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The Jiang–Su Absorption for Inclusions of Unital C*-algebras

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Abstract. We introduce the tracial Rokhlin property for a conditional expectation for an inclusion of unital C*-algebras $P \subset A$ with index finite, and show that an action α from a finite group G on a simple unital C*-algebra A has the tracial Rokhlin property in the sense of N. C. Phillips if and only if the canonical conditional expectation $E: A \to A^G$ has the tracial Rokhlin property. Let $\mathcal{C}$ be a class of infinite dimensional stably finite separable unital C*-algebras that is closed under the following conditions:

(1) If $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.
(2) If $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$.
(3) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Suppose that any C*-algebra in $\mathcal{C}$ is weakly semiprojective. We prove that if A is a local tracial $\mathcal{C}$-algebra in the sense of Fan and Fang and a conditional expectation $E: A \to P$ is of index-finite type with the tracial Rokhlin property, then P is a unital local tracial $\mathcal{C}$-algebra.

The main result is that if A is simple, separable, unital nuclear, Jiang–Su absorbing and $E: A \to P$ has the tracial Rokhlin property, then P is Jiang–Su absorbing. As an application, when an action α from a finite group G on a simple unital C*-algebra A has the tracial Rokhlin property, then for any subgroup H of G the fixed point algebra A^H and the crossed product algebra $A \rtimes_{\alpha_{|H}} H$ is Jiang–Su absorbing. We also show that the strict comparison property for a Cuntz semigroup $W(A)$ is hereditary to $W(P)$ if A is simple, separable, exact, unital, and $E: A \to P$ has the tracial Rokhlin property.

1 Introduction

The purpose of this paper is to introduce the tracial Rokhlin property for an inclusion of separable simple unital C*-algebras $P \subset A$ with finite index in the sense of [38], and prove theorems of the following type. Suppose that A belongs to a class of C*-algebras characterized by some structural property, such as tracial rank zero in the sense of [20]. Then P belongs to the same class. The classes we consider include:

- simple C*-algebras with real rank zero or stable rank one,
- simple C*-algebras with tracial rank zero or tracial rank less than or equal to one,
- simple C*-algebras with Jiang–Su algebra absorption,
- simple C*-algebras for which the order on projections is determined by traces,
- simple C*-algebras with the strict comparison property for the Cuntz semigroup.

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The third and fifth conditions are important properties related to Toms and Winter's conjecture, that is, the properties of strict comparison, finite nuclear dimension, and Z-absorption are equivalent for separable simple infinite-dimensional nuclear unital C*-algebras ([36, 39]).

We show that an action α from a finite group G on a simple unital C*-algebra A has the tracial Rokhlin property in the sense of [30] if and only if the canonical conditional expectation $E: A \to A^G$ has the tracial Rokhlin property for an inclusion $A^G \subset A$. When an action α from a finite group on a (not necessarily simple) unital C*-algebra has the Rokhlin property in the sense of [13], all of the above results are proved in [28, 29].

The essential observation was made in the proof of [30, Theorem 2.2] the crossed product $A \rtimes_\alpha G$ (= $C^*(G, A, \alpha)$ in [30]) has a local approximation property by C*-algebras stably isomorphic to homomorphic images of A. Since the Jiang–Su algebra $\mathcal{Z}$ belongs to classes of direct limits of semiprojective building blocks in [16], technical difficulties arise because we must treat arbitrary homomorphic images in the approximation property. (Homomorphic images of semiprojective C*-algebras need not be semiprojective.) In [40] they introduced the unital local tracial $\mathcal{C}$ property, which generalizes one of a local $\mathcal{C}$ property in [26], for proving that a C*-algebra with a local approximation property by homomorphic images of a suitable class $\mathcal{C}$ of semiprojective C*-algebras can be written as a direct limit of algebras in the class. When each homomorphism is injective, the unital local tracial $\mathcal{C}$ property is equivalent to the tracial approximation property in [7]. Note that when an action α from a finite group G on a simple unital C*-algebra A, the tracial approximation property for A is inherited to the crossed product algebra $A \rtimes_\alpha G$ (see [40] and Theorem 3.3).

We know of several results like those above for tracial approximation in the literature: for stable rank one ([7, Theorem 4.3] and [9]), for real rank zero ([9]), for the $\mathcal{Z}$-absorption ([11, Corollary 5.7]), for the order on projections determined by traces ([7, Theorem 4.12]).

The paper is organized as follows. In Section 2 we introduce the notion of a unital local tracial $\mathcal{C}$-algebra and a tracial approximation class (TA$\mathcal{C}$ class), and we show in Section 3 that when an action α from a finite group G on a simple unital C*-algebra A has the tracial Rokhlin property, the crossed product algebra $A \rtimes_\alpha G$ belongs to the class TA$\mathcal{C}$ for A in TA$\mathcal{C}$. In Section 4 we introduce the tracial Rokhlin property for an inclusion $P \subset A$ of unital C*-algebras and show that if A is a simple local tracial $\mathcal{C}$-algebra, then so is P (Theorem 4.11). In particular, if A has tracial topological rank zero (resp. less than or equal to one), so does P (Corollary 4.12). In Section 5 we present the main theorem: given an inclusion $P \subset A$ of separable simple nuclear unital C*-algebras of finite index type with the tracial Rokhlin property, if A is $\mathcal{Z}$-absorbing, then so is P (Theorem 5.4). As an application, any fixed point algebra A^H for any subgroup H of a finite group G is $\mathcal{Z}$-absorbing under the assumption that there exists an action α from G on a simple nuclear unital C*-algebra A such that A is $\mathcal{Z}$-absorbing (Corollary 5.5). Before treating the strict comparison for a Cuntz semigroup, we consider the Cuntz equivalent for positive elements and show that under the assumption that an inclusion $P \subset A$ of unital C*-algebras has the tracial Rokhlin property, for $n \in \mathbb{N}$ and positive elements $a, b \in M_n(P)$, if a is subequivalent to b

in $M_n(A)$, then a is subequivalent to b in $M_n(P)$ (Proposition 6.2). Finally, we consider the strict comparison property for a Cuntz semigroup and show that the strict comparison property is inherited to P when an inclusion $P \subset A$ of simple separable exact unital C*-algebras has the tracial Rokhlin property and A has the strict comparison (Theorem 7.2). Using a similar argument we show that if an inclusion $P \subset A$ of separable simple unital C*-algebras has the tracial Rokhlin property and the order on projections in A is determined by traces, then the order on projections in P is determined by traces (Corollary 7.3).

2 Local Tracial $\mathcal{C}$-algebra

We recall the definition of the local $\mathcal{C}$-property in [26, Definition 1.1].

Definition 2.1 Let $\mathcal{C}$ be a class of separable unital C*-algebras. Then $\mathcal{C}$ is *finitely saturated* if the following closure conditions hold:

(i) if $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$;
(ii) if $A_1, A_2, \dots, A_n \in \mathcal{C}$, then $\bigoplus_{k=1}^n A_k \in \mathcal{C}$;
(iii) if $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$;
(iv) if $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Moreover, the *finite saturation* of a class $\mathcal{C}$ is the smallest finitely saturated class that contains $\mathcal{C}$.

Definition 2.2 Let $\mathcal{C}$ be a class of separable unital C*-algebras. A *unital local $\mathcal{C}$-algebra* is a separable unital C*-algebra A such that for every finite set $F \subset A$ and every $\varepsilon > 0$, there is a C*-algebra B in the finite saturation of $\mathcal{C}$ and a unital *-homomorphism $\varphi: B \to A$ (not necessarily injective) such that $\operatorname{dist}(x, \varphi(B)) < \varepsilon$ for all $x \in F$.

When B in Definition 2.2 is non-unital, we perturb the condition as follows.

Definition 2.3 Let $\mathcal{C}$ be a class of separable unital C*-algebras.

