8
Fermions on a lattice

It turned out to be most difficult in the lattice approach to QCD to deal with fermions. Putting fermions on a lattice is an ambiguous procedure since the cubic symmetry of a lattice is less restrictive than the continuous Lorentz group.

The simplest chiral-invariant formulations of lattice fermions lead to a doubling of fermionic degrees of freedom, as was first noted by Wilson [Wil75], and describe from 16 to four relativistic continuum fermions, depending on the formulation. One-half of them have a positive axial charge and the other half have a negative one, so that the chiral anomaly cancels. There is a no-go theorem which says that the fermionic doubling is always present under natural assumptions concerning a lattice gauge theory.

A practical way out of this problem is to choose the fermionic lattice action to be explicitly noninvariant under the chiral transformation and to have, by tuning the mass of the lattice fermion, one relativistic fermion in the continuum and the masses of the doublers to be of the order of the inverse lattice spacing. The chiral anomaly is recovered in this way.

In this chapter we consider various formulations of lattice fermions and the doubling problem. We discuss briefly the results on spontaneous breaking of the chiral symmetry in QCD.

8.1 Chiral fermions

The quark fields are generically matter fields, the gauge transformation of which in the continuum is given by Eqs. (5.1) and (5.3), and can be put on a lattice according to Eq. (6.7). Then the lattice gauge transformation is

$$\psi_x \stackrel{\text{g.t.}}{\longrightarrow} \psi'_x = \Omega(x)\psi_x\,, \qquad \bar{\psi}_x \stackrel{\text{g.t.}}{\longrightarrow} \bar{\psi}'_x = \bar{\psi}_x\,\Omega^\dagger(x)\,. \tag{8.1}$$

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The lattice analog of the QCD action (5.13) is given as*

$$\begin{aligned} S[U,\bar\psi,\psi] &= \beta S_{\rm lat}[U] + M\sum_x \bar\psi_x\psi_x \\ &\quad + \frac{1}{2}\sum_{x,\mu>0}\Big[\bar\psi_x\gamma_\mu U_\mu^\dagger(x)\psi_{x+a\hat\mu} - \bar\psi_{x+a\hat\mu}\gamma_\mu U_\mu(x)\psi_x\Big]\,. \end{aligned} \tag{8.2}$$

The first term on the RHS is the pure gauge lattice action (6.16). The second term is a quark mass term on a lattice. The sum in the third term is over all lattice links (i.e. over all sites x and positive directions μ). This action is Hermitian and invariant under the lattice gauge transformation (6.13) and (8.1) with finite lattice spacing.

The partition function of lattice QCD with fermions is defined by

$$Z(\beta,M) = \int\prod_{x,\mu}{\rm d}U_\mu(x)\prod_x {\rm d}\bar\psi_x\,{\rm d}\psi_x\,{\rm e}^{-S[U,\bar\psi,\psi]}\,, \tag{8.3}$$

where the action is given by Eq. (8.2). The integration over $U_\mu(x)$ is as in Eq. (6.31), and the integral over the quark field is the Grassmann one. The averages are defined by

$$\begin{aligned} &\big\langle\, F[U,\bar\psi,\psi]\,\big\rangle \\ &= Z^{-1}(\beta,M)\int\prod_{x,\mu}{\rm d}U_\mu(x)\prod_x {\rm d}\bar\psi_x\,{\rm d}\psi_x\,{\rm e}^{-S[U,\bar\psi,\psi]}\,F[U,\bar\psi,\psi]\,, \end{aligned} \tag{8.4}$$

which extends Eq. (6.39) to the case of fermions. Since both the action and the measure in Eq. (8.4) are gauge invariant at finite lattice spacing, a nonvanishing result only occurs when the integrand, $F[U,\bar\psi,\psi]$, is gauge invariant as well.

In order to show how the lattice action (8.2) reproduces (5.13) in the naive continuum limit $a\to 0$, let us assume that the lattice quark field ψ_x varies slowly from site to site and substitute

$$\left.\begin{aligned} \psi_x &\to a^{3/2}\,\psi(x)\,, \\ \psi_{x+a\hat\mu} &\to a^{3/2}\left[\psi(x) + a\partial_\mu\psi(x)\right] \end{aligned}\right\} \tag{8.5}$$

in $d=4$. Here $\psi(x)$ is a continuum quark field and the power of a arises from the dimensional consideration (remember that ψ_x is dimensionless).

* The standard formula differs from this one by an interchange of U and $U^\dagger$ owing to the inverse ordering of matrices in the phase factors (see the footnote on p. 88). It does not matter how one defines $U_\mu(x)$ since the Haar measure is invariant under Hermitian conjugation.

