing spherical coordinates.

Note that we have also proved that

$$E \text{Vol}(A \setminus A_\eta) = E \text{Vol}(A_\eta \setminus A).$$

However, $\text{Vol}(A \setminus A_\eta)$ and $\text{Vol}(A_\eta \setminus A)$ are not equidistributed since the first random variable is bounded, and the second is not. As a direct corollary of identity (10), we obtain

$$\text{varVol}(A_\eta) = E(\text{Vol}(A \setminus A_\eta) - \text{Vol}(A_\eta \setminus A))^2, \tag{13}$$

which we shall use in the following.

4. Proofs

4.1. Asymptotics of the mean volume of the symmetric difference

In this section we give the proof of Theorem 1. The key step to prove it is the following relation between the Poisson–Voronoi approximation and the covariogram of a set A .

Lemma 1. *Let $g_A(x)$ be the covariogram of a Borel set A with $\text{Vol}(A) < \infty$. Then*

$$E \text{Vol}(A \Delta A_\eta) = -2 \int_0^\infty r^{d-1} e^{-\kappa_d r^d} \tilde{g}_A(\lambda^{-1/d} r) dr, \tag{14}$$

where

$$\tilde{g}_A(r) = \int_{\mathbb{S}^{d-1}} (g_A(ru) - g_A(0)) \mathcal{H}_{d-1}(du).$$

Proof. Replacing y in (11) by $x - \lambda^{-1/d} z$ we obtain

$$\begin{aligned} E \text{Vol}(A \Delta A_\eta) &= 2\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|y-x\|^d} \mathbf{1}(y \in A, x \in A^c) dy dx \\ &= 2 \int_{\mathbb{R}^d} e^{-\kappa_d \|z\|^d} \int_{\mathbb{R}^d} \mathbf{1}(x \in (A + \lambda^{-1/d} z) \cap A^c) dz dx \\ &= 2 \int_{\mathbb{R}^d} e^{-\kappa_d \|z\|^d} \text{Vol}((A + \lambda^{-1/d} z) \cap A^c) dz. \end{aligned}$$

By the definition of the covariogram,

$$\text{Vol}((A + \lambda^{-1/d} z) \cap A^c) = g_A(0) - g_A(\lambda^{-1/d} z).$$

We introduce spherical coordinates $z = ru$, where $r \in \mathbb{R}^+$ and $u \in \mathbb{S}^{d-1}$. This yields

$$E \text{Vol}(A \Delta A_\eta) = -2 \int_0^\infty r^{d-1} e^{-\kappa_d r^d} \left[\int_{\mathbb{S}^{d-1}} (g_A(\lambda^{-1/d} r u) - g_A(0)) \mathcal{H}_{d-1}(du) \right] dr.$$

Corollary 3. *For any measurable A with $\text{Vol}(A) < \infty$, it holds that*

$$E \text{Vol}(A \Delta A_\eta) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty.$$

Proof. The proof immediately follows from (14) and the continuity of the set covariogram.

Proof of Theorem 1. Using Lemma 1 and substituting t for $\kappa_d r^d$, we obtain

$$E \text{Vol}(A \Delta A_\eta) = -\frac{2}{d \kappa_d} \int_0^\infty e^{-t} \tilde{g}_A((\lambda \kappa_d)^{-1/d} t^{1/d}) dt.$$

It follows from (7) and the definition of \tilde{g}_A that

$$|\tilde{g}_A(r)| \leq \frac{1}{2} \mathcal{H}_{d-1}(\mathbb{S}^{d-1}) \text{Per}(A) r.$$

