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END BEHAVIOUR AND ERGODICITY FOR HOMEOMORPHISMS OF MANIFOLDS WITH FINITELY MANY ENDS

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1. Introduction. The recent paper of Berlanga and Epstein [**5**] demonstrated the significant role played by the "ends" of a noncompact manifold M in answering questions relating homeomorphisms of M to measures on M. In this paper we show that an analysis of the end behaviour of measure preserving homeomorphisms of a manifold also leads to an understanding of some of their ergodic properties, and allows results previously obtained for compact manifolds to be extended (with qualifications) to the noncompact case. We will show that ergodicity is typical (dense G_δ) with respect to various compact-open topology closed subsets of the space $\mathscr{H} = \mathscr{H}(M, \mu)$ consisting of all homeomorphisms of a manifold M which preserve a measure μ. It may be interesting for topologists to note that we prove when M is a σ-compact connected n-manifold, $n \geqq 2$, then M is the countable union of an increasing family of compact connected manifolds. If M is a PL or smooth manifold, this is well known and easy. If M is just, however, a topological n-manifold then we apply the recent results [**9**] and [**12**] to prove the result. The Borel measure μ is taken to be nonatomic, locally finite, positive on open sets, and zero for the manifold boundary of M.

The study of the space $\mathscr{H}(M, \mu)$ for compact connected manifolds was initiated by Oxtoby and Ulam [**10**] who showed that ergodicity is typical. Later Katok and Stepin [**8**] and Alpern [**1**, **2**] extended the ergodic theoretic generality of the Oxtoby-Ulam Theorem by showing that ergodicity could be replaced by weak mixing or indeed by any property which is typical in the purely measure theoretical context (i.e., for automorphisms of a Lebesgue space, with the coarse topology).

This paper also seeks to generalise the Oxtoby-Ulam Theorem, but in a different direction; namely by removing the assumption that the underlying manifold is compact. A complete generalisation in this direction was shown to be impossible by Alpern's example [**4**] of the unit shift

$$h(\theta, r) = (\theta, r + 1)$$

along the two dimensional cylinder $M = S^1 \times \mathbf{R}$. Any homeomorphism h' which is sufficiently close to h in the compact-open topology will map

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$S^1 \times [0, \infty)$ into a subset of itself, and therefore there is an open subset of $\mathscr{H}$ consisting entirely of nonergodics. This obstruction may be contrasted with the successful generalization of the Oxtoby-Ulam Theorem to all Euclidean spaces $\mathbf{R}^n$, $n \geqq 2$, achieved by Prasad [**11**]. In order to distinguish (i) between manifolds which are like Euclidean space or the cylinder in this respect (typicality or not of ergodicity) (ii) between various closed subspaces of $\mathscr{H}(M)$ for fixed M, and (iii) between individual homeomorphisms which are approximable by ergodic homeomorphisms ("approximately ergodic") or not, we must use the notions "end" and "drift".

Roughly speaking an end of M is a distinct way of "going to infinity" on M. More precisely, an end of M is a function e which assigns to each compact subset K of M a non-empty connected component $e(K)$ of $M - K$, in such a way that $K_1 \subset K_2$ implies that $e(K_2) \subset e(K_1)$. The manifold M may be compactified by adjoining the ends, $E(M)$, and defining for each compact K a typical neighbourhood $N_K(e_0)$ of an end e_0 as the set

$$e_0(K) \cup \{\text{ends } e{:}e(K) = e_0(K)\}$$

[**5**]. Every homeomorphism $h \in \mathscr{H}(M)$ induces a homeomorphism

$$\sigma_h{:}E(M) \rightarrow E(M)$$

such that

$$\sigma_h(e)(K) = h(e(h^{-1}(K)))$$

for all $e \in E(M)$ and compact K. For this paper we will assume that $E = E(M)$ is finite, so that σ_h is a permutation of ends (denoted usually by σ). We will say an end e has finite or infinite measure according as $\mu(e(K)) < \infty$ for some compact K or not. It is easily seen that the finite measured ends $F \subset E$ are invariant under any permutation σ induced by an $h \in \mathscr{H}$ (similarly for the infinite ends $E - F$).

A homeomorphism $h \in \mathscr{H}$ is called drift-free if for any cycle C of the induced permutation σ on E, and any sufficiently large compact set K,

$$\sum_{e \in C} [\mu(h(K) \cap e(K)) - \mu(K \cap he(K))] = 0.$$

We can now state our main result relating end behaviour to approximate ergodicity for $h \in \mathscr{H}$.

THEOREM 3. *The following five successively weaker conditions are each sufficient to ensure that $h \in \mathscr{H}(M)$ is approximately ergodic. The last* (v) *is also necessary.*

(i) *M is compact*

(ii) $\mu(M) < \infty$

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(iii) *M has at most one end of infinite measure*
(iv) *h induces a cyclic permutation of the ends of infinite measure*
(v) *h is drift free.*

Furthermore under assumptions (i)-(iii) *ergodicity may be replaced by any other measure theoretic property which is generic in the space of measure preserving bijections of a non-atomic Lebesgue space of measure* $\mu(M)$, *with the coarse topology. Since the identity map is drift free*, (M, μ) *always supports an ergodic homeomorphism.*

We note that the different category of ergodicity in the examples of $\mathbf{R}^n$ and the infinite cylinder, which motivated the work, is now explained by the presence and absence of condition (iii). More generally we find necessary and sufficient conditions for ergodicity to be generic in $\mathcal{H}(M)$ and for the closed subspace $\mathcal{H}_\sigma$ consisting of all $h \in \mathcal{H}(M)$ which induce the permutation σ on the ends.