(i) A unital C*-algebra A is said to be a *unital local tracial $\mathcal{C}$-algebra* if for every finite set $\mathcal{F} \subset A$ and every $\varepsilon > 0$, and any non-zero $a \in A^+$, there exist a non-zero projection $p \in A$, a C*-algebra $B \in \mathcal{C}$, and *-homomorphism $\varphi: B \to A$ such that $\varphi(1_B) = p$, and for all $x \in \mathcal{F}$:
 (a) $\|xp - px\| < \varepsilon$,
 (b) $\operatorname{dist}(pxp, \varphi(B)) < \varepsilon$,
 (c) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.
(ii) A unital C*-algebra A is said to belong to the *class $TA\mathcal{C}$* if for every finite set $\mathcal{F} \subset A$ and every $\varepsilon > 0$, and any non-zero $a \in A^+$, there exist a non-zero projection $p \in A$ and a sub C*-algebra $B \subset A$ such that $B \in \mathcal{C}$, $1_B = p$, and for all $x \in \mathcal{F}$:
 (a) $\|xp - px\| < \varepsilon$,
 (b) $\operatorname{dist}(pxp, B) < \varepsilon$,
 (c) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

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Note that (i) comes from [40, Definition 2.13], and (ii) is [7, Definition 2.2].

Remark 2.4 (i) When a unital C*-algebra A is a unital local tracial $\mathcal{C}$-algebra and each $\varphi(C) \in \mathcal{C}$, A belongs to the class TA$\mathcal{C}$.

(ii) If $\mathcal{C}$ is the class of finite dimensional C*-algebras $\mathscr{F}$, then a local TA$\mathscr{F}$-algebra belongs to the class of tracially AF C*-algebras ([20]).

(iii) If $\mathcal{C}$ is the class of interval algebras $\mathscr{I}$, then a local TA$\mathscr{I}$-algebra belongs to the class of C*-algebras of tracial topological one (TAI- algebras) ([23]) in the sense of Lin.

Recall that a C*-algebra A is said to have *Property (SP)* if any nonzero hereditary C*-subalgebra of A has a nonzero projection.

We have the following relation between the local $\mathcal{C}$ property and the local tracial approximational $\mathcal{C}$ property.

Proposition 2.5 *Let $\mathcal{C}$ be a finitely saturated class and let A be a local tracial $\mathcal{C}$-algebra. Then A has the property (SP) or A is a local $\mathcal{C}$ algebra.*

Proof Suppose that A does not have the Property (SP). Then there is a positive element $a \in A$ such that $\overline{aAa}$ has no non zero projection. Since A is a local TA$\mathcal{C}$-algebra, for every finite set $\mathcal{F} \subset A$ and every $\varepsilon > 0$, we conclude that there are a unital C*-algebra C in the class $\mathcal{C}$ and a unital *-homomorphism $\varphi\colon C \to A$ such that $\mathcal{F}$ can be approximated by a C*-algebra $\varphi(C)$ to within ε. Hence, A is a local $\mathcal{C}$-algebra. ∎

3 Tracial Rokhlin Property for Finite Group Actions

Inspired by the concept of the tracial AF C*-algebras in [20] Phillips defined the tracial Rokhlin property for a finite group action in [30, Lemma 1.16] as follows.

Definition 3.1 Let α be the action of a finite group G on a unital infinite dimensional finite simple separable unital C*-algebra A. The action α is said to have the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every nonzero positive $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(i) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$;

(ii) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$;

(iii) with $e = \sum_{g\in G} e_g$, the projection $1 - e$ is Murray–von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

It is obvious that the Rokhlin property is stronger than the tracial Rokhlin property. As pointed out in [13], the Rokhlin property gives rise to several K-theoretical constrains. For example, there is no action with the Rokhlin property on the noncommutative 2-torus. On the contrary, if A is a simple higher dimensional noncommutative torus with standard unitary generators $u_1, u_2, \dots, u_n$, then the automorphism that sends u_k to $\exp(2\pi/n)u_k$, and fixes u_j for $j \neq k$, generates an action $\mathbb{Z}/n\mathbb{Z}$ and has the tracial Rokhlin property, but for $n > 1$ never has the Rokhlin property ([30]).

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Lemma 3.2 ([26, Theorem 3.2], [1, Lemma 3.1]) *Let A be an infinite-dimensional, stably finite, simple, unital C*-algebra with Property (SP) such that the order on projections over A is determined by traces. Let G be a finite group of order n and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G with the tracial Rokhlin property. Then for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A \rtimes_\alpha G$, any $N \in \mathbb{N}$, and any non-zero $z \in (A \rtimes_\alpha G)^+$, there exist a non-zero projection $e \in A \subset A \rtimes_\alpha G$, a unital C*-subalgebra $D \subset e(A \rtimes_\alpha G)e$, a projection $f \in A$ and an isomorphism $\phi\colon M_n \otimes fAf \to D$, such that the following hold.*

(i) *With (e_{gh}) for $g, h \in G$ being a system of matrix units for M_n, we have $\phi(e_{11} \otimes a) = a$ for all $a \in fAf$ and $\phi(e_{gg} \otimes 1) \in A$ for $g \in G$.*
(ii) *With (e_{gg}) as in (i), we have $\|\phi(e_{gg} \otimes a) - \alpha_g(a)\| \le \varepsilon\|a\|$ for all $a \in fAf$.*
(iii) *For every $a \in F$, there exist $b_1, b_2 \in D$ such that $\|ea - b_1\| < \varepsilon$, $\|ae - b_2\| < \varepsilon$ and $\|b_1\|, \|b_2\| \le \|a\|$.*
(iv) $e = \sum_{g\in G} \phi(e_{gg} \otimes 1)$.
(v) $1 - e$ *is Murray–von Neumann equivalent to a projection in* $\overline{z(A \rtimes_\alpha G)z}$.
(vi) *There are N mutually orthogonal projections $f_1, f_2, \dots, f_N \in eDe$, each of which is Murray–von Neumann equivalent in $A \rtimes_\alpha G$ to $1 - e$.*

Proof In [1] the author assumed that A has real rank zero. But since A is simple and A has Property (SP), any nonzero positive element $z \in A \rtimes_\alpha G$ [14, Theorem 4.2] (with $N = 1$) supplies a nonzero projection $q \in A$ that is Murray–von Neumann equivalent in $A \rtimes_\alpha G$ to a projection in $\overline{z(A \rtimes_\alpha G)z}$. Moreover, [22, Lemma 3.5.7] provides nonzero orthogonal Murray–von Neumann equivalent projections $q_0, q_1, \dots, q_{2N} \in qAq$.

Therefore, the statement comes from the same argument as in [1, Lemma 3.1]. ∎

Theorem 3.3 *Let $\mathcal{C}$ be a class of infinite dimensional stably finite separable unital C*-algebras that is closed under the following conditions:*

(i) *$A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.*
(ii) *If $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$.*
(iii) *If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.*

Let $A \in TA\mathcal{C}$ be a simple C-algebra such that the order on projections over A is determined by traces. If α is an action of a finite group G on A with the tracial Rokhlin property, then $A \rtimes_\alpha G$ belongs to the class $TA\mathcal{C}$.*

Proof Since α is outer by [30, Lemma 1.5], $A \rtimes_\alpha G$ is simple by [18, Theorem 3.1].

By [30, Lemma 1.13], A has Property (SP) or α has the strict Rokhlin property.

Let $\mathcal{F} \subset A \rtimes_\alpha G$ be a finite set and let z be a positive nonzero element of $A \rtimes_\alpha G$ with $\|z\| \le 1$ and $\varepsilon > 0$.

If α has the strict Rokhlin property, then there are $n \in \mathbb{N}$, a projection $f \in A$, and a unital homomorphism $\phi\colon M_n \otimes fAf \to A \rtimes_\alpha G$ such that $\mathrm{dist}(a, \phi(M_n \otimes fAf)) < \varepsilon$ for all $a \in \mathcal{F}$ by [26, Theorem 3.2]. Since $M_n \otimes fAf \in \mathcal{C}$, from the simplicity of $M_n \otimes fAf$ we know $\phi(M_n \otimes fAf) \in \mathcal{C}$. Hence, $A \rtimes_\alpha G$ is unital local $\mathcal{C}$-algebra; that is, $A \rtimes_\alpha G$ belongs to $TA\mathcal{C}$.