Therefore, Lebesgue’s dominated convergence theorem, (6), and (8) yield

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} E \text{Vol}(A \Delta A_\eta) \lambda^{1/d} &= -\frac{2}{d} \kappa_d^{-1-1/d} \lim_{\lambda \rightarrow \infty} \int_0^\infty e^{-t} t^{1/d} \frac{\tilde{g}_A((\lambda \kappa_d)^{-1/d} t^{1/d})}{(\lambda \kappa_d)^{-1/d} t^{1/d}} dt \\ &= -\frac{2}{d} \kappa_d^{-1-1/d} \int_0^\infty e^{-t} t^{1/d} dt \int_{\mathbb{S}^{d-1}} \frac{\partial g_A}{\partial u}(0) \mathcal{H}_{d-1}(du) \\ &= \frac{2}{d} \kappa_{d-1} \kappa_d^{-1-1/d} \text{Per}(A) \int_0^\infty e^{-t} t^{1/d} dt \\ &= \frac{2}{d} \kappa_{d-1} \kappa_d^{-1-1/d} \Gamma\left(1 + \frac{1}{d}\right) \text{Per}(A). \end{aligned}$$

4.2. Asymptotics of higher moments

To prove Theorem 2 and Theorem 3, we need a number of lemmas. In this section, C is always some constant independent of λ and A . Our first statement is the following version of Hölder’s inequality.

Lemma 2. *For any events A_1, \dots, A_m , it holds that*

$$P\left(\bigcap_{r=1}^m A_r\right) \leq \prod_{r=1}^m (P(A_r))^{1/m}.$$

Lemma 3. *Let $x_0, y_0 \in \mathbb{R}^d$. For any $\varepsilon > 0$ and $m \in \mathbb{N}$, the following inequality holds:*

$$\begin{aligned} &\int_{(\mathbb{R}^d)^m} (P(x_0, x_1, \dots, x_m \in \nu_{\eta \cup \{y_0\}}(y_0)))^\varepsilon dx_1 \cdots dx_m \\ &\leq \exp\left(-\frac{\varepsilon \lambda \kappa_d \|x_0 - y_0\|^d}{m+1}\right) \left(\frac{m+1}{\varepsilon \lambda}\right)^m. \end{aligned}$$

Proof. By Lemma 2, we have

$$\begin{aligned} & \int_{(\mathbb{R}^d)^m} (\mathbb{P}(x_0, x_1, \dots, x_m \in \nu_{\eta \cup \{y_0\}}(y_0)))^\varepsilon dx_1 \cdots dx_m \\ & \leq (\mathbb{P}(x_0 \in \nu_{\eta \cup \{y_0\}}(y_0)))^{\varepsilon/(m+1)} \int_{(\mathbb{R}^d)^m} \prod_{i=1}^m (\mathbb{P}(x_i \in \nu_{\eta \cup \{y_0\}}(y_0)))^{\varepsilon/(m+1)} dx_1 \cdots dx_m \\ & = \exp\left(\frac{-\varepsilon\lambda\kappa_d \|x_0 - y_0\|^d}{m+1}\right) \left[\int_{\mathbb{R}^d} \exp\left(\frac{-\varepsilon\lambda\kappa_d \|x - y_0\|^d}{m+1}\right) dx \right]^m. \end{aligned}$$

Using (12) completes the proof.

Lemma 4. For any $a > 0$,

$$\int_{\mathbb{R}^d} \int_A e^{-a\lambda\|y-x\|^d} dy dx = \frac{\kappa_d \text{Vol}(A)}{a\lambda} \tag{15}$$

and

$$\int_{\mathbb{R}^d \setminus A} \int_A e^{-a\lambda\|y-x\|^d} dy dx \leq C \frac{\text{Per}(A)}{\lambda^{1+1/d}} \text{ as } \lambda \rightarrow \infty. \tag{16}$$

Proof. The first equation follows from (10) after replacing λ by $\lambda'a/\kappa_d$. The second estimate follows from (11) after replacing λ by $\lambda'a/\kappa_d$ and then applying Theorem 1.

Introduce the notation B_r^x for the closed ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$ in the Euclidean metric.