Equation (8.5) together with Eq. (6.10) yields

$$\bar{\psi}_x \gamma_\mu U^\dagger_\mu(x) \psi_{x+a\hat{\mu}} \quad \to \quad a^3 \bar{\psi}\gamma_\mu\psi + a^4 \bar{\psi}\nabla^{\rm fun}_\mu \gamma_\mu \psi + \mathcal{O}\big(a^5\big)\,, \tag{8.6}$$

where there is no summation over μ in the second term on the RHS as earlier in this part. The first term cancels when substituted into Eq. (8.2), while the second one reproduces the fermionic part of the continuum action. The mass term is also reproduced if $M = am$.

The fermionic lattice action (8.2) was proposed in [Wil74]. For $M = 0$ it is invariant under the global chiral transformation

$$\psi_x \;\xrightarrow{\text{c.t.}}\; {\rm e}^{{\rm i}\alpha\gamma_5}\psi_x\,, \qquad \bar{\psi}_x \;\xrightarrow{\text{c.t.}}\; \bar{\psi}_x\, {\rm e}^{{\rm i}\alpha\gamma_5}\,. \tag{8.7}$$

For this reason, these lattice fermions are called *chiral fermions*. Since the lattice action is both gauge and chiral invariant, there is no Adler–Bell–Jackiw anomaly according to the general arguments of Chapter 3.

Problem 8.1 Show that the lattice action (8.2) is invariant under

$$\psi_x \quad \to \quad {\rm i}\gamma_4\gamma_5\,(-1)^{t/a}\,\psi_x\,. \tag{8.8}$$

Find 15 further similar transformations.

Solution Let us define the generators T_A by

$$\psi_x \;\to\; T_A\psi_x\,, \qquad \bar{\psi}_x \;\to\; \bar{\psi}_x T^\dagger_A\,. \tag{8.9}$$

The transformation (8.8) can be performed for each of the $d = 4$ axes which gives

$$T_A \;=\; {\rm i}\gamma_\mu\gamma_5\,(-1)^{x_\mu/a}\,. \tag{8.10}$$

The other generators are given by products of (8.10). Their explicit form is [KS81a]

$$\begin{aligned} T_A \;=\; & \mathbb{I},\; {\rm i}\gamma_\mu\gamma_5\,(-1)^{x_\mu/a},\; {\rm i}\gamma_\mu\gamma_\nu\,(-1)^{(x_\mu+x_\nu)/a}\;(\mu>\nu)\,, \\ & \gamma_4\,(-1)^{(x+y+z)/a},\;\ldots,\;\gamma_1\,(-1)^{(y+z+t)/a},\;\gamma_5\,(-1)^{(x+y+z+t)/a}\,. \end{aligned} \tag{8.11}$$

All together there are $1+4+6+4+1 = 16$ independent transformations which form a discrete subgroup of the $U(4)$ group.

8.2 Fermion doubling

As was pointed out at the end of the previous section, the lattice fermionic action (8.2) is both gauge and chiral invariant (for $M = 0$) so that there is no chiral anomaly in the continuum. Since the anomaly is present for one continuum fermion, this suggests that the action (8.2) is associated with more than one species of continuum fermions.

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In order to verify this explicitly, let us calculate the poles of the lattice fermionic propagator.

As usual, it is easier to work with the Fourier image of ψ_x:

$$\psi_k = a^{5/2} \sum_x \psi_x \,\mathrm{e}^{-\mathrm{i}kx} . \tag{8.12}$$

The free fermionic action then reads as

$$S_0[\bar{\psi},\psi] = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \,\bar{\psi}_k \,G^{-1}(k)\,\psi_k \tag{8.13}$$

with

$$G^{-1}(k) = \frac{1}{a} \sum_{\mu=1}^{4} \mathrm{i}\gamma_\mu \sin k_\mu a \tag{8.14}$$

for $M = 0$.

In the naive continuum limit, the sin function in Eq. (8.14) can be expanded as a power series in a, which results in the free (inverse) continuum propagator

$$G^{-1}(k) \;\to\; \mathrm{i} \sum_{\mu=1}^{4} \gamma_\mu k_\mu = \mathrm{i}\,\hat{k} . \tag{8.15}$$

Lorentz invariance has been restored after summing over μ.

When passing from the lattice expression (8.14) to the continuum one (8.15), it was implicitly assumed that the momentum k_μ is not of the order of $1/a$ because otherwise the sin function cannot be expanded in a. The doubling of relativistic continuum fermionic states occurs exactly for this reason.

To find the poles of the propagator, let us return to Minkowski space by substituting $k_4 = \mathrm{i}E$, where E is the energy. The poles are then determined from the dispersion law

$$\sinh^2 Ea = \sum_{\mu=1}^{3} \sin^2 p_\mu a . \tag{8.16}$$

Let us look for solutions of Eq. (8.16) with positive energy $E > 0$ (solutions with negative energy are associated as usual with antiparticles). Suppose that a particle moves along the z-axis so that components of the four-momentum

$$p^{(1)} = (E, 0, 0, p_z) \tag{8.17}$$

are related by

$$\sinh Ea \;=\; \sin p_z a\,, \tag{8.18}$$

which follows from the substitution of (8.17) into the dispersion law.