THEOREM 4. (i) *Ergodicity is* (*compact-open*) *generic in* $\mathcal{H}(M)$ *if and only if M has at most one end of infinite measure.*

(ii) *Ergodicity is generic in* $\mathcal{H}_\sigma$ *if and only if* σ *is a cyclic permutation of the ends of infinite measure.*

The paper is organized as follows. In Section 2 we outline the general setting and ideas of the proof, particularly the relation between the measure theoretic and topological aspects. We quote there two results established for the compact and finite measure contexts that we will need later. In Section 3 using the recent results of Quinn [**12**] and Kirby and Siebenmann [**9**] we show that every σ-compact connected n-manifold is the increasing union of compact connected n-manifolds. Using this structure for M we introduce, for each homeomorphism, the drift (of the homeomorphism) into a collection of ends; roughly speaking it measures the net flow of mass under the homeomorphism into a collection of ends. The drift seems to be related to Thurston's flux homomorphism of [**14**], if the manifold is oriented by the ends (see also [**7**] and [**13**]). Section 4 contains the main approximation results of the paper. Roughly speaking, to approximate arbitrarily well in the compact open topology a homeomorphism h, by an ergodic measure preserving homeomorphism, we first go outside of $\mathcal{H}$ by approximating h by a measure preserving transformation f (not necessarily a homeomorphism) which is ergodic. This is the content of Propositions 1 and 2. Then we come back into $\mathcal{H}$ by approximating (in two different senses simultaneously) the measure preserving transformation $h^{-1}f$ by a measure preserving homeomorphism. This is the Luzin Theorem; Proposition 3. A Baire category argument used in Theorem 1 (Section 5) completes the approximation of the homeomorphism h by an ergodic measure preserving homeomorphism. The reader is urged to skim over Sections 2 and 3 at a first reading.

2. Definitions and results from previous papers. In this section we define terms and state known results.

As in [**1**] and [**4**] our approach is to consider $\mathscr{H} = \mathscr{H}(M)$ as a subset of $\mathscr{G}(M)$, the group of all invertible (Borel measurable) μ-preserving transformations of M. We will use three topologies on $\mathscr{G}(M)$: the coarse (sometimes called weak), uniform and compact-open. These are defined by the following basic neighbourhoods, in which g and f belong to $\mathscr{G}$, B and B_i are finite measured subsets of M, K is a compact set, ϵ and λ are positive numbers, d denotes the metric on M and Δ denotes symmetric difference.

$$\text{coarse: } \mathscr{V}(g, \lambda, B_1, \ldots, B_n) = \{f:\mu(fB_i \Delta gB_i) < \lambda \quad i = 1, \ldots, n\}$$

$$\text{uniform: } \mathscr{U}(g, B, \lambda) = \{f:\mu\{x \in B:fx \neq gx\} < \lambda\}$$

$$\text{compact-open: } \mathscr{C}(g, K, \epsilon) = \{f:d(f(x), g(x)) < \epsilon \text{ for a.e. } x \text{ in } K\}.$$

The coarse topology is coarser than the other two and ($\mathscr{G}$, coarse) and ($\mathscr{H}$, compact-open) are each topologically complete spaces.

The following result comes from Theorems 2 and 3 and the proof of Theorem 3 in [**3**].

THEOREM A. *Let $\mathscr{G}$ denote the group of all μ-preserving bijections of a finite Lebesgue space (X, Σ, μ). Let τ_1 and τ_2 belong to $\mathscr{G}$, with τ_2 antiperiodic. Let $\mathscr{A}$ be a finite subalgebra of Σ such that $\tau_1/\mathscr{A}$ has no non-trivial periodic point (set). (This means there is no $A \in \mathscr{A} - \{\emptyset, X\}$ with $\tau_1^i(A) \in \mathscr{A}$ for $i = 1, \ldots, k$, and $\tau_1^k(A) = A$.) Then there is a conjugate $\tau_2' = \tau^{-1}\tau_2\tau$, $\tau \in \mathscr{G}$, of τ_2 such that*

$$\tau_2'(A) = \tau_1(A) \quad \textit{for all } A \in \mathscr{A}.$$

Suppose further that d is a metric on X such that μ is positive on open sets. Let Y denote the union of all atoms of $\mathscr{A}$ whose image under τ_1 is relatively compact and connected. Then given any $\delta > 0$ we may further assume that

$$d(\tau_2'(x), \tau_1(x)) < \delta \quad \textit{for } \mu - \text{a.e. } x \textit{ in } Y.$$

The proof of the next result is in [**1**], Theorem 5.1.

THEOREM B. *Let (R, d) be a compact connected n-manifold such that if $d(x, y) < \epsilon$ there is a connected open set containing x and y with diameter less than ϵ. Let μ be a finite Borel measure on R which is nonatomic, locally positive and zero for the boundary of R. Then given any μ-preserving bijection $g:R \to R$ with*

$$\text{ess sup } d(x, g(x)) < \epsilon,$$

and any $\lambda > 0$, there is a μ-preserving homeomorphism $h^*:R \to R$ which fixes the boundary of R and such that

$$\sup d(x, h^*(x)) < \epsilon \quad \textit{and}$$

$$\mu\{x:h^*(x) \neq g(x)\} < \lambda.$$

These results are used as follows: Given $h \in \mathscr{H}(M)$ and compact K, to find a homeomorphism near h on K which is almost ergodic, we will use Theorem A to find a μ-preserving automorphism (not necessarily continuous) of $K \cup hK$ onto itself which is ergodic, and near (in the manifold metric d) h on K. That this automorphism is extendible to an ergodic $f \in \mathscr{G}(M)$ is the content of Propositions 1 and 2. Then by using Theorem B (the Luzin Theorem) the automorphism $h^{-1}f$ is approximated by homeomorphism h^* which is close enough to $h^{-1}f$ so that hh^* is near h on K and hh^* is "almost" ergodic on M.

3. A decomposition theorem, separating sets and drift. Given the σ-compact connected n-manifold M, we would like to write M as the increasing union of compact connected n-manifolds. If the manifold is smooth or PL this presents no problem; for the general case we use the results in [**12**] and [**9**].

THEOREM 0. *Let M be a σ-compact connected n-manifold, $n \geqq 2$. If $K \subset M$ is a compact set then there is a compact connected n-manifold R containing K in its interior.*

Proof. If $n \neq 4$, then this is a consequence of the fact that M possesses a handlebody decomposition; for $n \geqq 6$ this is due to Kirby and Siebenmann [**9**] III, 2.1, for $n = 5$ to Quinn [**12**], (Theorem 2.3.1) and $n = 2, 3$ this follows from the triangulation theorem of Moise.