Lemma 5. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$. If $B_{\|x_1-y_1\|}^{x_1} \cap B_{\|x_2-y_2\|}^{x_2} \neq \emptyset$ then

$$\mathbb{P}(B_{\|x_1-y_1\|}^{x_1} \cap \eta = \emptyset, B_{\|x_2-y_2\|}^{x_2} \cap \eta = \emptyset) \leq 2 \exp\left(-\frac{\lambda\kappa_d}{2^{2d+1}}(\|x_1 - y_2\|^d + \|x_2 - y_1\|^d)\right).$$

Proof. Since $B_{\|x_1-y_1\|}^{x_1} \cap B_{\|x_2-y_2\|}^{x_2} \neq \emptyset$, it follows from the triangle inequality that

$$\frac{\|x_1 - y_2\|}{4}, \frac{\|x_2 - y_1\|}{4} \leq \max(\|x_1 - y_1\|, \|x_2 - y_2\|).$$

Therefore, by Lemma 2 and the stationarity of η , we have

$$\begin{aligned} & \mathbb{P}(B_{\|x_1-y_1\|}^{x_1} \cap \eta = \emptyset, B_{\|x_2-y_2\|}^{x_2} \cap \eta = \emptyset) \\ & \leq \mathbb{P}(B_{\|x_1-y_2\|/4}^{x_1} \cap \eta = \emptyset, B_{\|x_2-y_1\|/4}^{x_1} \cap \eta = \emptyset) \\ & \quad \text{or } \mathbb{P}(B_{\|x_1-y_2\|/4}^{x_2} \cap \eta = \emptyset, B_{\|x_2-y_1\|/4}^{x_2} \cap \eta = \emptyset) \\ & \leq \sum_{i=1}^2 \mathbb{P}(B_{\|x_1-y_2\|/4}^{x_i} \cap \eta = \emptyset, B_{\|x_2-y_1\|/4}^{x_i} \cap \eta = \emptyset) \\ & \leq \sum_{i=1}^2 (\mathbb{P}(B_{\|x_1-y_2\|/4}^{x_i} \cap \eta = \emptyset) \mathbb{P}(B_{\|x_2-y_1\|/4}^{x_i} \cap \eta = \emptyset))^{1/2} \\ & = 2 \exp\left(-\frac{\lambda\kappa_d}{2^{2d+1}}(\|x_1 - y_2\|^d + \|x_2 - y_1\|^d)\right). \end{aligned}$$

Lemma 6. For any $x_1, y_1, \dots, x_n, y_n \in \mathbb{R}^d$, it holds that

$$\begin{aligned} & \mathbb{P}(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n) \\ & \leq \exp\left(-\lambda\kappa_d \sum_{r=1}^n \|x_r - y_r\|^d\right) \\ & \quad + 2 \sum_{s < t} \exp\left(-\frac{\lambda\kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d\right) \\ & \quad \times \exp\left(-\frac{\lambda\kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right). \end{aligned}$$

Proof. If the balls $B_{\|x_r - y_r\|}^{x_r}$, $r = 1, \dots, n$, are pairwise disjoint then we obviously have

$$\mathbb{P}(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n) = \exp\left(-\lambda\kappa_d \sum_{r=1}^n \|x_r - y_r\|^d\right).$$

Suppose that, for some indices $s \neq t$, it holds that $B_{\|x_s - y_s\|}^{x_s} \cap B_{\|x_t - y_t\|}^{x_t} \neq \emptyset$. Applying Lemma 2, we obtain

$$\begin{aligned} & \mathbb{P}(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n) \\ & \leq (\mathbb{P}(B_{\|x_s - y_s\|}^{x_s} \cap \eta = \emptyset, B_{\|x_t - y_t\|}^{x_t} \cap \eta = \emptyset))^{1/(n+1)} \prod_{r=1}^n (\mathbb{P}(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset))^{1/(n+1)} \\ & = \exp\left(-\frac{\lambda\kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d\right) (\mathbb{P}(B_{\|x_s - y_s\|}^{x_s} \cap \eta = \emptyset, B_{\|x_t - y_t\|}^{x_t} \cap \eta = \emptyset))^{1/(n+1)}. \end{aligned}$$

It remains to apply Lemma 5 to complete the proof.