Since sin is a periodic function, the four-vector

$$p^{(2)} \;=\; \left(E, 0, 0, \frac{\pi}{a} - p_z\right) \tag{8.19}$$

is also a solution of Eq. (8.18) if (8.17) is. Quite analogously, the four-vectors

$$\left.\begin{aligned} p^{(3)} \;&=\; \left(E, \frac{\pi}{a}, 0, p_z\right),\\ &\;\vdots\quad ,\\ p^{(8)} \;&=\; \left(E, \frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a} - p_z\right), \end{aligned}\right\} \tag{8.20}$$

which are obtained from $p^{(1)}$ and $p^{(2)}$ by changing zeros for π/a, also satisfy Eq. (8.18). Therefore, a quark state with energy E is eightfold degenerate.

The quark states with four-momenta $p^{(1)}, \ldots, p^{(8)}$ are different states. Their wave functions equal

$$\Psi^{(j)}(t,x,y,z) \;\propto\; \exp\left[\mathrm{i}Et - \mathrm{i}p_x^{(j)}x - \mathrm{i}p_y^{(j)}y - \mathrm{i}p_z^{(j)}z\right]. \tag{8.21}$$

The wave function in the state with momentum $p^{(3)}$ differs, say, from the wave function in the state $p^{(1)}$ by an extra factor of $(-1)^{x/a}$. In other words, it changes strongly as $a \to 0$ with one step along the lattice in the x-direction. One more step returns the wave function to the initial value.

For such functions, the naive continuum limit of the lattice action (8.2) is as good as for the slowly varying functions when Eq. (8.5) holds. In order to see that, let us rewrite the action (8.2) as

$$\begin{aligned} S[U,\bar\psi,\psi] \;&=\; \beta S_{\mathrm{lat}}[U] + M\sum_x \bar\psi_x\psi_x \\ &\quad + \frac{1}{2}\sum_{x,\mu>0}\left\{\bar\psi_x\gamma_\mu\left[U_\mu^\dagger(x)\psi_{x+a\hat\mu} - U_{-\mu}^\dagger(x)\psi_{x-a\hat\mu}\right]\right\}. \end{aligned} \tag{8.22}$$

Even if ψ_x has opposite signs at neighboring lattice sites along the μ-axis,

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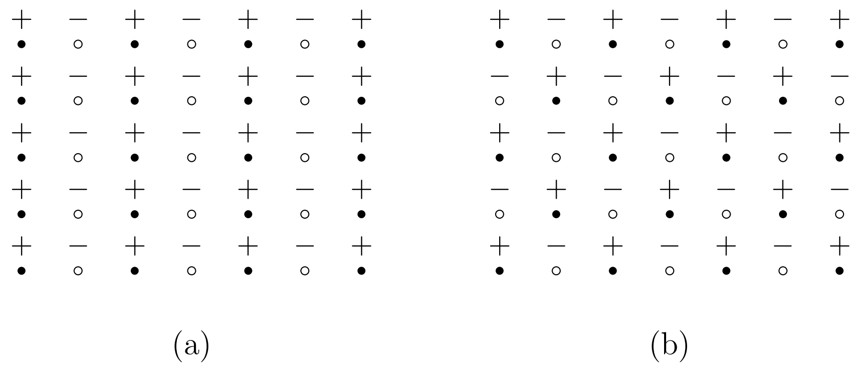

Fig. 8.1. Altering signs of ψ_x on a lattice along (a) one axis and (b) two axes.

as illustrated by Fig. 8.1a, i.e.

$$\psi_{x+a\hat{\mu}} \quad \to \quad -\psi_x \,, \tag{8.23}$$

then the difference $\psi_{x+a\hat{\mu}} - \psi_{x-a\hat{\mu}}$ on the RHS of Eq. (8.22) is still of the correct order in a:

$$\left.\begin{aligned} \psi_x \quad &\to \quad a^{3/2}\,\psi(x)\,, \\ \psi_{x+a\hat{\mu}} - \psi_{x-a\hat{\mu}} \quad &\to \quad -2a^{5/2}\partial_\mu\psi(x)\,, \end{aligned}\right\} \tag{8.24}$$

so that the continuum fermionic action is reproduced except for the sign of the γ_μ-matrix which is opposite to that in Eq. (5.13).

This extra minus sign can be absorbed in the redefinition of the continuum fermionic field $\psi(x) \to \mathrm{i}\gamma_\mu\gamma_5\psi(x)$, which changes its chirality. Therefore, the axial charge of the doublers is opposite. Analogously, four of the eight doublers have a positive axial charge and the four others have a negative one dependent on whether the sign of ψ_x alters at neighboring sites along an even or odd number of axes (see Fig. 8.1). In Euclidean space the doubling also occurs along the temporal axis, so the number of doublers is equal to $2^d = 16$: eight of them with positive and eight with negative axial charge. This explains why the chiral anomaly cancels.