Next, suppose that A has Property (SP). Then there exists a non-zero projection $q \in A$ that is Murray–von Neumann equivalent in $A \rtimes_\alpha G$ to a projection in $\overline{z(A \rtimes_\alpha G)z}$ by [25, Theorem 2.1]. Since A is simple, take orthogonal nonzero projections q_1, q_2

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with $q_1, q_2 \le q$ by [22, Lemma 3.5.7]. Set $n = \mathrm{card}(G)$, and set $\varepsilon_0 = \frac{1}{12}\varepsilon$. By Lemma 3.2 for n as given, for ε_0 in place of ε, and for q_1 in place of z there exist a non-zero projection $e \in A \subset A \rtimes_\alpha G$, a unital C*-subalgebra $D \subset e(A \rtimes_\alpha G)e$, a projection $f \in A$, and an isomorphism $\phi\colon M_n \otimes fAf \to D$, such that the following hold:

(a) With (e_{gh}) for $g, h \in G$ being a system of matrix units for M_n, we have $\phi(e_{11} \otimes a) = a$ for all $a \in fAf$ and $\phi(e_{gg} \otimes 1) \in A$ for $g \in G$;
(b) with (e_{gg}) as in (a), we have $\|\phi(e_{gg} \otimes a) - \alpha_g(a)\| \le \varepsilon_0$ for all $a \in fAf$;
(c) for every $a \in \mathcal{F}$ there exist $d_1, d_2 \in D$ such that $\|ea - d_1\| < \varepsilon_0$, $\|ae - d_2\| < \varepsilon_0$ and $\|d_1\|, \|d_2\| \le 1$;
(d) $e = \sum_{g \in G} \phi(e_{gg} \otimes 1)$;
(e) $1 - e$ is Murray–von Neumann equivalent to a projection in $\overline{q_1(A \rtimes_\alpha G)q_1}$.

We note that there is a finite set T in the closed unit ball of $M_n \otimes fAf$ such that for every $a \in \mathcal{F}$ there are $b_1, b_2 \in T$ such that $\|ea - \phi(b_1)\| < \varepsilon_0$ and $\|ae - \phi(b_2)\| < \varepsilon_0$. Moreover, $\|ea - ae\| < 8\varepsilon_0$. Indeed, the condition that $\|ea - \phi(b_1)\| < \varepsilon_0$ and $\phi(b_1)e = \phi(b_1)$ implies that $\|eae - \phi(b_1)\| < \varepsilon_0$. Similarly, the condition that $\|ae - \phi(b_2)\| < \varepsilon_0$ implies that $\|ea^* - ea^*e\| < 2\varepsilon_0$. Hence, $\|ea - eae\| < 2\varepsilon_0$.

Since A is simple and has Property (SP), we choose equivalent nonzero projections $f_1, f_2 \in A$ such that $f_1 \le f$ and $f_2 \le q_2$ by [22, Lemma 3.5.6]. Since $M_n \otimes fAf \in TA\mathcal{C}$ by [7, Lemma 2.3], there is a projection $p_0 \in M_n \otimes fAf$ and a C*-subalgebra $C \subset M_n \otimes fAf$ such that $C \in \mathcal{C}$, $1_C = p_0$ such that $\|p_0 b - b p_0\| < \frac{1}{4}\varepsilon$ for all $b \in T$, such that for every $b \in T$, there exists $c \in C$ with $\|p_0 b p_0 - c\| < \frac{1}{4}\varepsilon$, and such that $1 - p_0 \le e_{11} \otimes f_1$ in $M_n \otimes fAf$.

Set $p = \phi(p_0)$, and set $E = \phi(C)$, which is a unital subalgebra of $p(A \rtimes_\alpha G)p$ and belongs to $\mathcal{C}$. Note that $e - p = \phi(1 - p_0) \le \phi(e_{11} \otimes f_1) = f_1$.

Let $a \in \mathcal{F}$. Then we can take $b \in T$ such that $\|\phi(b) - eae\| < \frac{1}{4}\varepsilon$. Indeed, since condition (c) implies that there is an element $b \in T$ such that $\|ea - \phi(b)\| < \varepsilon_0$ and $\|eae - ea\| < 2\varepsilon_0$, we have

$$\begin{aligned}\|\phi(b) - eae\| &= \|\phi(b) - ea + ea - eae\| < \|\phi(b) - ea\| + \|eae - ea\| \\ &< (2+1)\varepsilon_0 = \frac{3}{12}\varepsilon = \frac{1}{4}\varepsilon.\end{aligned}$$

Then, using $pe = ep = p$,

$$\begin{aligned}\|pa - ap\| &\le 2\|ea - ae\| + \|peae - eaep\| \\ &\le 2\|ea - ae\| + 2\|eae - \phi(b)\| + \|p_0 b - b p_0\| \\ &< 4\varepsilon_0 + 6\varepsilon_0 + \varepsilon_0 = 11\varepsilon_0 < \varepsilon.\end{aligned}$$

Choosing $c \in C$ such that $\|p_0 b p_0 - c\| < \frac{1}{4}\varepsilon$, the element $\phi(c)$ is in E and satisfies

$$\begin{aligned}\|pap - \phi(c)\| &= \|peaep - \phi(c)\| \\ &= \|p(eae - \phi(b))p + p\phi(b)p - \phi(c)\| \\ &\le \|eae - \phi(b)\| + \|p_0 b p_0 - c\| < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon < \varepsilon.\end{aligned}$$

Finally, in $A \rtimes_\alpha G$ we have

$$1 - p = (1 - e) + (e - p) \le q_1 + q_2 \le q$$

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and q is Murray–von Neumann equivalent to a projection in $\overline{z(A \rtimes_\alpha G)z}$. ■

By using Theorem 3.3 we will provide a new proof of [9, Theorem 3.1].

Theorem 3.4 ([9, Theorem 3.1]) *Let A be an infinite dimensional simple separable unital C*-algebra with stable rank one and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ has stable rank one.*

Proof Let $\mathcal{C}$ be the set of unital C*-algebras with stable rank one. Then $\mathcal{C}$ is closed under three conditions in Theorem 3.3 from [31, Theorem 3.3] and [2, Theorem 4.5]. Then from Theorem 3.3 $A \rtimes_\alpha G$ belongs to the class TA$\mathcal{C}$.

Hence from [7, Theorem 4.3], $A \rtimes_\alpha G$ has stable rank one. ■

Theorem 3.5 *Let $\mathcal{C}$ be the class of unital separable C*-algebras with real rank zero. Then any simple unital stably finite C*-algebra in the class TA$\mathcal{C}$ has real rank zero.*

Proof We can deduce this from the same argument as in the proof of [7, Theorem 4.3]. ■

Using Theorems 3.3 and 3.5 we will provide a new proof of [9, Theorem 3.2].

Corollary 3.6 ([9, Theorem 3.2]) *Let A be an infinite dimensional simple separable unital C*-algebra with real rank zero and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ has real rank zero.*

Proof Let $\mathcal{C}$ be the set of unital C*-algebras with real rank zero. Then $\mathcal{C}$ is closed under the three conditions in Theorem 3.3, from [4, Corollary 2.8 and Theorem 2.10]. Then from Theorem 3.3, $A \rtimes_\alpha G$ belongs to the class TA$\mathcal{C}$.

Hence from Theorem 3.5, $A \rtimes_\alpha G$ has real rank zero. ■

Theorem 3.7 *Let A be an infinite-dimensional simple separable unital C*-algebra such that the order on projections over A is determined by traces, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G with tracial Rokhlin property. Then the order on projections over $A \rtimes_\alpha G$ is determined by traces.*

Proof Let $\mathcal{C}$ be the set of unital C*-algebras such that the order on projections over them is determined by traces. Then $\mathcal{C}$ is closed under three conditions in Theorem 3.3. Indeed, conditions (i) and (ii) are obvious from the definition. We will check condition (iii). Let r be a projection in A and suppose that the order of projections on A is determined by traces over A. Let p, q be projections in rAr and assume for any tracial state τ on rAr, that $\tau(p) < \tau(q)$. Then, for any tracial state ρ on A, the restriction $\rho(r)^{-1}\rho_{|rAr}$ of ρ on rAr is also a tracial state on rAr. Hence,

$$\rho(p) = \rho(r)\{\rho(r)^{-1}\rho_{|rAr}(p)\} < \rho(r)\{\rho(r)^{-1}\rho_{|rAr}(q)\} = \rho(q).$$

Since the order of projections on A is determined by traces, $p \precsim q$ in A. That is, there is a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* \leq q$. Set $w = rur$. Then $w \in rAr$, and $w^*w = ru^*rrur = ru^*rur \leq ru^*ur = rpr = p$. Since $q \leq r$, $u^*qu \leq u^*ru$. Hence

$$p = u^*(uu^*)u \leq u^*qu \leq u^*ru.$$

Therefore, $p = rpr \le ru^*rur = w^*w$. Then we have, $w^*w = p$.