Proof of Theorem 2. We have

$$\begin{aligned} \mathbb{E} \text{Vol}^n(A_\eta) &= \mathbb{E} \int_{(\mathbb{R}^d)^n} \mathbf{1}(\text{there exists } (y_1, \dots, y_n) \in (\eta \cap A)^n : x_i \in \nu_\eta(y_i), i = 1, \dots, n) \\ & \quad \times dx_1 \cdots dx_n \\ &= \sum_{i=1}^n \sum_{m_1 + \dots + m_i = n} B_{n,i,m_1, \dots, m_i} \beta_{i,m_1, \dots, m_i}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \beta_{i,m_1, \dots, m_i} &= \int_{(\mathbb{R}^d)^n} \mathbb{E} \sum_{(y_1, \dots, y_i) \in (\eta \cap A)^i} \mathbf{1}(x_1, \dots, x_{m_1} \in \nu_\eta(y_1), \dots, x_{n-m_i+1}, \dots, x_n \in \nu_\eta(y_i)) \\ & \quad \times dx_1 \cdots dx_n \end{aligned}$$

and B_{n,i,m_1, \dots, m_i} denotes the number of ways to divide the set $\{1, 2, \dots, n\}$ into i subsets of size m_1, \dots, m_i . It is clear that

$$B_{n,n,1, \dots, 1} = 1. \tag{18}$$

Fix some i and m_1, \dots, m_i . Using the Slivnyak–Mecke formula (9), we obtain

$$\beta_{i,m_1,\dots,m_i} = \lambda^i \int_{(\mathbb{R}^d)^n} \int_{A^i} \mathbb{P}(x_1, \dots, x_{m_1} \in \nu_{\eta \cup \tilde{y}_i}(y_1), \dots, x_{n-m_i+1}, \dots, x_n \in \nu_{\eta \cup \tilde{y}_i}(y_i)) \times dy_1 \cdots dy_i dx_1 \cdots dx_n,$$

where $\tilde{y}_i = \{y_1, \dots, y_i\}$. Taking into account the fact that $\nu_{\eta \cup \tilde{y}_i}(y_r) \subset \nu_{\eta \cup \{y_r\}}(y_r)$, and using Fubini’s theorem, Lemma 2, and Lemma 3, we obtain

$$\begin{aligned} \beta_{i,m_1,\dots,m_i} &\leq \lambda^i \int_{A^i} \prod_{r=1}^i \int_{(\mathbb{R}^d)^{m_r}} (\mathbb{P}(x_1, \dots, x_{m_r} \in \nu_{\eta \cup \{y_r\}}(y_r)))^{1/i} dx_1 \cdots dx_{m_r} dy_1 \cdots dy_i \\ &\leq \lambda^i \int_{A^i} \prod_{r=1}^i \left(\frac{im_r}{\lambda}\right)^{m_r-1} \int_{\mathbb{R}^d} \left(\exp\left(\frac{-\lambda\kappa_d \|x_1 - y_r\|^d}{im_r}\right)\right) dx_1 dy_1 \cdots dy_i. \end{aligned}$$

By (15) we obtain

$$\beta_{i,m_1,\dots,m_i} \leq C \text{Vol}^i(A) \lambda^{i - \sum_{r=1}^i m_r} = C \text{Vol}^i(A) \lambda^{i-n}.$$

The maximum order of λ is achieved for $i = n$, which together with (17) and (18) implies that

$$\begin{aligned} \mathbb{E} \text{Vol}^n(A_\eta) &\leq \lambda^n \int_{(\mathbb{R}^d)^n} \int_{A^n} \mathbb{P}(x_r \in \nu_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) dy_1 \cdots dy_n dx_1 \cdots dx_n \\ &\quad + C(\text{Vol}(A))^{n-1} \lambda^{-1} \quad \text{as } \lambda \rightarrow \infty. \end{aligned}$$

It is clear that

$$\mathbb{P}(x_r \in \nu_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) \leq \mathbb{P}(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n).$$

Therefore, by Lemma 6,

$$\mathbb{E} \text{Vol}^n(A_\eta) \leq v_n + 2 \sum_{s < t} v_{n,s,t} + C(\text{Vol}(A))^{n-1} \lambda^{-1} \quad \text{as } \lambda \rightarrow \infty, \tag{19}$$

where

$$v_n = \lambda^n \int_{(\mathbb{R}^d)^n} \int_{A^n} \exp\left(-\lambda\kappa_d \sum_{r=1}^n \|x_r - y_r\|^d\right) dy_1 \cdots dy_n dx_1 \cdots dx_n$$

and

$$\begin{aligned} v_{n,s,t} &= \lambda^n \int_{(\mathbb{R}^d)^n} \int_{A^n} \exp\left(-\frac{\lambda\kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d\right) \\ &\quad \times \exp\left(-\frac{\lambda\kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) \\ &\quad \times dy_1 \cdots dy_n dx_1 \cdots dx_n. \end{aligned}$$