For $n = 4$ the following argument is based on the recent result of Quinn [**12**] (Theorem 2.2.3) that M has a smooth structure in the complement of a point p. First we find a compact connected manifold containing p. Let $B(p)$ be a Euclidean neighbourhood of p. The boundary of $B(p)$ is a compact set contained in $M - \{p\}$, and so it is contained in a smooth manifold R_1. The set

$$R^* = R_1 \cup \{\text{the component of } M - R_1 \text{ containing } p\}$$

is a compact connected n-manifold containing p. It is now a simple matter to extend R^* to a compact connected n-manifold containing K, since away from p, M has a smooth structure.

F. Quinn has suggested to us the following more usual way to obtain this result: Pick $x \in M$,

(1) let $f:M \to [0, \infty)$ be a proper function;

(2) approximate f so that each integer is a regular value;

(3) let M_i = the component of $f^{-1}[0, i]$ containing x.

Then

$$M = \bigcup_{i=1}^{\infty} M_i$$

and each M_i is compact and connected. If M is a topological manifold then

(2) requires the transversality theorem of Kirby-Siebenmann [9], III, 1.1 or III 3.1 for $n \neq 4, 5$ and Theorem 2.4.1 of Quinn, [12] if $n = 4, 5$.

LEMMA 0. *Let $K \subset M$ be a compact set. Then there is a compact connected n-dimensional manifold R, containing K in its interior, such that $M - R$ has no bounded (relatively compact) components and R has boundary μ measure zero.*

Proof. Suppose R_1 is a compact connected n-manifold, from Theorem 0 containing K in its interior. Then

$$R^* = R_1 \cup \{\text{the bounded components of } M - R_1\}$$

is a compact connected n-manifold having no bounded components in its complement. Next perturb ∂R^*, the boundary of R^*, by a small homeomorphism of M, h, so that

$$\mu(h(\partial R^*)) = 0.$$

Then $R = hR^*$ will still contain K in its interior if h is small enough. That such small homeomorphisms exist (due to Oxtoby and Ulam) can be found for example in [5].

In the earlier papers, [4] and [11], for the case $M = \mathbf{R}^n$ the fact that K, in the compact-open neighbourhoods $\mathscr{C}(h, K, \epsilon)$ could be restricted to n-cubes was heavily exploited. The analogue for general manifolds will be to restrict K to be a "strictly h-separating set". The rest of this section will be devoted to the definition (in stages) and elementary properties of these sets. We will also define our previously mentioned condition of "drift-free".

Definition. A subset $K \subset M$ is called *separating* if it is a compact connected n-manifold with boundary measure zero, its complement has no bounded (relatively compact) components, and $e(K) = e'(K)$ if and only if $e = e'$.

LEMMA 1. *Every compact set is contained in a separating set.*

Proof. Let an arbitrary compact set K_1 be given. For any pair e, e' of distinct ends of M there is a compact set $K_{e,e'}$ such that

$$e(K_{e,e'}) \neq e'(K_{e,e'}).$$

Let K_2 be the (finite) union of these sets. Then K_2, and any larger compact set, will separate any pair of ends. By Lemma 0, there is a compact connected n-manifold R containing the compact set $K = K_1 \cup K_2$. R is a separating set.

Definition. A separating set $K \subset M$ is called *h-separating*, for some $h \in \mathscr{H}$, if

$$h(e(K)) \subset K \cup (\sigma(e))(K) \quad \text{for every end } e,$$

where σ is the permutation of ends induced by h.

LEMMA 2. *Every compact set is contained in an h-separating set.*

Proof. By Lemma 1 we may assume the given compact set K_1 is already separating. Let R be the separating set obtained by applying Lemma 1 to $K = K_1 \cup hK_1$. To establish that R is h-separating it suffices to show that

(1) $h(e(R)) \cap e'(R) = \emptyset$ for all $e' \neq \sigma e$.

Since $h(e(R)) = [\sigma e](h(R))$ (1) is equivalent to

(2) $[\sigma e](h(R)) \cap e'(R) = \emptyset$ for $e' \neq \sigma e$.

But as $hK_1 \subset h(R) \cap R$, the monotonicity property of ends shows that (2) is implied by

(3) $[\sigma e](h(K_1)) \cap e'(hK_1) = \emptyset$ for $e' \neq \sigma e$.

However (3) is true because hK_1 is a separating set (since K_1 is).

In order to define the "drift-free" condition mentioned in the introduction, we consider homeomorphisms $h \in \mathscr{H}$ inducing a given permutation σ on the ends of M. Denote these collectively by $\mathscr{H}_\sigma$ and consider the following real valued functions on $\mathscr{H}_\sigma$.

Definition. For any cycle of ends C permuted by σ and any h-separating set K, $h \in \mathscr{H}_\sigma$, let

$$\delta_C(h, K) = \sum_{e \in C} \mu(hK \cap eK) - \mu(K \cap e(hK)).$$

Observe that if we set

$$C(K) = \bigcup_{e \in C} e(K)$$

then

$$h(C(K)) = C(h(K)) \quad \text{and}$$

$$\delta_C(h, K) = \mu(hK \cap C(K)) - \mu(K \cap h(C(K)).$$

LEMMA 3. *$\delta_C(h, K)$ is independent of K, and henceforth denoted simply $\delta_C(h)$.*

Proof. Let R be any h-separating set containing $K \cup hK \cup h^{-1}K$. We will show that

$$\delta_C(h, R) = \delta_C(h, K).$$

This will establish the lemma since we can always find such an R which simultaneously has this relation to any two given h-separating sets.