On the contrary,

$$ww^* = ruru^*r \le ruu^*r \le rqr = q.$$

This implies that $p \le q$ in rAr. Hence, the order of projections on rAr is determined by traces over rAr. That is, $\mathcal{C}$ satisfies condition (iii).

Then from Theorem 3.3 $A \rtimes_\alpha G$ belongs to the class $\mathrm{TA}\mathcal{C}$.

Hence, from [7, Theorem 4.12], the order on projections over $A \rtimes_\alpha G$ is determined by traces. ∎

Definition 3.8 ([21, Theorem 6.13]) Let $\mathcal{T}^{(0)}$ be the class of all finite-dimensional C*-algebras and let $\mathcal{T}^{(k)}$ be the class of all C*-algebras with the form $pM_n(C(X))p$, where X is a finite CW complex with dimension k and $p \in M_n(C(X))$ is a projection.

A simple unital C*-algebra A is said to have tracial topological rank no more than k if for any set $\mathcal{F} \subset A$, and $\varepsilon > 0$ and any nonzero positive element $a \in A$, there exists a C*-subalgebra $B \subset A$ with $B \in \mathcal{T}^{(k)}$ and $\mathrm{id}_B = p$ such that

(i) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
(ii) $pxp \in_\varepsilon B$, for all $x \in \mathcal{F}$,
(iii) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The following is proved in [27], but we will provide its proof.

Theorem 3.9 ([27]) *Let A be an infinite-dimensional simple unital C*-algebra with tracial topological rank no more than (resp. equal to) k, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G with tracial Rokhlin property. Then $A \rtimes_\alpha G$ has tracial topological rank more than resp. equal to) k.*

Proof Let $\mathcal{C}$ be the set $\mathcal{T}^{(k)}$. Then $\mathcal{T}^{(k)}$ is closed under the three conditions in Theorem 3.3 from [21, Remark 3.6, Theorems 5.3 and 5.8]. Then from Theorem 3.3, $A \rtimes_\alpha G$ belongs to the class $\mathrm{TA}\mathcal{C}$. This means that $A \rtimes_\alpha G$ has tracial topological rank no more than (respectively equal to) k from the Definition 3.8. ∎

4 The Tracial Rokhlin Property for an Inclusion of Unital C*-algebras

Let $P \subset A$ be an inclusion of unital C*-algebras and let $E: A \to P$ be a conditional expectation of index-finite as defined in [38, Definition 1.2.2]. Note that E is faithful and satisfies that

$$E(b_1ab_2) = b_1E(a)b_2 \tag{4.1}$$

for any $a \in A$ and $b_1, b_2 \in P$.

As in the case of the Rokhlin property in [19, Definition 3.1], we can define the tracial Rokhlin property for a conditional expectation for an inclusion of unital C*-algebras.

Recall that an inclusion of unital C*-algebras $P \subset A$ with a conditional expectation E from A to P has finite index in the sense of Watatani [38] if there is a finite set

$\{(u_i, v_i)\}_{i=1}^n \subset A \times A$ such that for every $a \in A$,

$$a = \sum_{i=1}^n u_i E(v_i a) = \sum_{i=1}^n E(a u_i) v_i.$$

Set $\mathrm{Index}\, E = \sum_{i=1}^n u_i v_i$.

We give several remarks about the above definitions.

(a) $\mathrm{Index}\, E$ does not depend on the choice of the quasi-basis in the above formula, and it is a central element of A [38, Proposition 1.2.8].
(b) Once we know that there exists a quasi-basis, we can choose one of the form $\{(w_i, w_i^*)\}_{i=1}^m$, which shows that $\mathrm{Index}\, E$ is a positive element [38, Lemma 2.1.6].
(c) By the above statements, if A is a simple C^*-algebra, then $\mathrm{Index}\, E$ is a positive scalar.
(d) If $\mathrm{Index}\, E < \infty$, then E is faithful; that is, $E(x^* x) = 0$ implies $x = 0$ for $x \in A$.

Let $A_P (= A)$ be a pre-Hilbert module over P with a P-valued inner product

$$\langle x, y \rangle_P = E(x^* y), \quad x, y \in A_P.$$

We denote by $\mathcal{E}_E$ and η_E the Hilbert P-module completion of A by the norm $\|x\|_P = \|\langle x, x\rangle_P\|^{\frac{1}{2}}$ for x in A and the natural inclusion map from A into $\mathcal{E}_E$. Then $\mathcal{E}_E$ is a Hilbert C^*-module over P. Since E is faithful, the inclusion map η_E from A to $\mathcal{E}_E$ is injective. Let $L_P(\mathcal{E}_E)$ be the set of all (right) P-module homomorphisms $T\colon \mathcal{E}_E \to \mathcal{E}_E$ with an adjoint right P-module homomorphism $T^*\colon \mathcal{E}_E \to \mathcal{E}_E$ such that

$$\langle T\xi, \zeta\rangle = \langle \xi, T^*\zeta\rangle \quad \xi, \zeta \in \mathcal{E}_E.$$

Then $L_P(\mathcal{E}_E)$ is a C^*-algebra with the operator norm $\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}$. There is an injective $*$-homomorphism $\lambda\colon A \to L_P(\mathcal{E}_E)$ defined by

$$\lambda(a)\eta_E(x) = \eta_E(ax)$$

for $x \in A_P$ and $a \in A$, so that A can be viewed as a C^*-subalgebra of $L_P(\mathcal{E}_E)$. Note that the map $e_P\colon A_P \to A_P$ defined by

$$e_P \eta_E(x) = \eta_E(E(x)), \quad x \in A_P$$

is bounded, and thus it can be extended to a bounded linear operator, denoted by e_P again, on $\mathcal{E}_E$. Then $e_P \in L_P(\mathcal{E}_E)$ and $e_P = e_P^2 = e_P^*$; that is, e_P is a projection in $L_P(\mathcal{E}_E)$. A projection e_P is called the *Jones projection* of E.

The *(reduced)* C^**-basic construction* is a C^*-subalgebra of $L_P(\mathcal{E}_E)$ defined as

$$C_r^*\langle A, e_P\rangle = \overline{\mathrm{span}\{\lambda(x) e_P \lambda(y) \in L_P(\mathcal{E}_E) : x,\ y \in A\}}^{\|\cdot\|}$$

If $\mathrm{Index}\, E$ is finite, $C_r^*\langle A, e_P\rangle$ has the certain universality ([38, Proposition 2.2.9]), so we call it *the* C^**-basic construction* and denote it by $C^*\langle A, e_P\rangle$ by identifying $\lambda(A)$ with A in $C^*\langle A, e_P\rangle$; that is,

$$C^*\langle A, e_P\rangle = \Big\{ \sum_{i=1}^n x_i e_P y_i : x_i, y_i \in A, n \in \mathbb{N} \Big\}.$$

Note that by [38, Lemma 2.1.1],

$$e_P a e_P = E(a) e_P \tag{4.2}$$

for any $a \in A$.

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Then there exists a dual conditional expectation $\widehat{E}: C^*\langle A, e_P\rangle \to A$ such that

$$\widehat{E}(xe_Py) = (\operatorname{Index} E)^{-1}xy \tag{4.3}$$

and $\operatorname{Index}\widehat{E} = \operatorname{Index} E$ ([38, Proposition 2.3.4]). Note that the basic construction $C^*\langle A, e_P\rangle$ is isomorphic to $qM_n(P)q$ for some $n \in \mathbb{N}$ and a projection $q \in M_n(P)$ ([38, Lemma 3.3.4]).