Problem 8.2 Calculate the vector and axial charges of the doublers deriving the vector and axial currents on a lattice.

Solution The vector and axial currents on a lattice can be derived using a lattice analog of the Noether theorem. The invariance of the lattice fermionic action under

$$\psi_x \;\to\; \mathrm{e}^{\mathrm{i}\alpha(x)}\,\psi_x\,, \qquad \bar{\psi}_x \;\to\; \bar{\psi}_x\,\mathrm{e}^{-\mathrm{i}\alpha(x)} \tag{8.25}$$

results in the lattice vector current

$$J^{\mathrm{V}}_\mu(x) \;=\; \frac{1}{2}\left[\bar{\psi}_x\gamma_\mu U^\dagger_\mu(x)\psi_{x+a\hat{\mu}} + \bar{\psi}_{x+a\hat{\mu}}\gamma_\mu U_\mu(x)\psi_x\right], \tag{8.26}$$

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which is conserved in the sense that

$$\sum_{\mu>0} \left[J_\mu^{\rm V}(x) - J_\mu^{\rm V}(x-a\hat\mu)\right] \;=\; 0\,. \tag{8.27}$$

This can be proven using the lattice (quantum) Dirac equation

$$\frac{1}{2}\sum_{\mu>0}\gamma_\mu\left[U_\mu^\dagger(x)\psi_{x+a\hat\mu} - U_{-\mu}^\dagger(x)\psi_{x-a\hat\mu}\right] + M\psi_x \;\stackrel{\rm w.s.}{=}\; \frac{\delta}{\delta\bar\psi_x}\,, \tag{8.28}$$

which is the lattice analog of Eq. (3.19).

Analogously, the lattice chiral transformation

$$\psi_x \;\to\; {\rm e}^{{\rm i}\alpha(x)\gamma_5}\,\psi_x\,, \qquad \bar\psi_x \;\to\; \bar\psi_x\,{\rm e}^{{\rm i}\alpha(x)\gamma_5} \tag{8.29}$$

results in the lattice axial current

$$J_\mu^{\rm A}(x) \;=\; \frac{\rm i}{2}\left[\bar\psi_x\gamma_\mu\gamma_5 U_\mu^\dagger(x)\psi_{x+a\hat\mu} + \bar\psi_{x+a\hat\mu}\gamma_\mu\gamma_5 U_\mu(x)\psi_x\right], \tag{8.30}$$

which reproduces (3.10) as $a\to 0$. The current (8.30) is conserved for $M=0$.

It is now easy to verify that 16 generators (8.11) commute with the lattice $U(1)$ transformation (8.25) so that the lattice vector current (8.26) is left invariant. Analogously, the lattice axial $U(1)$ transformation (8.29) commutes only with $1+6+1=8$ of 16 generators (8.11) which are constructed from the products of an even number of the generators (8.10) and does not commute with the $4+4=8$ other generators. Therefore, the axial current (8.30) is invariant under the $1+6+1=8$ transformations, which are the products of an even number of the generators (8.10), and alters its sign under the other $4+4=8$ transformations, which are the products of an odd one. Thus, the vector charge of all the doublers is the same, while the axial charge is positive for eight and negative for the other eight doublers.

It is worth noting that the mass term in Eq. (8.2) is not γ_5 invariant, but does not remove the fermion doubling.

One might think of removing the doubling problem by modifying the expression for the inverse lattice propagator $G^{-1}(k)$ in the free fermionic lattice action (8.13), for instance, by adding next-to-neighbor terms. It is easy to see that this does not help if the function $G^{-1}(k)$ is periodic as it should be on a lattice. A typical form of $G^{-1}(k)$ as a function of, say, k_4 is depicted in Fig. 8.2. The behavior around $k_4=0$ is prescribed by Eq. (8.15) and is just a straight line with a positive slope. Therefore, $G^{-1}(k)$ will inevitably have another zero at $k_4=\pi/a$ owing to periodicity.

This is the difference between the fermionic and bosonic cases. For bosons $G^{-1}(k)$ is quadratic in k_4 near $k_4=0$ rather than linear as for fermions. The typical behavior of $G^{-1}(k)$ for bosons is shown in Fig. 8.3. There is no doubling of states in the bosonic case.

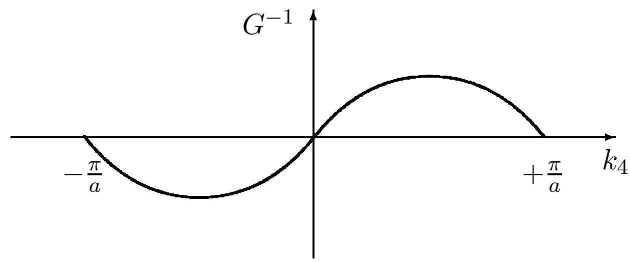

Fig. 8.2. Momentum dependence of G^{-1} for the chiral lattice fermions. The periodicity leads to an extra zero at $k_4 = \pi/a$.