Let $Q = R \cap C(K)$ and observe that

$$hQ - Q = [h(R) \cap C(R)] \underset{\text{disj.}}{\cup} [K \cap h(C(K))] \quad \text{and}$$

$$Q - hQ = [h(K) \cap C(K)] \underset{\text{disj.}}{\cup} [R \cap hC(R)].$$

Hence

$$\delta_C(R) - \delta_C(K) = \mu(hQ - Q) - \mu(Q - hQ) = 0.$$

LEMMA 4. $\sum_C \delta_C(h) = 0$, *where the sum is taken over all cycles of ends induced by h.*

Proof. Fix any h-separating set K and write

$$\delta_C(h) = \mu(hK \cap C(K)) - \mu(K \cap C(hK)).$$

Summing over all cycles C, we obtain

$$\sum_C \delta_C(h) = \mu(hK - K) - \mu(K - hK) = 0$$

since h preserves μ.

Definitions. A homeomorphism $h \in \mathscr{H}_\sigma$ (with induced end permutation σ) is called *drift-free* if $\delta_C(h) = 0$ for every cycle C of σ. For fixed σ, the set of all such homeomorphisms is denoted $\mathscr{H}^0_\sigma$.

LEMMA 5. $\delta_C:\mathscr{H}_\sigma \to \mathbf{R}$ *is continuous in the compact-open topology, for every cycle C of* σ. *Hence* $\mathscr{H}^0_\sigma$ *is a closed subset of* $\mathscr{H}_\sigma$ *and hence of* $\mathscr{H}$. *Furthermore* $\mathscr{H}^0_\sigma$ *is invariant under right composition with any homeomorphism of compact support.*

Proof. We show that $\delta_C(\cdot, K)$ is continuous with respect to the weaker coarse topology for any fixed K. If $h_j \to h$ in the coarse topology then

$$\mu(h_jK\Delta hK) + \mu(h_j^{-1}K\Delta h^{-1}K) \to 0,$$

and consequently

$$\mu(h_jK \cap C(K)) \to \mu(hK \cap C(K)) \quad \text{and}$$
$$\begin{aligned}\mu(K \cap C(h_jK)) &= \mu(K \cap h_j(C(K))) \\ &= \mu(h_j^{-1}K \cap C(K)) \to \mu(h^{-1}K \cap C(K)) \\ &= \mu(K \cap hC(K)).\end{aligned}$$

Therefore $\delta_C(h_j) \to \delta_C(h)$.

Let $h_1 \in \mathscr{H}^0_\sigma$ and suppose for some R, h_2 satisfies

$$h_2(R) = R \quad \text{and} \quad h_2(C(R)) = C(R).$$

Then,

$$\begin{aligned}\delta_C(h_1h_2) &= \delta_C(h_1h_2, R) \\ &= \mu(h_1h_2(R) \cap C(R)) - \mu(R \cap h_1h_2(C(R))) \\ &= \mu(h_1R \cap C(R)) - \mu(R \cap h_1(C(R)) = \delta_C(h_1) = 0.\end{aligned}$$

For Propositions 1 and 2 we will need a slightly sharper version of h-separation, namely the following.

Definition. An h-separating set K is called *strictly h-separating* if

$$\mu(K \cap h(C(K))) > 0$$

for every cycle of ends C induced by h.

This definition excludes the possibility that $C(K)$ might be h-invariant, for then it would also have to be f-invariant for the ergodic approximation given in Propositions 1 and 2. (See next section.)

LEMMA 6. *Let $h \in \mathscr{H}^0_\sigma$, $\epsilon > 0$, and an h-separating set K be given. Then there is an $h^* \in \mathscr{H}$ with compact support and*

$$\sup_{x \in M} d(x, h^*(x)) < \epsilon$$

such that K is strictly hh^*-separating. Consequently there is a subbasic family of compact-open neighbourhoods of the form $\mathscr{C}(h, K, \epsilon)$ for which K is strictly h-separating.

Proof. For any cycle C such that $h(C(K)) = C(K)$ choose an n-ball B_C with diameter less than ϵ such that

$$\mu(B_C \cap K) > 0 \quad \text{and} \quad \mu(B_C \cap C(K)) > 0.$$

Define a homeomorphism $h_C \in \mathscr{H}$ with support in B_C such that $B_C \cap K$ is not invariant under h_C. The homeomorphism h_C can be taken to be any ergodic homeomorphism of B_C fixing the boundary, which exists by [**10**]. Or h_C can be constructed using Theorem B, or by specific construction. Take h^* to be the composition of the h_C's. It follows from Lemma 5 that $hh^* \in \mathscr{H}^0_\sigma$ and by construction that K is hh^*-separating.

4. Approximation results.

PROPOSITION 1. *Assume $\mu(M) < \infty$. Let h be a μ-preserving homeomorphism of M and let $K \subset M$ be a strictly h-separating (compact) set. If M is compact let $K = M$. Then for any antiperiodic μ-preserving bijection $\theta \in \mathscr{G}(M)$, and any $\delta > 0$, there is a conjugate*

$$f = \tau^{-1}\theta\tau, \quad \tau \in \mathscr{G}(M),$$

of θ satisfying

(i) $d(f^{-1}(y), h^{-1}(y)) < \delta$ *for* μ – a.e. y *in* $h(K)$,

(ii) $f(K) = h(K)$,

(iii) $f(e(K)) = h(e(K))$ *for every end e of M.*

Proof. Assume that M is non compact. Apply Theorem A with $X = M$, $\tau_1 = h^{-1}$, $\tau_2 = \theta^{-1}$, and $\mathscr{A}$ is the finite algebra whose atoms are the sets $h(K) = Y$ and $h(e(K))$ as e varies over the ends of M. The assumption that K is strictly h-separating ensures that $\tau_1/\mathscr{A}$ has no non-trivial periodic point (set). The automorphism $f \in \mathscr{G}(M)$ defined by $f^{-1} = \tau_2'$, where τ_2' is the conjugate of $\tau_2 = \theta^{-1}$ given by Theorem A, satisfies the requirements of the Proposition.

In the case when $K = M$, is compact, the above proof works trivially with $\mathscr{A} = \{K, \emptyset\}$.