For a C^*-algebra A, we set

$$c_0(A) = \{(a_n) \in l^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\},$$
$$A^\infty = l^\infty(\mathbb{N}, A)/c_0(A).$$

We identify A with the C^*-subalgebra of A^∞ consisting of the equivalence classes of constant sequences and set $A_\infty = A^\infty \cap A'$. For an automorphism $\alpha \in \operatorname{Aut}(A)$, we denote by α^∞ and α_∞ the automorphisms of A^∞ and A_∞ induced by α, respectively.

Example 4.1 Let A be an unital C*-algebra and let α be an action from a finite group G on $\operatorname{Aut}(A)$.

(a) An inclusion of $A \subset A \rtimes_\alpha G$ is of index-finite type and $\operatorname{Index} F = |G|$, where F is a canonical conditional expectation from $A \rtimes_\alpha G$ onto A such that $F(\sum_{g\in G} a_g u_g) = a_e$. Indeed, $\{(u_g^*, u_g)\}_{g\in G}$ is a quasi-basis for F and $\operatorname{Index} F = \sum_{g\in G} u_g^* u_g = |G|$.

(b) If A is simple and α is outer, then an inclusion $A^\alpha \subset A$ is of index-finite type and $\operatorname{Index} E = |G|$, where E is the canonical conditional expectation from A onto A^α such that $E(a) = \frac{1}{|G|}\sum_{g\in G} \alpha_g(a)$. Indeed, since A is simple and α is outer, α is saturated by [15, Remark 4.6]. Then by [15, Theorem 4.1] an inclusion $A^\alpha \subset A$ is of index-finite type and $\operatorname{Index} E = |G|$. Note that the crossed product $A \rtimes_\alpha G$ is equal to the basic construction $C^*\langle A^\alpha, e_P\rangle$, where $e_P = \frac{1}{|G|}\sum_{g\in G} u_g$. See the detail in [15, Sections 3 and 4]. Note that $A \rtimes_\alpha G$ is isomorphic to $pM_{|G|}(A^\alpha)p$ for some projection $p \in M_{|G|}(A^\alpha)$ by [38, Lemma 3.3.4].

Definition 4.2 Let $P \subset A$ be an inclusion of unital C*-algebras and let $E: A \to P$ be a conditional expectation of index-finite type. We denote by E^∞ the canonical conditional expectation from A^∞ to P^∞ induced by E. A conditional expectation E is said to have the *tracial Rokhlin property* if for any nonzero positive $z \in A^\infty$ there exists a projection $e \in A' \cap A^\infty$ satisfying that $(\operatorname{Index} E)E^\infty(e) = g$ is a projection, and $1 - g$ is Murray–von Neumann equivalent to a projection in the hereditary subalgebra of A^∞ generated by z, and a map $A \ni x \mapsto xe$ is injective. We call e a *Rokhlin projection*.

As in the case of an action with the tracial Roklin property ([30, Lemma 1.13]), if $E: A \to P$ is a conditional expectation of index-finite type for an inclusion of unital C*-algebras $P \subset A$ and E has the tracial Rokhlin property, then A has Property (SP) or E has the Rokhlin property; that is, there is Rokhlin projection $e \in A' \cap A^\infty$ such that $(\operatorname{Index} E)E^\infty(e) = 1$.

Lemma 4.3 *Let $P \subset A$ be an inclusion of unital C*-algebras and let $E: A \to P$ be a conditional expectation of index-finite type. Suppose that E has the tracial Rokhlin property; then A has Property (SP) or E has the Rokhlin property.*

Proof If A does not have Property (SP), then A^∞ does not have Property (SP); that is, there is a nonzero positive element $x \in A^\infty$ that generates a hereditary subalgebra that contains no nonzero projection. Since E has the tracial Rokhlin property, there exists a projection $e \in A_\infty$ such that $1 - (\text{Index}E)E^\infty(e)$ is equivalent to some projection in $\overline{xA^\infty x}$. Hence, $1 - (\text{Index}E)E^\infty(e) = 0$. This implies that E has the Rokhlin property. ■

Remark 4.4 (i) A projection g in Definition 4.2 is not zero, because that E^∞ is faithful.

(ii) A projection g in Definition 4.2 satisfies that $g \in P' \cap P^\infty$. Indeed, for any $x \in P$, since $ex = xe$ and E^∞ has norm one, we have

$$\begin{aligned}
xg &= x(\text{Index}E)E^\infty(e) = (\text{Index}E)x\big(E(e_1), E(e_n), \dots\big)\ \big(e = (e_n)\big)\\
&= (\text{Index}E)\big(xE(e_1), xE(e_2), \dots\big) = (\text{Index}E)\big(E(xe_1), E(xe_2), \dots\big)\\
&= (\text{Index}\,E)E^\infty(xe) = (\text{Index}\,E)E^\infty(ex)\\
&= (\text{Index}E)\big(E(e_1x), E(e_2x), \dots\big) = (\text{Index}\,E)\big(E(e_1)x, E(e_2)x, \dots\big) \qquad (4.1)\\
&= (\text{Index}\,E)\big(E(e_1), E(e_2), \dots\big)x = (\text{Index}E)E^\infty(e)x = gx.
\end{aligned}$$

Remark 4.5 In Definition 4.2 when A is simple, the following hold:

(i) We do not need the injectivity of the map $A \ni x \mapsto xe$.

(ii) We have $ege = e$. Indeed, since A is simple, Index$E$ is scalar by [38, Remark 2.3.6] and from [38, Lemma 2.1.5 (2)], there exists a constant $C > 0$ such that

$$E(e) \geq \frac{C}{(\text{Index}E)^2}e.$$

Thus, $g \geq e$, which implies that $e \in \overline{gA^\infty g}$. Therefore, $(1-g)e = 0$; that is,

$$ge = e. \tag{4.4}$$

Hence, $e = ege$.

Lemma 4.6 *Let $E\colon A \to P$ be of index-finite type with the tracial Rokhlin property and consider the basic extension $P \subset A \subset B$. Then the Rokhlin projection $e \in A' \cap A^\infty$ satisfies $eBe = Ae$.*

Proof Let e_p be the Jones projection for the inclusion $A \supset P$ as in [38, 2.1.1]. Set $f = (\text{Index}\,E)ee_pe$. Then, since $g = (\text{Index}\,E)E^\infty(e)$ is a projection such that $ge = e$ by (4.4) we have

$$\begin{aligned}
f^2 &= (\text{Index}\,E)^2ee_pee_pe = (\text{Index}\,E)^2ee_P(e_n)e_Pe\ (e = (e_n))\\
&= (\text{Index}\,E)^2e(e_Pe_ne_P)e = (\text{Index}\,E)^2e(E(e_n)e_P)e \qquad (4.2)\\
&= (\text{Index}\,E)^2e(E(e_n))e_pe = (\text{Index}\,E)^2eE^\infty(e)e_pe = (\text{Index}\,E)ege_pe\\
&= (\text{Index}\,E)ee_pe \qquad (4.4)\\
&= f.
\end{aligned}$$

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Let $\widehat{E}$ be the dual conditional expectation for E. Using [19, 2.3 (4)],

$$\widehat{E}^{\infty}(e-f) = e - \mathrm{Index}E\widehat{E}^{\infty}(ee_pe) = e - e = 0.$$

Thus, since $\widehat{E}$ is faithful, we have $e = f = (\mathrm{Index}\, E)ee_pe$; that is,

$$ee_Pe = (\mathrm{Index}E)^{-1}e. \tag{4.5}$$

Then since we have for any $x, y \in A$,

$$e(xe_py)e = xee_pey = (\mathrm{Index}E)^{-1}xey = (\mathrm{Index}E)^{-1}xye \in Ae.$$

Since B is the linear span of $\{xe_Py | x, y \in A\}$, we have $eBe \subset Ae$. Conversely, since $A \subset B$, $Ae \subset eBe$, and we conclude that $eBe = Ae$. ■

The following is the heredity of Property (SP) for an inclusion of unital C*-algebras.