By (10),

$$v_n = \text{Vol}^n(A). \tag{20}$$

Let us estimate $v_{n,s,t}$. Using Fubini’s theorem, it follows from (15) that

$$\begin{aligned}
 v_{n,s,t} &\leq C \text{Vol}^{n-2}(A) \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_A \int_A \exp\left(-\frac{\lambda \kappa_d}{n+1} (\|x_s - y_s\|^d + \|x_t - y_t\|^d)\right) \\
 &\quad \times \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) \\
 &\quad \times dy_t dy_s dx_t dx_s \\
 &\leq C \text{Vol}^{n-2}(A) \lambda^2 \int_{\mathbb{R}^d} \int_A \exp\left(-\frac{\lambda \kappa_d}{n+1} (\|x_s - y_s\|^d)\right) \\
 &\quad \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) \\
 &\quad \times dy_t dx_t dy_s dx_s.
 \end{aligned}$$

Furthermore, by (12),

$$v_{n,s,t} \leq C \text{Vol}^{n-2}(A) \int_{\mathbb{R}^d} \int_A \exp\left(-\frac{\lambda \kappa_d}{n+1} \|x_s - y_s\|^d\right) dy_s dx_s,$$

and applying (15) again, we obtain

$$v_{n,s,t} \leq C \text{Vol}^{n-1}(A) \lambda^{-1}.$$

Combining this with estimate (19) and (20), we obtain

$$E \text{Vol}^n(A_\eta) \leq \text{Vol}^n(A) + C \text{Vol}^{n-1}(A) \lambda^{-1} \quad \text{as } \lambda \rightarrow \infty.$$

The application of Lyapunov’s inequality

$$E \text{Vol}^n(A_\eta) \geq (E \text{Vol}(A_\eta))^n = \text{Vol}^n(A)$$

completes the proof.

Proof of Theorem 3. We have

$$E \text{Vol}^n(A \Delta A_\eta) = E(\text{Vol}(A \setminus A_\eta) + \text{Vol}(A_\eta \setminus A))^n = \sum_{k=0}^n \binom{n}{k} u_k, \tag{21}$$

where

$$u_k = E \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \mathbf{1}(x_1, \dots, x_k \in A_\eta, x_{k+1}, \dots, x_n \notin A_\eta) dx_1 \cdots dx_n.$$

Fix some k . We have

$$\begin{aligned}
 u_k &= E \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \mathbf{1}(\text{there exist } (y_1, \dots, y_k) \in (\eta \cap A)^k, \\
 &\quad (y_{k+1}, \dots, y_n) \in (\eta \setminus A)^{n-k}: x_i \in v_\eta(y_i), i = 1, \dots, n) dx_1 \cdots dx_n \\
 &= \sum_{i=1}^k \sum_{j=1}^{n-k} \sum_{m_1+\dots+m_i=k} \sum_{l_1+\dots+l_j=n-k} B_{k,i,m_1,\dots,m_i} B_{n-k,j,l_1,\dots,l_j} \beta_{i,j,m_1,\dots,m_i,l_1,\dots,l_j}, \tag{22}
 \end{aligned}$$

where

$$\begin{aligned} &\beta_{i,j,m_1,\dots,m_i,l_1,\dots,l_j} \\ &= \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \mathbb{E} \sum_{(y_1,\dots,y_i) \in (\eta \cap A)^i} \sum_{(y_{i+1},\dots,y_{i+j}) \in (\eta \setminus A)^j} \mathbf{1}(x_1, \dots, x_{m_1} \in \nu_\eta(y_1), \dots, \\ &\hspace{20em} x_{n-l_j+1}, \dots, x_n \in \nu_\eta(y_{i+j})) \\ &\hspace{20em} \times dx_1 \cdots dx_n \end{aligned}$$

and $B_{k,i,m_1,\dots,m_i}, B_{n-k,j,l_1,\dots,l_j}$ are the same combinatorial coefficients as in the proof of Theorem 2.