PROPOSITION 2. *Assume $\mu(M) = \infty$. Let $h \in \mathscr{H}_\sigma^0$ and let $K \subset M$ be any strictly h-separating set. Then given any $\delta > 0$ there is an ergodic transformation $f \in \mathscr{G}$ satisfying*

(i) $d(f^{-1}(y), h^{-1}(y)) < \delta$ *for* μ − a.e. y *in* $h(K)$,

(ii) $f(K) = h(K)$,

(iii) $f(e(K)) = h(e(K))$ *for every end e of M.*

Proof. Let C_i, $i = 1, \ldots, p$, denote all the infinite measure cycles of ends under σ, and recall that F denotes the set of ends of finite measure. Ends of finite measure will be treated (Step 1) roughly the same as in the proof of Proposition 1 and infinite measured ends will first be treated together in cycles (Step 1) then individually (Step 2). For each cycle C_i of infinite measured ends define the "arrival" and "departure" sets A_i and D_i, by the formulae

$$A_i = \bigcup_{e \in C_i} (K \cap h(e(K))) \quad \text{and} \quad D_i = \bigcup_{e \in C_i} (e(K) \cap h(K)).$$

A_i consists of all points which have just arrived in K from

$$C_i(K) = \bigcup_{e \in C_i} e(K)$$

and D_i consists of all points which have just departed from K and are now in $C_i(K)$. The assumptions that K is strictly h-separating and h is drift-free (the ^{0} in $\mathscr{H}_\sigma^0$) ensure that

$$\mu(A_i) = \mu(D_i) > 0 \quad \text{for } i = 1, \ldots, p.$$

The construction of the ergodic approximation f is done in two steps. In Step 1, using finite measure techniques, we construct an ergodic μ-preserving automorphism $\tilde{f}: X \to X$ where

$$X = K \cup h(K) \cup \left(\bigcup_{e \in F} e(K) \right).$$

The automorphism $\tilde{f}$ will satisfy

(i′) $d(\tilde{f}^{-1}(y), h^{-1}(y)) < \delta$ for μ − a.e. y in $h(K)$,

(ii′) $\tilde{f}(K) = h(K)$,

(iii′) $\tilde{f}(e(K)) = h(e(K))$

for all finite measured ends $e \in F$, and

$$\tilde{f}(D_i) = A_i, \quad i = 1, \ldots, p.$$

In Step 2 we utilize a skyscraper construction to extend the restriction of $\tilde{f}$ to

$$X - \bigcup_{i=1}^{p} D_i$$

to the required ergodic automorphism $f \in \mathcal{G}$ using the ergodic transformation T induced by $\tilde{f}$ on the base $X - \tilde{f}(X)$ of the skyscraper.

Step 1. Let X be the finite measured (but possibly noncompact) set

$$K \cup h(K) \cup \left(\bigcup_{e \in F} e(K)\right).$$

Let $\tilde{h}:X \to X$ be a μ-preserving automorphism which agrees with h on

$$K \cup \left(\bigcup_{e \in F} e(K)\right)$$

and maps each set D_i onto A_i, $i = 1, \dots, p$. We apply Theorem A to $\tau_1 = \tilde{h}^{-1}$ with τ_2 ergodic and $\mathcal{A}$ the finite algebra whose atoms are $h(K)$, $h(eK)$ for $e \in F$, and A_i, $i = 1, \dots, p$. We take Y to be the atom $h(K)$ whose $\tilde{h}^{-1}$ image is the compact connected set K. As in the proof of Proposition 1 the condition that $\tilde{h}^{-1}$ restricted to $\mathcal{A}$ has no nontrivial periodic points (sets) follows from the assumption that K is a strictly h-separating set (otherwise we might have $\tilde{h}^{-1}(C(K)) = C(K)$ where C is a cycle of finite measure ends). If we take $\tilde{f}^{-1}:X \to X$ to be the ergodic (conjugate to the ergodic τ_2) transformation τ_2' given by Theorem A, then $\tilde{f}$ is ergodic and has the properties listed above as (i′)-(iii′).

Step 2. Let

$$A = \bigcup_{i=1}^{p} A_i \quad \text{and} \quad D = \bigcup_{i=1}^{p} D_i.$$

Let T be the transformation induced by $\tilde{f}$ on A. T is ergodic because $\tilde{f}$ is ergodic. The transformation $\tilde{f}:X \to X$ may be represented as in figure 1 by a skyscraper construction with A as the base, D as the union of tops, and T as the base transformation.

The final stage of the construction of f is to stack with subsets of $M - X$ additional levels on top of the skyscraper to fill up all of the space M. Once this is done, the required automorphism f in $\mathcal{G}(M)$ is defined as follows. All points not on a top level are mapped by f onto the point directly above (one level up). For a point x lying on a top level of the skyscraper define $f(x) = T(y)$ where y is the point on the base A lying below x. The ergodicity of T will guarantee that the resulting automorphism of M is ergodic and that any point leaving X by D_i eventually returns to X through A_i. So if we identify all levels stacked above an A_i with points in $C_i(K)$ the resulting transformation f will satisfy not quite

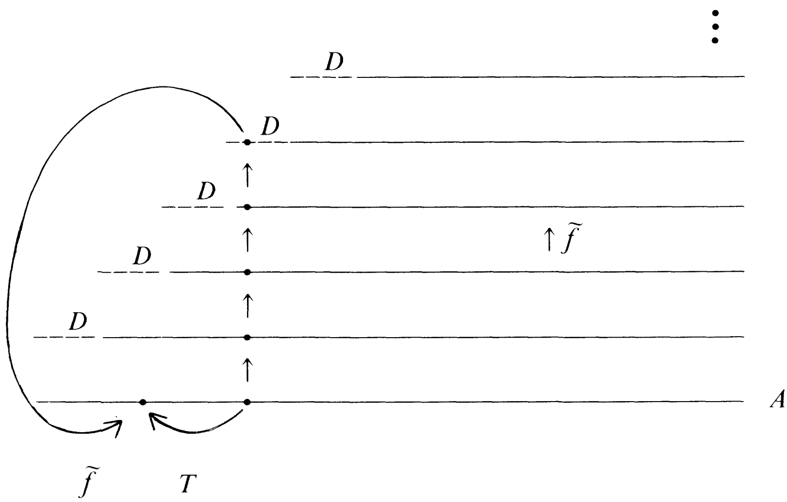

Figure 1. Skyscraper built on A.

condition (iii) but at least the weaker condition $f(C(K)) = h(C(K))$ for every cycle C of infinite measured ends. (Step 1 already guarantees (iii) for all $e \in F$.)