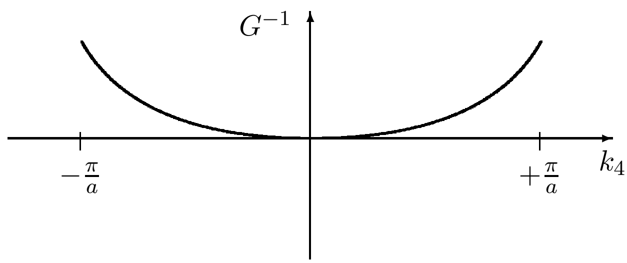

Fig. 8.3. Momentum dependence of G^{-1} for the lattice bosons. No doubling of states is associated with this behavior.

Remark on the Nielsen–Ninomiya theorem

A general proof of the theorem which states that there is no way to avoid fermion doubling under natural assumptions concerning the structure of a lattice gauge theory was given by Nielsen and Ninomiya [NN81]. It is sometimes formulated as an absence of neutrinos on the lattice. In other words, this is a no-go theorem for putting theories with an unequal number of left- and right-handed massless Weyl particles on a lattice, such as in the standard electroweak theory.

A naive way to bypass the Nielsen–Ninomiya theorem is, say, to choose a fermionic lattice action which is highly nonlocal. Then it is possible to replace $\sin k_\mu a$ in Eq. (8.14) by $k_\mu a$ itself to obtain an expression which is similar to the continuum propagator (8.15). However, such a nonlocal modification is useless in practice.

Some recent progress [Neu98, Lus98] in formulating chiral gauge theories on the lattice has been based on the idea of modifying the lattice chiral transformation in the spirit of Ginsparg and Wilson [GW82] and using a sophisticated lattice approximation of the Dirac operator which has no doublers but is manifestly invariant under such a modified lattice chiral transformation. Ordinary chiral symmetry is then reproduced in the continuum limit.

8.3 Kogut–Susskind fermions

The number of continuum fermion species is not necessarily equal to 16. It can be reduced down to four by a trick which was proposed for the Hamiltonian formulation in [KS75, Sus77] and elaborated for the Euclidean formulation in [STW81, KS81b].

Let us substitute

$$\psi_x = \gamma_1^{x/a}\gamma_2^{y/a}\gamma_3^{z/a}\gamma_4^{t/a}\phi_x \tag{8.31}$$

into the free fermionic action. Then it takes the form

$$S_0[\bar{\psi},\psi] = \frac{1}{2}\sum_x\sum_i\sum_{\mu>0}\eta_\mu(x)\left([\phi_x^\dagger]^i[\phi_{x+a\hat{\mu}}]^i - [\phi_{x+a\hat{\mu}}^\dagger]^i[\phi_x]^i\right), \tag{8.32}$$

which is diagonal with respect to the spinor indices, since

$$\eta_\mu(x) = (-1)^{(x_1+\cdots+x_{\mu-1})/a}, \tag{8.33}$$

or explicitly

$$\left.\begin{aligned} \eta_1(x) &= 1, \\ \eta_2(x) &= (-1)^{x_1/a}, \\ &\vdots\;, \\ \eta_d(x) &= (-1)^{(x_1+\cdots+x_{d-1})/a}, \end{aligned}\right\} \tag{8.34}$$

does not depend on spinor indices.

The idea is to leave only one component of ϕ_x^i in order to reduce the degeneracy:

$$\phi_x^i = \begin{pmatrix} \chi_x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \bar{\phi}_x^i = \begin{pmatrix} \bar{\chi}_x \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8.35}$$

These lattice fermions are known as the *staggered fermions*, since $\eta_\mu(x)$ is staggering from one lattice site to another. They are also often called the *Kogut–Susskind fermions* because of their relation to those of [KS75, Sus77].

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The action of the Kogut–Susskind fermions is

$$\begin{aligned} S[U,\bar\psi,\psi] &= \beta S_{\rm lat}[U] + M\sum_x \bar\chi_x\chi_x \\ &\quad + \frac{1}{2}\sum_{x,\mu>0}\eta_\mu(x)\left[\bar\chi_x\,U^\dagger_\mu(x)\,\chi_{x+a\hat\mu} - \bar\chi_{x+a\hat\mu}\,U_\mu(x)\,\chi_x\right]. \end{aligned} \tag{8.36}$$

It describes $2^{d/2} = 4$ species for complex χ_x or $2^{d/2-1} = 2$ species for Majorana χ_x. Components of a continuum bispinor are distributed in this approach over four lattice sites.

There is no chiral anomaly for the Kogut–Susskind fermions as with the chiral fermions.

Remark on four generations

It might seem plausible to identify four species of Kogut–Susskind fermions with four generations of quarks and leptons (see, for example, [KMN83]). Remember that one of the motivations for adding the fourth generation to the standard model is to cancel the anomaly. However, there are problems with this idea concerning the splitting of fermion masses for the four generations.