Proposition 4.7 *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type. Suppose that A is simple and $E: A \to P$ has the tracial Rokhlin property. Then we have that*

(i) *P is simple;*
(ii) *A has Property (SP) if and only if P has Property (SP).*

Proof (i): Let e be a Rokhlin projection for E and $P \subset A \subset B$ be the basic extension. Since P is stably isomorphic to B by [38, Lemma 3.3.4], we will show that B is simple. By [12, Theorem 3.3] B can be written as finite direct sums of simple C*-algebras. Moreover, each simple C*-subalgebra has the form of Bz for some projection $z \in B \cap B'$. To show the simplicity of B it is enough to show that $B' \cap B = \mathbb{C}$.

Since $e = [(e_n)] \in A' \cap A^{\infty}$, for any $x \in A' \cap B$, we have

$$ex = [(e_n)]x = [(e_nx)] = [(xe_n)] = xe.$$

We can assume that $x = a_1e_pa_2$, where e_p is the Jones projection for E. Then

$$\begin{aligned} xe = exe = e(a_1e_pa_2)e &= a_1ee_pea_2 \\ &= (\mathrm{Index}E)^{-1}a_1a_2e \ ((4.5)) = \widehat{E}(x)e, \end{aligned}$$

where $\widehat{E}: B \to A$ be the dual conditional expectation of E. Note that $\widehat{E}(x) \in A'$. Hence, we have $xe \in (A' \cap A)e$. Therefore, $(A' \cap B)e \subset (A' \cap A)e$.

Since A is simple and $(B' \cap B)e \subset (A' \cap B)e \subset (A' \cap A)e$, we have $(B' \cap B)e = \mathbb{C}e$. Since the map $\rho: A' \cap B \to (A' \cap B)e$ by $\rho(x) = xe$ is an isomorphism, $B' \cap B = \mathbb{C}$; that is, B is simple, and P is simple.

(ii) This follows from [25, Corollary, Section 5]. ■

Proposition 4.8 *Let G be a finite group, α an action of G on an infinite dimensional finite simple separable unital C*-algebra A, and E the canonical conditional expectation from A onto the fixed point algebra $P = A^{\alpha}$ defined by*

$$E(x) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(x) \quad \text{for } x \in A,$$

where $|G|$ is the order of G. Then α has the tracial Rokhlin property if and only if E has the tracial Rokhlin property.

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Proof Suppose that α has the tracial Rokhlin property. Since A is separable, there is an increasing sequence of finite sets $\{F_n\}_{n\in\mathbb{N}} \subset A$ such that $\overline{\bigcup_{n\in\mathbb{N}} F_n} = A$. Let any nonzero positive element $x = (x_n) \in A^\infty$. Then we can assume that each x_n is a nonzero positive element. The simplicity of A implies that the map $A \ni x \mapsto xe$ is injective.

Since α has the tracial Rokhlin property, for each n there are mutually orthogonal projections $\{e_{g,n}\}_{g\in G}$ such that the following hold:

(i) $\|\alpha_h(e_{g,n}) - e_{hg,n}\| < \frac{1}{n}$ for all $g, h \in G$,

(ii) $\|[e_{g,n}, a]\| < \frac{1}{n}$ for all $g \in G$, and $a \in F_n$ with $\|a\| \le 1$,

(iii) $1 - \sum_{g\in G} e_{g,n}$ is equivalent to a projection q_n in $\overline{x_n A x_n}$.

Set $e_g = [(e_{g,n})] \in A^\infty$ for $g \in G$. Then for all $g, h \in G$,

$$\|\alpha_h^\infty(e_g) - e_{hg}\| = \limsup \|\alpha_h(e_{g,n}) - e_{hg,n}\| = 0;$$

hence, $\alpha_h^\infty(e_g) = e_{hg}$ for all $g, h \in G$.

For all $a \in \bigcup_{n\in\mathbb{N}} F_n$ with $\|a\| \le 1$ and all $g \in G$, we have

$$\|[e_g, a]\| = \limsup \|[e_{g,n}, a]\| = 0;$$

hence, $e_g \in A_\infty$ for all $g \in G$.

Set $q = (q_n) \in A^\infty$. Then q is a projection in $\overline{xA^\infty x}$ and

$$1 - \sum_{g\in G} e_g = \Big(1 - \sum_{g\in G} e_{g,n}\Big) \sim (q_n) = q.$$

Therefore, if we set $e = e_1$ for the identity element 1 in G, then $e \in A' \cap A^\infty$ and

$$E^\infty(e) = \frac{1}{|G|}\sum_{g\in G} \alpha_g^\infty(e) = \frac{1}{|G|}\sum_{g\in G} e_g,$$
$$1 - |G|E^\infty(e) = 1 - \sum_{g\in G} e_g \sim q \in \overline{xA^\infty x}.$$

Note that $\operatorname{Index} E = |G|$ by Example 4.1. It follows that E has the tracial Rokhlin property.

Conversely, suppose that E has the tracial Rokhlin property. From Lemma 4.3 A has Property (SP) or E has the Rokhlin property. If E has the Rokhlin property, then α has the Rokhlin property by [19, Proposition 3.2]; hence, α has the tracial Rokhlin property from the definition.

We can assume that A has Property (SP). Then for any finite set $F \subset A$, $\varepsilon > 0$, and any nonzero positive element $x \in A$ there is a projection $e \in A_\infty$ such that $|G|E^\infty(e)(= g)$ is a projection and $1 - g$ is equivalent to a projection $q \in \overline{xA^\infty x}$. We note that $g \ne 0$ by Remark 4.4, and $e \ne 0$. When we write $e = (e_n)$ and $q = (q_n)$, we can assume that for each $n \in \mathbb{N}$ e_n is projection and $1 - e_n$ is equivalent to q_n.

Define $e_g = \alpha_g^\infty(e) \in A_\infty$ for $g \in G$; write $e_g = [(\alpha_g(e_n))] = [(e_{g,n})]$ for $g \in G$. Then since we have

$$\sum_{g\in G} e_g = \sum_{g\in G} \alpha_g(e) = |G|E^\infty(e) = g$$

and g is projection, we can assume that $\{e_{g,n}\}_{g\in G}$ are mutually orthogonal projections for each $n \in \mathbb{N}$ by [22, Lemma 2.5.6].

Then $\alpha_h^\infty(e_g) = e_{hg}$ for all $g, h \in G$, $\|[e_g, a]\| = 0$ for all $a \in F$ and all $g \in G$, and

$$1 - \sum_{g\in G} e_g = 1 - \sum_{g\in G} \alpha_g^\infty(e) = 1 - |G|E^\infty(e) = 1 - g \sim q \in \overline{xA^\infty x},$$

Then there exists $n \in \mathbb{N}$ such that $\|\alpha_h(e_{g,n}) - e_{hg,n}\| < \varepsilon$ for all $g, h \in G$, $\|[e_{g,n}, a]\| < \varepsilon$ for all $a \in F$ and $g \in G$, and

$$1 - \sum_{g\in G} e_{g,n} \sim q_n \in \overline{xAx}.$$

Set $f_g = e_{g,n}$ for $g \in G$; then we have

$$\|\alpha_h(f_g) - f_{hg}\| < \varepsilon,$$

for all $g, h \in G$, $\|[f_g, a]\| < \varepsilon$ for all $a \in F$ and $g \in G$, and

$$1 - \sum_{g\in G} f_g \sim q_n \in \overline{xAx}.$$

Hence, α has the tracial Rokhlin property. ∎

The following lemma is key to proving the main theorem in this section.

Lemma 4.9 *Let $A \supset P$ be an inclusion of unital C*-algebras and let E be a conditional expectation from A onto P with index-finite type. Suppose that A is simple. If E has the tracial Rokhlin property with a Rokhlin projection $e \in A_\infty$ and a projection $g = (\mathrm{Index}E)E^\infty(e)$, then there is a unital linear map $\beta\colon A^\infty \to P^\infty g$ such that for any $x \in A^\infty$ there exists the unique element y of P^∞ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty g$. In particular, $\beta_{|A}$ is a unital injective *-homomorphism and $\beta(x) = xg$ for all $x \in P$.*

Proof Since E has the tracial Rokhlin property, A has Property (SP) or E has the Rokhlin property by Lemma 4.3. If E has the Rokhlin property, then the conclusion comes from [28, Lemma 2.5] with $g = 1$. Therefore, we can assume that A has Property (SP).