Fix some i, j , and $m_1, \dots, m_i, l_1, \dots, l_j$. Using the Slivnyak–Mecke formula (9) twice, we obtain

$$\begin{aligned} &\beta_{i,j,m_1,\dots,m_i,l_1,\dots,l_j} \\ &= \lambda^{i+j} \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^j} \int_{A^i} \mathbb{P}(x_1, \dots, x_{m_1} \in \nu_{\eta \cup \tilde{y}_{i+j}}(y_1), \dots, \\ &\hspace{15em} x_{n-l_j+1}, \dots, x_n \in \nu_{\eta \cup \tilde{y}_{i+j}}(y_{i+j})) \\ &\hspace{15em} \times dy_1 \cdots dy_{i+j} dx_1 \cdots dx_n, \end{aligned}$$

where $\tilde{y}_{i+j} = \{y_1, \dots, y_{i+j}\}$. By Fubini’s theorem and Lemma 2,

$$\begin{aligned} &\beta_{i,j,m_1,\dots,m_i,l_1,\dots,l_j} \\ &\leq \lambda^{i+j} \int_{(\mathbb{R}^d \setminus A)^j} \int_{A^i} \prod_{r=1}^i \int_{(\mathbb{R}^d \setminus A)^{m_r}} (\mathbb{P}(x_1, \dots, x_{m_r} \in \nu_{\eta \cup \{y_r\}}(y_r)))^{1/(i+j)} dx_1 \cdots dx_{m_r} \\ &\hspace{10em} \times \prod_{r=1}^j \int_{A^{l_r}} (\mathbb{P}(x_1, \dots, x_{l_r} \in \nu_{\eta \cup \{y_{i+r}\}}(y_{i+r})))^{1/(i+j)} dx_1 \cdots dx_{l_r} \\ &\hspace{10em} \times dy_1 \cdots dy_{i+j}. \end{aligned}$$

Using Lemma 3 and (16), we obtain, asymptotically as $\lambda \rightarrow \infty$,

$$\begin{aligned} \beta_{i,j,m_1,\dots,m_i,l_1,\dots,l_j} &\leq C\text{Per}(A)^{i+j} \lambda^{i+j+\sum_{r=1}^i(-m_r-1/d)+\sum_{r=1}^j(-l_r-1/d)} \\ &= C\text{Per}(A)^{i+j} \lambda^{-n+i+j-(i+j)/d}. \end{aligned}$$

The maximum order of λ is achieved for $i = k, j = n - k$, and the next term of maximum order is achieved for $i + j = n - 1$, which together with (22) and (18) implies that, asymptotically as $\lambda \rightarrow \infty$,

$$\begin{aligned} u_k &\leq \lambda^n \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^{n-k}} \int_{A^k} \mathbb{P}(x_r \in \nu_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) \\ &\hspace{15em} \times dy_1 \cdots dy_n dx_1 \cdots dx_n \\ &\quad + C\text{Per}(A)^{n-1} \lambda^{-1-(n-1)/d}. \end{aligned}$$

It is clear that

$$P(x_r \in \cup_{\eta \in \tilde{y}_n} (y_r), r = 1, \dots, n) \leq P(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n).$$

Therefore, by Lemma 6, asymptotically as $\lambda \rightarrow \infty$,

$$u_k \leq v_k + 2 \sum_{s < t} v_{k,s,t} + C \text{Per}(A)^{n-1} \lambda^{-1-(n-1)/d}, \tag{23}$$

where

$$v_k = \lambda^n \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^{n-k}} \int_{A^k} \exp\left(-\lambda \kappa_d \sum_{r=1}^n \|x_r - y_r\|^d\right) dy_1 \cdots dy_n dx_1 \cdots dx_n$$

and

$$\begin{aligned} v_{k,s,t} = \lambda^n \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^{n-k}} \int_{A^k} \exp\left(-\frac{\lambda \kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d\right) \\ \times \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) \\ \times dy_1 \cdots dy_n dx_1 \cdots dx_n. \end{aligned}$$