8.4 Wilson fermions

The chiral lattice fermions were proposed by Wilson [Wil74]. Soon after that he recognized [Wil75] the problem of fermion doubling and proposed a lattice fermionic action that describes only one relativistic fermion in the continuum. The latter fermions are called *Wilson fermions*.

The lattice action for the Wilson fermions reads

$$\begin{aligned} S[U,\bar\psi,\psi] &= \beta S_{\rm lat}[U] + M\sum_x \bar\psi_x\psi_x \\ &\quad - \frac{1}{2}\sum_{x,\mu>0}\left[\bar\psi_x\,(1-\gamma_\mu)\,U^\dagger_\mu(x)\,\psi_{x+a\hat\mu} + \bar\psi_{x+a\hat\mu}\,(1+\gamma_\mu)\,U_\mu(x)\,\psi_x\right]. \end{aligned} \tag{8.37}$$

The difference between this action and the action (8.2) for chiral fermions arises from the projectors $(1\pm\gamma_\mu)$ which pick only one fermionic state.

Substituting the expansion (8.5) in the action (8.37), we obtain, in the naive continuum limit, the continuum fermionic action (5.13) with the mass being

$$m = \frac{M-4}{a}. \tag{8.38}$$

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Therefore, the Wilson lattice fermions describe a relativistic fermion of the mass m in the continuum when

$$M \;\rightarrow\; 4 + ma\,. \tag{8.39}$$

In order to see that there are no other relativistic fermion states in the limit (8.39), let us consider the fermionic propagator which is given by

$$G^{-1}(k) \;=\; M - \frac{1}{2}\sum_{\mu=1}^{4}\Big[(1-\gamma_\mu)\,\mathrm{e}^{\mathrm{i}k_\mu a} + (1+\gamma_\mu)\,\mathrm{e}^{-\mathrm{i}k_\mu a}\Big]. \tag{8.40}$$

Introducing the Minkowski-space energy $E = -\mathrm{i}k_4$, we obtain the following dispersion law:

$$\cosh Ea \;=\; \frac{1 + \Big(M - \sum_{\mu=1}^{3}\cos p_\mu a\Big)^2 + \sum_{\mu=1}^{3}\sin^2 p_\mu a}{2\Big(M - \sum_{\mu=1}^{3}\cos p_\mu a\Big)}\,. \tag{8.41}$$

Let a particle be at rest, i.e. $p_1 = p_2 = p_3 = 0$ and $E = m > 0$. Then Eq. (8.41) reduces for $ma \ll 1$ to the relation (8.39). It is easy to show that a particle at rest is the only solution to Eq. (8.41) with finite energy as $a \rightarrow 0$.

The difference between the dispersion laws for the chiral and Wilson fermions is because the function on the RHS of Eq. (8.41) is no longer periodic. It reduces for $a \rightarrow 0$ and $M \rightarrow 4$ to a usual relation

$$E^2 \;=\; \vec{p}^{\,2} + m^2 \tag{8.42}$$

between the energy and momentum of a relativistic particle.

Problem 8.3 Show that the solution to the dispersion law (8.41) is unique for $M \approx 4$.

Solution For $M \approx 4$, we can replace the LHS of Eq. (8.41) by 1 and substitute $M = 4$ on the RHS. Then Eq. (8.41) reduces to the equation for spatial components of the four-momentum:

$$\Big(3 - \sum_{\mu=1}^{3}\cos p_\mu a\Big)^2 + \Big(3 - \sum_{\mu=1}^{3}\cos^2 p_\mu a\Big) \;=\; 0\,, \tag{8.43}$$

for which the only solution is $p_1 = p_2 = p_3 = 0$, since both terms on the LHS are nonnegative.

It is instructive to discuss what happens with the fermion doublers under the change of $\pm\gamma_\mu$ by $(1 \pm \gamma_\mu)$ in the lattice fermionic action. Let us consider one such state, such as that with $p_1 = \pi/a$, $p_2 = p_3 = 0$. Its

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energy is determined by Eq. (8.41) to be $\sim 1/a$ so that this state is not essential as $a \to 0$.

The chiral anomaly is correctly recovered using the Wilson fermions. The 15 states of the mass $\sim 1/a$ play the role of regulators, which results in an anomaly as $a \to 0$.

Problem 8.4 Calculate the masses of all 16 fermionic states.

Solution Substituting Eqs. (8.23), (8.24) and so on into the action (8.37), we obtain

$$m = \frac{M - \sum_{\mu=1}^{4} s_\mu}{a}, \tag{8.44}$$

where

$$s_\mu = e^{ip_\mu a} \begin{cases} +1 & p_\mu = 0 \\ -1 & p_\mu = \dfrac{\pi}{a} \,. \end{cases} \tag{8.45}$$

Therefore, one state is relativistic as $M \to 4$, while 15 others have masses $\sim 1/a$.