Since A has Property (SP), g and e are nonzero projections by Remark 4.4. As in the same argument in the proof of [28, Lemma 2.5], we have for any element x in A^∞ there exists a unique element $y = (\mathrm{Index}E)E^\infty(xe) \in P^\infty$ such that $xe = ye$. Indeed, by Lemma 4.6 we have $ee_Pe = (\mathrm{Index}E)^{-1}e$. Then

$$\begin{aligned} xe &= (\mathrm{Index}E)\widehat{E}^\infty(e_Pxe) = (\mathrm{Index}E)^2\widehat{E}^\infty(e_Pxee_Pe) && (4.3)\\ &= (\mathrm{Index}E)^2\widehat{E}^\infty(E^\infty(xe)e_Pe) && (4.2)\\ &= (\mathrm{Index}E)E^\infty(xe)e, \end{aligned}$$

where $\widehat{E}$ is the dual conditional expectation for E. Put $y = (\mathrm{Index}E)E^\infty(xe) \in P^\infty$. Then we have $xe = ye$. Note that since $eg = e$ by Remark 4.5, we have

$$yg = (\mathrm{Index}E)E^\infty(xe)g = (\mathrm{Index}E)E^\infty(xeg) \;\; (g \in P^\infty) = (\mathrm{Index}E)E^\infty(xe) = y.$$

Therefore, we can define a unital map $\beta\colon A^\infty \to P^\infty g$ $\beta(x) = (\mathrm{Index}E)E^\infty(xe)$ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty g$. Indeed, from the definition of β we know that $\beta(A' \cap A^\infty) \subset P^\infty g$. On the contrary, for any $x \in A' \cap A$ and $a \in P$ we have

$$\begin{aligned} a\beta(x) &= a(\mathrm{Index}E)E^\infty(xe) = \mathrm{Index}E)E^\infty(axe) && (4.1)\\ &= \mathrm{Index}E)E^\infty(xea) = \mathrm{Index}E)E^\infty(xe)a && (4.1)\\ &= \beta(x)a \end{aligned}$$

Hence $\beta(A' \cap A^\infty) \subset P'$. Therefore, $\beta(A' \cap A^\infty) \subset P' \cap P^\infty g$.

Note that β is injective. Indeed, if $\beta(x) = 0$ for $x \in A$, then $xe = 0$. Hence, from the definition of the tracial Rokhlin property for E, $x = 0$.

Since for any $x \in A$

$$\begin{aligned} \beta(x)g &= (\mathrm{Index}E)E^\infty(xe)g = (\mathrm{Index}E)E^\infty(xeg)\\ &= (\mathrm{Index}E)E^\infty(ex)\ (= \beta(x)) = (\mathrm{Index}E)E^\infty(gex)\\ &= g(\mathrm{Index}E)E^\infty(xe) = g\beta(x), \end{aligned}$$

we know that $\beta_{|A}$ is a unital *-homomorphism from A to $gP^\infty g$ from the same argument as in the proof of [28, Lemma 2.5]. In particular for any $x \in P$, we have

$$\beta(x) = (\mathrm{Index}E)E^\infty(xe) = x(\mathrm{Index}E)E^\infty(e) = xg(= gx). \qquad \blacksquare$$

The following lemma is important to prove the heredity of the local tracial $\mathcal{C}$-property for an inclusion of unital C*-algebras.

Lemma 4.10 *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type, and $E: A \to P$ has the tracial Rokhlin property. Suppose that projections $p, q \in P^\infty$ satisfy $ep = pe$ and $q \le ep$ in A^∞, where e is a Rokhlin projection for E. Then $q \le p$ in P^∞.*

Proof Let s be a partial isometry in A^∞ such that $s^*s = q$ and $ss^* \le ep$.

Set $v = (\mathrm{Index}E)^{1/2}E^\infty(s)$. Then

$$\begin{aligned} v^*ve_p &= (\mathrm{Index}E)E^\infty(s)^*E^\infty(s)e_p = (\mathrm{Index}E)E^\infty(s^*e)E^\infty(es)e_p\\ &= (\mathrm{Index}E)e_ps^*ee_pese_p = e_ps^*ese_p \quad ((\mathrm{Index}E)ee_pe = e; (4.5))\\ &= E^\infty(s^*es)e_p = E^\infty(s^*s)e_p = E^\infty(q)e_p = qe_p. \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{E}^\infty(v^*ve_p) &= \widehat{E}^\infty(qe_p),\\ (\mathrm{Index}E)^{-1}v^*v &= (\mathrm{Index}E)^{-1}q,\\ v^*v &= q. \end{aligned}$$

Since

$$pv = p(\mathrm{Index}E)^{1/2}E^\infty(s), = (\mathrm{Index}E)^{1/2}E^\infty(ps), = (\mathrm{Index}E)^{1/2}E^\infty(s) = v,$$

we have $q \le p$ in P^∞. $\blacksquare$

Theorem 4.11 *Let $\mathcal{C}$ be a class of weakly semiprojective C*-algebras satisfying conditions in Theorem 3.3. Let $A \supset P$ be an inclusion of unital C*-algebras and E a conditional* *expectation from A onto P with index-finite type. Suppose that A is a simple, local tracial $\mathcal{C}$-algebra and E has the tracial Rokhlin property. Then P is a local tracial $\mathcal{C}$-algebra.*

Proof We will prove that for every finite set $F \subset P$, every $\varepsilon > 0$, and $z \in P^+\backslash 0$ there are C*-algebra $Q \in \mathcal{C}$ with $q = 1_Q$ and *-homomorphism $\pi\colon Q \to A$ such that $\|\pi(q)x - x\pi(q)\| < \varepsilon$ for all $x \in F$, $\pi(q)S\pi(q) \subset_\varepsilon \pi(Q)$, and $1 - \pi(q)$ is equivalent to some non-zero projection in $\overline{zPz}$.

Since E has the tracial Rokhlin property, A has Property (SP) or E has the Rokhlin property by Lemma 4.3.

Suppose that E has the Rokhlin property. Them we have from [28, Lemma 2.5] that there is a unital *-homomorphism $\beta\colon A \to P^\infty$ such that $\beta(x) = x$ for all $x \in P$. Since A is a local tracial $\mathcal{C}$-algebra, there are an algebra $B \in \mathcal{C}$ with $1_B = p$ and a *-homomorphism $\pi\colon B \to A$ such that $\|x\pi(p) - \pi(p)x\| < \varepsilon$ for all $x \in F$, $\pi(p)F\pi(p) \subset_\varepsilon \pi(B)$, such that $1 - \pi(p)$ is equivalent to a non-zero projection $q \in \overline{zAz}$. Since E has the Rokhlin property, there exists a non-zero projection $e \in A' \cap A^\infty$ such that $E^\infty(e) = \frac{1}{\operatorname{Index} E}$.

Since B is weakly semiprojective, there exists $k \in \mathbb{N}$ and $\overline{\beta \circ \pi}\colon B \to \prod_{n=k}^\infty P$ such that $\beta \circ \pi = \pi_k \circ \overline{\beta \circ \pi}$, where $\pi_k((b_k, b_{k+1}, \dots)) = (0, \dots, 0, b_k, b_{k+1}, \dots)$. For each $l \in \mathbb{N}$ with $l \geq k$ let β_l be a *-homomorphism from B to P so that $\overline{\beta \circ \pi}(b) = (\beta_n(b))_{n=k}^\infty$ for $b \in B$. Then $\beta \circ \pi(b) = (0, \dots, 0, \beta_k(b), \beta_{k+1}(b), \dots) + C_0(P)$ for $b \in B$, and β_l is a *-homomorphism for $l \geq k$.