To ensure property (iii) we proceed with the stacking as follows. Fix any cycle C_i of infinite measured ends, and label them $e_1, e_2, \ldots, e_m$, where $\sigma(e_j) = e_{j+1}$ (with arithmetic on j done mod m). Fix any column of the skyscraper (figure 1) determined by a base of the form $A^s \cap \tilde{f}^{-k}D^t$ where $A^s \subset A_i$ and $D^t \subset D_i$ are defined by

$$A^s = K \cap h(e_s(K)) \quad \text{and} \quad D^t = e_t(K) \cup h(K).$$

See figure 2.

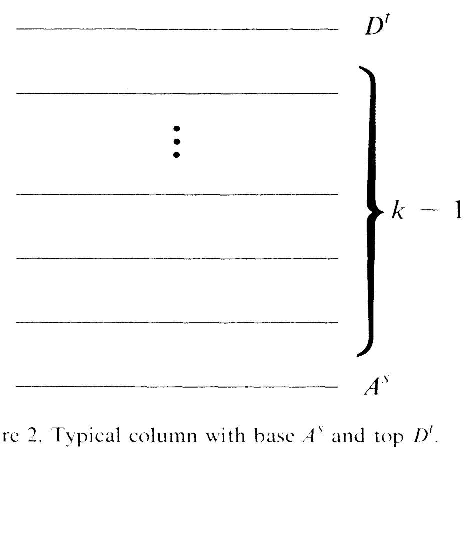

Figure 2. Typical column with base A^s and top D^t.

Let

$$w_0 = \mu(A^s \cap \tilde{f}^{-k} D^t)$$

be the width of this column. On top of this column place $s - t - 1$ (mod m) sets (called column levels) of measure (width) w_0 and identify them with subsets of $e_{t+1}(K) - h(K), e_{t+2}(K) - h(K), \ldots, e_{s-1}(K) - h(K)$ as in figure 3. We call this a parity-corrected column.

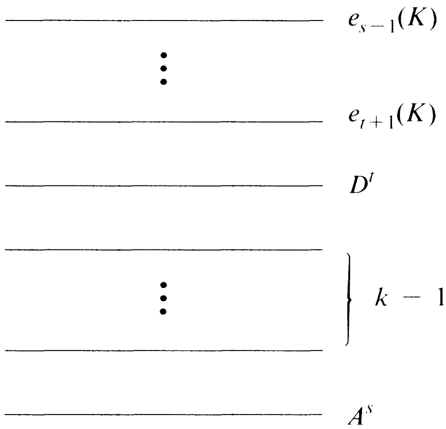

Figure 3. Typical column after parity correction.

If we perform a parity-correction on every column, keep the same ergodic transformation T on the base, and call the union of all columns $\tilde{\tilde{\tilde{X}}}$, then the resulting transformation

$$\tilde{\tilde{\tilde{f}}}:\tilde{\tilde{\tilde{X}}} \rightarrow \tilde{\tilde{\tilde{X}}}$$

satisfies (i), (ii) and

$$\tilde{\tilde{\tilde{f}}}(e(K) \cap \tilde{\tilde{\tilde{X}}}) \subset h(e(K)).$$

The only problem with $\tilde{\tilde{\tilde{f}}}$ is that

$$\mu(\tilde{\tilde{\tilde{X}}}) < \infty$$

so that $\tilde{\tilde{\tilde{f}}}$ is not defined everywhere on M. This is easily corrected as follows. For each infinite measure cycle C_i, pick a single typical (parity-corrected) column of $\tilde{\tilde{\tilde{f}}}$ restricted to

$$\tilde{\tilde{\tilde{X}}} - \tilde{\tilde{\tilde{f}}}^{-1}(A),$$

that is a column as in figure 3. Let $w_0, w_1, w_2, \ldots$ be a sequence decreasing to 0 and with infinite sum. On top of the parity-corrected column of figure 3 stack $m = m_i$ (length of cycle C_i) sets of width w_1, then m sets of width w_2, etc. In each of these groups of m the lowest set is chosen from (or identified with a subset of) $e_s(K)$, and the highest from $e_{s-1}(K)$. This

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results in a column of infinite measure with a top set drawn entirely from $e_{s-1}(K)$. See figure 4. If we perform this construction on one column from each cycle, the resulting automorphism $f:M \to M$ satisfies the remaining requirement (iii) of the proposition exactly.

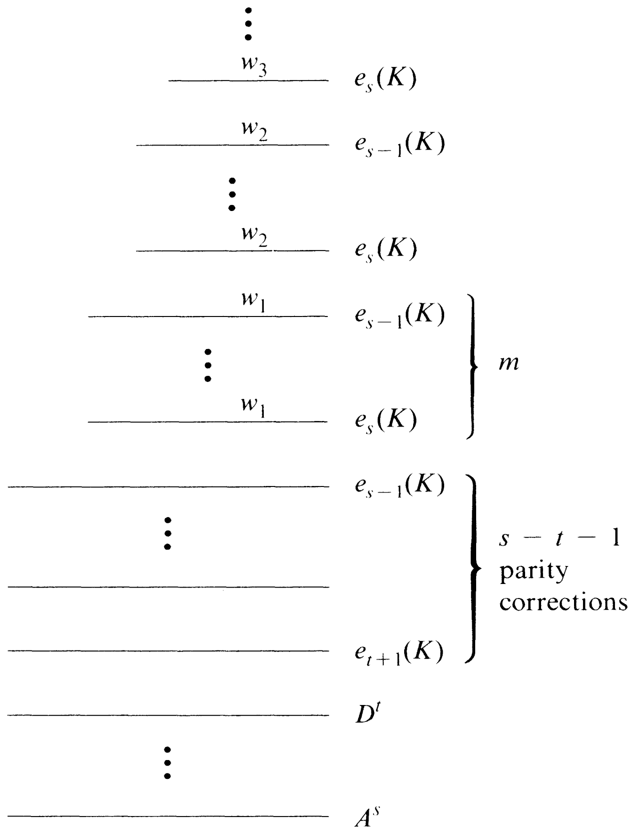

Figure 4. A fully extended column.