Remark on backtrackings for Wilson fermions

Another way to understand why the doubling problem is removed for the Wilson fermions is to consider how they propagate on a lattice. The projectors

$$P_\mu^\pm = \frac{1 \pm \gamma_\mu}{2} \qquad \boxed{\text{Wilson fermions}} \tag{8.46}$$

restrict the propagation of the Wilson fermions. One-half of the states propagate only in positive directions and the other half propagate only in negative directions. In particular, there are no backtrackings in the (lattice) sum over paths, since

$$P_\mu^+ P_\mu^- = 0\,. \tag{8.47}$$

This removes the doubling.

Problem 8.5 Represent the fermion propagator in an external Yang–Mills field as a sum over paths on a lattice, performing an expansion in $1/M$.

Solution Let us rescale the fermion field, absorbing the parameter M in front of the mass term. The fermionic part of the new action is given by

$$S_\psi = \sum_x \bar\psi_x \psi_x - \kappa \sum_{x,\mu>0} \left[\bar\psi_x P_\mu^- U_\mu^\dagger(x) \psi_{x+a\hat\mu} + \bar\psi_{x+a\hat\mu} P_\mu^+ U_\mu(x) \psi_x\right], \tag{8.48}$$

where $\kappa = 1/M$ is usually called the *hopping parameter*. The large-mass expansion in $1/M$ is now represented as the hopping parameter expansion in κ.

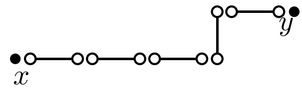

Fig. 8.4. A path Γ_{yx} made out of the string bits, which leads to a nonvanishing term of the hopping parameter expansion for the quark propagator (8.49) on a lattice. Each site involves at least two quark fields (depicted by the circles). Otherwise the Grassmann integral at a given site vanishes.

It is convenient to depict each of the two terms in square brackets in Eq. (8.48) by a string bit as in Fig. 6.2 on p. 101 with the quark fields at the ends and the gauge variable at the link. The first term corresponds to the negative direction of the link, and the second term corresponds to the positive direction. Substituting Eq. (8.48) into definition (8.4) and expanding the exponential in κ, we obtain a combination of terms constructed from the string bits. A nonvanishing contribution to the quark propagator

$$G^{ij}_{mn}(x,y;U) \quad = \quad \left\langle \psi^i_m(x)\bar\psi^j_n(y)\right\rangle_\psi \,, \tag{8.49}$$

where i,j and m,n represent, respectively, color and spinor indices, emerges when the links form a path Γ_{yx} that connects x and y on the lattice as depicted in Fig. 8.4. Otherwise, the average over ψ and $\bar\psi$ vanishes owing to the rules of integration over Grassmann variables described in Problem 2.2 on p. 37.

Therefore, we obtain

$$G^{ij}_{mn}(x,y;U) \quad = \quad \sum_{\Gamma_{yx}} \frac{1}{M^{L(\Gamma)+1}} U^{ij}[\Gamma_{yx}] \left[\prod_{\Gamma_{yx}} P^{\pm}_\mu\right]_{mn} , \tag{8.50}$$

where P^+_μ or P^-_μ are associated with the positive or negative direction of a given link $\in \Gamma_{yx}$. For the Wilson fermions, they are given by Eq. (8.46), while

$$P^{\pm}_\mu \quad = \quad \pm\frac{\gamma_\mu}{2} \qquad \boxed{\text{chiral fermions}} \tag{8.51}$$

for chiral fermions. The sum in Eq. (8.50) runs over all the paths between x and y on the lattice, while $L(\Gamma)$ denotes the length of the path Γ_{yx} in the lattice units. A continuum counterpart of Eq. (8.50) is derived in Problem 12.1.

Problem 8.6 Represent the integral over fermions in Eq. (8.3) as a sum over closed paths on a lattice, performing an expansion in $1/M$.

Solution The calculation is analogous to that of the previous Problem. The result can be written as

$$\int \prod_x \mathrm{d}\bar\psi_x \,\mathrm{d}\psi_x \,\mathrm{e}^{-S_\psi} \quad = \quad \mathrm{e}^{-S_{\rm ind}[U]} \tag{8.52}$$

with

$$S_{\rm ind}[U] \quad = \quad -\sum_\Gamma \frac{\operatorname{tr} U\,[\Gamma]}{L(\Gamma)\,M^{L(\Gamma)}}\,\mathrm{sp}\prod_\Gamma P^{\pm}_\mu , \tag{8.53}$$

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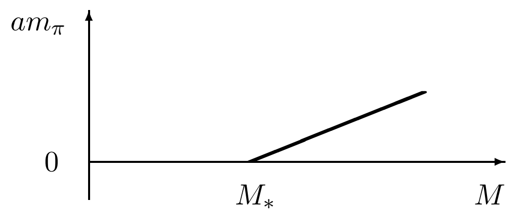

Fig. 8.5. Dependence of the π-meson mass on the lattice quark mass M. At $M = M_*$ the π-meson becomes massless and the chiral symmetry is restored.

where the combinatoric factor $1/L(\Gamma)$ arises from the identity of L links forming the closed contour Γ, and the minus sign is because of fermions.