By identity (11),

$$v_k = 2^{-n} (\text{E Vol}(A \Delta A_\eta))^n. \tag{24}$$

Let us estimate $v_{k,s,t}$. For instance, we assume that $s \leq k$ and $t \geq k + 1$ (other cases are treated in the same way). In the same way as in the proof of Theorem 2, we obtain, by inequality (16),

$$\begin{aligned} v_{k,s,t} \leq C \text{Per}(A)^{n-2} \lambda^{2-(n-2)/d} \\ \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus A} \int_{\mathbb{R}^d} \int_A \exp\left(-\frac{\lambda \kappa_d}{n+1} (\|x_s - y_s\|^d)\right) \\ \times \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) \\ \times dy_s dy_t dx_s dx_t \end{aligned}$$

as $\lambda \rightarrow \infty$. Furthermore, by (12),

$$v_{k,s,t} \leq C \text{Per}(A)^{n-2} \lambda^{-(n-2)/d} \int_{\mathbb{R}^d \setminus A} \int_A \exp\left(-\frac{\lambda \kappa_d}{n+1} \|x_s - y_s\|^d\right) dy_s dx_s \quad \text{as } \lambda \rightarrow \infty,$$

and applying (16) again, we obtain

$$v_{k,s,t} \leq C \text{Per}(A)^{n-1} \lambda^{-1-(n-1)/d} \quad \text{as } \lambda \rightarrow \infty.$$

Combining this with (23) and (24), we obtain

$$u_k \leq 2^{-n} (\mathbb{E} \text{Vol}(A \Delta A_\eta))^n + C \text{Per}(A)^{n-1} \lambda^{-1-(n-1)/d} \quad \text{as } \lambda \rightarrow \infty. \tag{25}$$

Inserting this into (21) we obtain

$$\mathbb{E} \text{Vol}^n(A \Delta A_\eta) \leq (\mathbb{E} \text{Vol}(A \Delta A_\eta))^n + C \text{Per}(A)^{n-1} \lambda^{-1-(n-1)/d} \quad \text{as } \lambda \rightarrow \infty.$$

The application of Lyapunov’s inequality

$$\mathbb{E} \text{Vol}^n(A \Delta A_\eta) \geq (\mathbb{E} \text{Vol}(A \Delta A_\eta))^n$$

completes the proof.

Proof of Corollary 2. As mentioned above, the second inequality immediately follows from Theorem 3. To prove the first inequality, let us again combine (21) and (25) now for $n = 2$ and $k = 0, 1, 2$. We obtain, for sufficiently large λ ,

$$\mathbb{E} \text{Vol}^2(A \setminus A_\eta) + \mathbb{E} \text{Vol}^2(A_\eta \setminus A) \leq \frac{1}{2} (\mathbb{E} \text{Vol}(A \Delta A_\eta))^2 + 2C \text{Per}(A) \lambda^{-1-1/d}$$

and

$$2 \mathbb{E}(\text{Vol}(A \setminus A_\eta) \text{Vol}(A_\eta \setminus A)) \leq \frac{1}{2} (\mathbb{E} \text{Vol}(A \Delta A_\eta))^2 + 2C \text{Per}(A) \lambda^{-1-1/d}.$$

Combining this with Lyapunov’s inequality

$$\mathbb{E} \text{Vol}^2(A \setminus A_\eta) + \mathbb{E} \text{Vol}^2(A_\eta \setminus A) + 2 \mathbb{E}(\text{Vol}(A \setminus A_\eta) \text{Vol}(A_\eta \setminus A)) \geq (\mathbb{E} \text{Vol}(A \Delta A_\eta))^2,$$

we obtain, for sufficiently large λ ,

$$\mathbb{E} \text{Vol}^2(A \setminus A_\eta) + \mathbb{E} \text{Vol}^2(A_\eta \setminus A) - 2 \mathbb{E}(\text{Vol}(A \setminus A_\eta) \text{Vol}(A_\eta \setminus A)) \leq 4C \text{Per}(A) \lambda^{-1-1/d},$$

which together with (13) completes the proof.

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