PROPOSITION 3. *Let $g \in \mathcal{G}(M)$ satisfy the following conditions for some separating set K.*

(i) $g(K) = K$,

(ii) $d(x, g(x)) < \delta$ *for some* $\delta > 0$ *and* μ − a.e. x *in* K,

(iii) $g(e(K)) = e(K)$ *for every end e.*

Then any uniform (and hence coarse) topology neighbourhood $\mathcal{U}$ of g contains a homeomorphism $h \in \mathcal{H}$ with compact support satisfying (i)-(iii) (*with h replacing g*).

Proof. Let

$$\mathcal{U}(g, B, \lambda) = \{f \in \mathcal{G}:\mu\{x \in B:f(x) \neq g(x)\} < \lambda\}$$

be a uniform topology neighbourhood of g, where $\mu(B) < \infty$ and $\lambda > 0$. For every end e of M define $B_e = B \cap e(K)$ and choose $\lambda_e > 0$ so that

$$\sum_{e \in E} \lambda_e \leqq \lambda.$$

For each end e choose $B'_e \subset B_e$ so that

$$\mu(B_e - B'_e) < \lambda_e/2$$

and $B'_e \cup gB'_e$ is a relatively compact subset of $e(K)$. It is hypothesis (iii) that ensures that this is possible. By Lemma 0 there is a compact connected n-manifold $K_e \subset e(K)$ with boundary measure zero which contains $B'_e \cup gB'_e$ in its interior. Let g_e be any μ-preserving automorphism of K_e onto itself which agrees with g on B'_e. Now apply the Luzin Theorem for measure preserving homeomorphisms, Theorem B, to the restriction of g to K, with norm bound δ, and measure error $\lambda/4$. This yields a μ-preserving homeomorphism $h_K:K \to K$ which is the identity on the boundary of K, satisfying conditions (i) and (ii), and

$$\mu\{x \in K:h_K(x) \neq g(x)\} < \lambda/4.$$

Similarly we apply Theorem B to each $g_e:K_e \to K_e$, but without any δ. This yields μ-preserving homeomorphisms of K_e which fix the boundary of K_e and satisfy

$$\mu\{x \in K_e:h_e(x) \neq g_e(x)\} < \lambda_e/4.$$

Let $h \in \mathscr{H}$ be the μ-preserving homeomorphism which agrees with h_K on K and each h_e on K_e and is the identity elsewhere. It follows that h has compact support and satisfies (i)-(iii). To see that h belongs to $\mathscr{U}(g, B, \lambda)$ observe that B is the disjoint union of the sets

$$B - B' - K, \; B \cap K \quad \text{and} \quad B' = \bigcup_e B'_e.$$

The first of these has measure less than $\lambda/2$. On each of the latter two sets h differs from g only for points of total measure less than $\lambda/4$. Hence

$$\mu\{x \in B:h(x) \neq g(x)\} < \lambda/2 + \lambda/4 + \lambda/4 = \lambda.$$

5. Category theorems.

THEOREM 1. *The ergodic homeomorphisms of M form a dense G_δ subset of $\mathscr{H}_\sigma^0$ (drift-free μ-preserving homeomorphisms inducing the permutation σ on the ends) with respect to the compact-open topology.*

Proof. This statement is vacuous if $\mathscr{H}_\sigma^0$ is empty, which may happen for some σ, so we exclude this possibility. We may assume $\mu(M) = \infty$, since Theorem 2 proves a stronger result when $\mu(M) < \infty$.

Since $\mathscr{E}$ (ergodic automorphisms) is a G_δ subset of $\mathscr{G}$ in the coarse topology **[6]** we may write

$$\mathscr{E} = \bigcap_{i=1}^{\infty} \mathscr{V}_i$$

where each $\mathscr{V}_i$ is a coarse topology open set containing $\mathscr{E}$. Since the compact-open topology is finer than the coarse topology, each set $\mathscr{V}_i \cap \mathscr{H}_\sigma^0$ is open in the compact-open topology on $\mathscr{H}_\sigma^0$. Since $\mathscr{H}_\sigma^0$ is a closed (Lemma 6) and hence Baire subset of the topologically complete space $\mathscr{H}$, in the compact-open topology, it is enough to prove that each set $\mathscr{V}_i \cap \mathscr{H}_\sigma^0$ is compact-open dense in $\mathscr{H}_\sigma^0$. So we must show that

$$\mathscr{V} \cap \mathscr{H}_\sigma^0 \cap \mathscr{C} \neq \emptyset$$

for any compact-open neighbourhood $\mathscr{C}(h, K, \epsilon)$ with $h \in \mathscr{H}_\sigma^0$ and any coarse topology open set $\mathscr{V}$ containing $\mathscr{E}$. By Lemmas 2 and 6 we may assume without loss of generality that K is strictly h-separating. Let $\delta = \omega(\epsilon)$ where ω is the uniform modulus of continuity of h on K. Let $f \in \mathscr{G}$ be the ergodic automorphism of M given by Proposition 2, satisfying

(i) $d(f^{-1}(y), h^{-1}(y)) < \delta$ for μ − a.e. y in $h(K)$,

(ii) $f(K) = h(K)$,

(iii) $f(e(K)) = h(e(K))$ for every end e of M.