Equation (8.53) defines an effective (or induced) action of a pure lattice gauge theory, which is nonlocal since it involves arbitrarily large loops. However, it can be made the single-plaquette lattice action (6.16) by introducing many flavors of lattice fermions [Ban83, Ham83].

8.5 Quark condensate

The lattice action (8.37) is not invariant under the chiral transformation. Therefore, the chiral symmetry is broken explicitly for the Wilson fermions.

Nevertheless, one expects a restoration of chiral symmetry as $a \to 0$ when the relativistic fermion is massless (say, for $M = 4$ in the free case), while heavy states with $m \sim 1/a$ play the role of regulators. For the interaction theory, this restoration happens at some value $M = M_*$, which is no longer equal to 4. A signal of this restoration is the vanishing of the mass of the π-meson (as illustrated by Fig. 8.5). $m_\pi = 0$ is usually associated with the fact that the chiral symmetry is realized in a spontaneously broken phase and the π-meson is the corresponding Goldstone boson.

For the chiral or Kogut–Susskind fermions with $M = 0$, the lattice action is invariant under the global chiral transformation (8.7). The order parameter for breaking the chiral symmetry is

$$\bar\psi\psi \quad \stackrel{\text{c.t.}}{\longrightarrow} \quad \bar\psi\, \mathrm{e}^{2\mathrm{i}\alpha\gamma_5}\psi\,, \tag{8.54}$$

which is not invariant under the chiral transformation. Therefore, the average of $\bar\psi\psi$ must vanish if the symmetry is not broken spontaneously.*

* Spontaneous symmetry breaking usually occurs when the vacuum state is not invariant under the symmetry transformation.

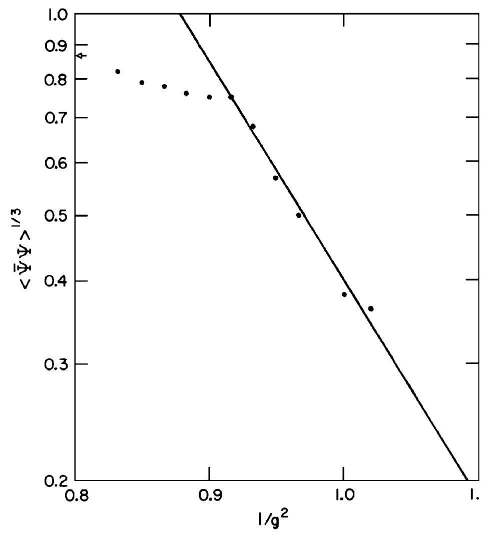

Fig. 8.6. Monte Carlo data from Hamber and Parisi [HP81] for the quark condensate in the quenched approximation.

Such spontaneous breaking results in

$$\langle \bar{\psi}\psi \rangle \neq 0\,. \tag{8.55}$$

This nonvanishing value of the average of $\bar{\psi}_x \psi_x$ does not depend on x owing to translational invariance and is called the *quark condensate*.

The spontaneous breaking of the chiral symmetry in QCD was demonstrated using Monte Carlo calculations of the quark condensate. This quantity has a dimension of $[\text{mass}]^3$ and should depend on g^2 at small g^2 as prescribed by the asymptotic scaling. The Monte Carlo data for the quark condensate from the pioneering paper by Hamber and Parisi [HP81] are shown in Fig. 8.6. Its agreement with asymptotic scaling demonstrates that the chiral symmetry is spontaneously broken in the continuum QCD.

Remark on Monte Carlo simulations with fermions

Monte Carlo simulations with quarks are much more difficult than in a pure gauge theory. Integrating over the quark fields using Eq. (2.15), one is left with the determinant, say for the Kogut–Susskind fermions, of the matrix

$$D[U] \;=\; M\delta_{xy} + \frac{1}{2}\sum_{\mu>0}\Big[\eta_\mu(x)\,U^\dagger_\mu(x)\,\delta_{x(x+a\hat{\mu})} - \eta_\mu(x)\,U_\mu(x)\,\delta_{x(x-a\hat{\mu})}\Big] \tag{8.56}$$

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for a given configuration of the gluon field $U_\mu(x)$. This results in a pure gauge-field problem with the effective action given by

$$e^{-\beta S_{\text{eff}}[U]} = \det D[U]\, e^{-\beta S[U]}\,. \qquad (8.57)$$

The matrix that appears in this determinant has at least $NL^4 \times NL^4$ elements, and is to be calculated at each Monte Carlo upgrading of $U_\mu(x)$.

Several methods are proposed to manage the quark determinant exactly or approximately. The simplest one is not to take it into account at all. This approximation is known as the *quenched* approximation when only valence quarks are considered, while the effects of virtual quark loops are disregarded. Recently, progress in the full theory has been achieved using some tricks to evaluate the quark determinants (see, for example, [Aok00] for a review of the subject).