Set $g = h^{-1}f$. Since $f \in \mathscr{E} \subset \mathscr{V}$, $g \in h^{-1}\mathscr{V}$, which is open in the coarse topology. Furthermore $g(K) = K$ and $g(e(K)) = e(K)$ for every end e. Also for x in K,

$$\begin{aligned} d(x, g(x)) &= d(x, h^{-1}f(x)) \\ &= d(f^{-1}f(x), h^{-1}f(x)) < \delta, \end{aligned}$$

since $f(x) \in f(K) = h(K)$. Therefore we may apply Proposition 3 to assert the existence of a homeomorphism h^*, with compact support, belonging to the coarse open set $h^{-1}\mathscr{V}$, and with

$$d(x, h^*(x)) < \delta \quad \text{for all } x \text{ in } K.$$

The homeomorphism hh^* is trivially in $\mathscr{V}$, is in $\mathscr{C}(h, K, \epsilon)$ by choice of δ, and belongs to $\mathscr{H}_\sigma^0$ by Lemma 6. Hence

$$\mathscr{V} \cap \mathscr{H}_\sigma^0 \cap \mathscr{C} \neq \emptyset,$$

as required to complete the proof.

THEOREM 2. *Assume $\mu(M) < \infty$. Then for any conjugate-invariant dense G_δ subset $\mathscr{P} \subset \mathscr{G}$ with respect to the coarse topology, $\mathscr{P} \cap \mathscr{H}$ is a dense G_δ subset of $\mathscr{H} = \mathscr{H}(M)$ with respect to the compact-open topology.*

Proof. Since $\mathscr{P}$ and $\mathscr{E}$ are dense G_δ subsets of $\mathscr{G}$, so is $\mathscr{P} \cap \mathscr{E}$. Hence $\mathscr{P}$ contains an ergodic automorphism $\theta \in \mathscr{G}$ and consequently the entire conjugacy class of θ. Consequently we may imitate the proof of Theorem 1 except that Proposition 1 may be used in place of Proposition 2.

THEOREM 3. *Let h be a μ-preserving homeomorphism of the manifold M. Sufficient conditions that h be approximately ergodic include the following successively weaker assumptions, the last of which* (v) *is necessary.*

(i) *M is compact,*

(ii) $\mu(M) < \infty$,

(iii) *M has at most one end of infinite measure,*

(iv) *h induces a cyclic permutation of the ends of infinite measure,*

(v) *h is drift-free.*

For the first two assumptions h is approximately $\mathscr{P}$ (in the compact-open closure of $\mathscr{P} \cap \mathscr{H}$) for any $\mathscr{P} \subset \mathscr{G}$ which is dense G_δ in the coarse topology and conjugate-invariant.

Proof. Since it is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) (the last implication follows from Lemma 4), the first part of the theorem follows if we show that (v) is necessary and sufficient for h to be approximately ergodic.

For necessity, observe that if h is not drift-free then there is a cycle C of the induced permutation σ such that $\delta_C(h') \neq 0$ for all h' in some compact-open neighbourhood of h in $\mathscr{H}_\sigma$. However if h' is even recurrent (much less ergodic), then we must have $\delta_C(h') = 0$. An easy way to see this (arrived at in a discussion with Steve Kalikow) is to observe that if K is h'-separating and T is the transformation induced by the recurrent h' on the set

$$[h'(K) \cap C(K)] \cup [K \cap h'C(K)]$$

then T transposes the two sets in this union. Since the induced transformation is μ-preserving this implies that

$$\delta_C(h') = \mu(h'(K) \cap C(K)) - \mu(K \cap h'C(K)) = 0.$$

To see that condition (v) is sufficient, observe that h belongs to the subset $\mathscr{H}_\sigma^0$ of $\mathscr{H}_\sigma$, in which ergodicity is generic by Theorem 1.

Theorem 2 already states the second part of the present theorem, under assumptions (i) or (ii). Under assumption (iii) the proof contained in [**4**] goes over if n-cubes are replaced by strictly h-separating sets and any ends of finite measure are first dealt with as in Step 1 of the proof of our Proposition 2.

THEOREM 4. (i) *Ergodicity is generic in $\mathscr{H}(M)$ if and only if M has at most one end of infinite measure.*

(ii) *Ergodicity is generic in $\mathscr{H}_\sigma$ if and only if σ is a cyclic permutation of the ends of infinite measure.*

Proof. Consider the following variation of the example of the non-ergodic homeomorphism described in the introduction. The manifold is the tube $\mathbf{R} \times I^{n-1}$ where I^{n-1} is the $n - 1$ dimensional unit cube. The homeomorphism h moves points "to the right" by one unit in the centre,

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tapering off to the identity on the boundary of I^{n-1}. Then h is an end preserving homeomorphism with positive drift to the right, where $\mathbf{R} \times I^{n-1}$ has Lebesgue measure.

Given a manifold M with two distinct ends of infinite measure we can find a tube $\mathbf{R} \times I^{n-1}$ connecting them. Furthermore this tube (with Lebesgue measure) may be embedded into (M, μ) in a measure preserving manner [5] so that the above homeomorphism can also be extended to a μ-preserving homeomorphism h^* of M. It is clear that any homeomorphism in $\mathscr{H}(M)$ close enough to h^* will also be nonergodic. In case (ii) we only need to join two infinite measure ends belonging to distinct cycles. The composition of h^* with a homeomorphism in $\mathscr{H}_\sigma^0$ will yield a homeomorphism in $\mathscr{H}_\sigma$ with non-zero drift; in fact we get an open set of homeomorphisms that are non ergodic.

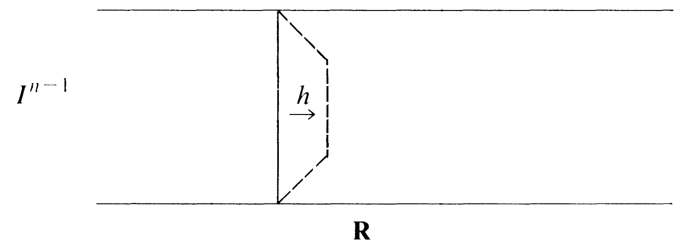

Figure 5. h moves points in the centre, one unit to the right, tapering off to the identity along the boundary of I^{n-1}.

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