Since $1 - p \sim q \in \overline{zAz}$,

$$\begin{aligned} 1 - \beta \circ \pi(p) &= 1 - \beta(1 - \pi(p)) \\ &= \beta(1 - \pi(p)) \sim \beta \circ \pi(q) \in \beta(\overline{zAz}), \\ [(1 - \beta_k(\pi(p)))] &\sim [(q_k)] \in \overline{zP^\infty z}, \end{aligned}$$

where each q_k are projections in P. Taking the sufficient large k, since

$$\lim_k \|\beta_k(x) - x\| = 0$$

for $x \in P$, we have

(a) $\|x\beta_k(p) - \beta_k(p)x\| < 2\varepsilon$ for any $x \in F$,
(b) $\beta_k(p)F\beta_k(p) \subset_\varepsilon \beta_k(p)\beta_k(B)\beta_k(p)$,
(c) $1 - \beta_k(p) = \beta_k(1 - p) \sim q_k \in \overline{zPz}$.

Hence, P is a local tracially $\mathcal{C}$-algebra.

Suppose that A has Property (SP). Since A is simple, from Proposition 4.7, P also has Property (SP). Let $F \subset P$ be a finite set, $\varepsilon > 0$, and $z \in P^+\backslash 0$. Since P is simple and has Property (SP), there are orthogonal non-zero projections $r_1, r_2 \in \overline{zPz}$.

Since A is a local tracial $\mathcal{C}$-algebra, there are an algebra $B \in \mathcal{C}$ with $1_B = p$ and a *-homomorphism $\pi\colon B \to A$ such that $\|x\pi(p) - \pi(p)x\| < \varepsilon$ for all $x \in F$, $\pi(p)F\pi(p) \subset_\varepsilon B$, and $1 - \pi(p)$ is equivalent to a non-zero projection $q \in \overline{r_1Ar_1}$. Since E has the tracial Rokhlin property, there exist the Rokhlin projection $e' \in A' \cap A^\infty$. Take another Rokhlin projection $e \in A' \cap A^\infty$ for a projection $e'r_2$ such that $g = \operatorname{Index} E E(e)$ satisfies $1 - g$ is equivalent to a projection $\overline{e'r_2A^\infty e'r_2}$; that is, $1 - g \preceq e'r_2$ in A^∞. Then

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by Lemma 4.10, we know that $1-g \le r_2$ in P^∞; that is, there is a projection $s \le r_2 \in P^\infty$ such that $1-g \sim s$.

Write $g = [(g_n)]$ for some projections $\{g_k\}_{k\in\mathbb{N}} \subset P$. From Lemma 4.9, there exists an injective *-homomorphism $\beta\colon A \to gP^\infty g$ such that $\beta(x) = xg$ for all $x \in P$, and $\overline{\beta\circ\pi}\colon B \to \prod_{n=k}^\infty P$ such that $\beta\circ\pi = \pi_k \circ \overline{\beta\circ\pi}$, where

$$\pi_k(b_k, b_k+1, \dots) = (0,\dots,0,b_k,b_{k+1},\dots) + C_0(P).$$

For each $l \in \mathbb{N}$ with $l \ge k$, let β_l be a map from B to $g_l P g_l$ so that $\overline{\beta\circ\pi}(b) = (\beta_l(b))_{l=k}^\infty$ for $b \in B$. Then $\beta\circ\pi(b) = (0,\dots,0,\beta_k(b),\beta_{k+1}(b),\dots) + C_0(P)$ for all $b \in B$ and β_l is a *-homomorphism for $l \ge k$.

Since $1-\pi(p) \sim q \in \overline{r_1 A r_1}$,

$$\begin{aligned} 1-(\beta\circ\pi)(p) &= 1-g+g-\beta(\pi(p)) = 1-g+\beta(1-\pi(p)) \\ &\sim s+\beta(q) \in r_2P^\infty r_2 + \iota\circ\beta(\overline{r_1Ar_1}) \\ &\subset r_2P^\infty r_2 + r_1P^\infty r_1 \subset \overline{zP^\infty z}, \end{aligned}$$

we have $[(1-\beta_k(p)))] \sim [(q_k)] \in \overline{zP^\infty z}$, where each q_k is projection in P. Taking a sufficiently large k, since $\lim_k \|\beta_k(x) - x\| = 0$ for $x \in P$, we have

(a) $\|x\beta_k(p) - \beta_k(p)x\| < 2\varepsilon$ for any $x \in F$,
(b) $\beta_k(p)F\beta_k(p) \subset_\varepsilon \underline{\beta_k}(p)\beta_k(B)\beta_k(p)$,
(c) $1-\beta_k(p) \sim q_k \in \overline{zPz}$.

Hence, P is a local tracially $\mathcal{C}$-algebra. ■

Corollary 4.12 *Let $P \subset A$ be an inclusion of unital C*-algebras and let E be a conditional expectation from A onto P with index-finite type. Suppose that A is an infinite-dimensional simple C*-algebra with tracial topological rank zero (resp. less than or equal to one) and E has the tracial Rokhlin property. Then P has tracial rank zero (resp. less than or equal to one).*

Proof Since the classes $\mathcal{T}^{(k)}$ $(k = 0,1)$ are semiprojective with respect to a class of unital C*-algebras [24] and finitely saturated [26, Examples 2.1 & 2.2, and Lemma 1.6], the conclusion comes from Theorem 4.11 and Definition 3.8. ■

Finally, in this section we give the heredity of stable rank one and real rank zero for an inclusion of unital C*-algebras.

Proposition 4.13 *Let $P \subset A$ be an inclusion of unital C*-algebras with index finite-type. Suppose that $E\colon A \to P$ has the tracial Rokhlin property and A is simple with* $\mathrm{tsr}(A) = 1$. *Then* $\mathrm{tsr}(P) = 1$.

Proof Since E has the tracial Rokhlin property, E has the Rokhlin property or A has Property (SP) by Lemma 4.3. If E has the Rokhlin property, we conclude that $\mathrm{tsr}(P) = 1$ by [19, 5.9]. Therefore, we assume that A has the Property (SP). Then we know that P is simple and has Property (SP) by Proposition 4.7.

Note that $\mathrm{tsr}(A) = 1$, and A is stably finite by [31, Theorem 3.3]. Since an inclusion $P \subset A$ is of index-finite type, P is stably finite. Hence, using the idea in [32] we

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have only to show that any two sided zero divisor in P is approximated by invertible elements in P.

Let $x \in P$ be a two sided zero divisor. From [32, Lemma 3.5] we can assume that there is a positive element $y \in P$ and a unitary $u \in P$ such that $yux = 0 = uxy$. If we show that ux can be approximated by invertible elements, so does x. Hence, we can assume that $yx = 0 = xy$. Since P has Property (SP), there is a non-zero projection $e \in \overline{yPy}$. Since P is simple, we can take orthogonal projections e_1 and e_2 in P such that $e = e_1 + e_2$ and $e_2 \leq e_1$ by [22, Lemma 3.5.6 (2)]. Note that $x \in (1-e_1)A(1-e_1)$. Since $\mathrm{tsr}((1-e_1)A(1-e_1)) = 1$, there is an invertible element b in $(1-e_1)A(1-e_1)$ such that $\|x - b\| < \frac{1}{3}\varepsilon$.

Since E has the tracial Rokhlin property, there is a projection $g \in P' \cap P^\infty$ such that $1 - g \leq e_2$. That is, there is a partial isometry $w \in P^\infty$ such that $w^*w = 1 - g$ and $ww^* \leq e_1$ by Lemma 4.10 (see also Corollary 6.3.). Moreover, $\|\beta(x) - \beta(b)\| < \frac{1}{3}\varepsilon$ by Lemma 4.9. Note that $\beta(b)$ is invertible in $g(1-e_1)P^\infty(1-e_1)g$.

Set $v = w(1-e_1)$. Then

$$v^*v = (1-e_1)w^*w(1-e_1) = (1-g)(1-e_1),$$
$$vv^* = w(1-e_1)w^* \leq ww^* \leq e_1.$$

Set

$$z = \frac{\varepsilon}{3}(e_1 - vv^*) + \frac{\varepsilon}{3}v + \frac{\varepsilon}{3}v^* + (1-g)x(1-g).$$

Hence, z is invertible in $e_1P^\infty e_1 + (1-g)(1-e_1)P^\infty(1-e_1)(1-g)$ and $\|z-(1-g)x(1-g)\| < \frac{\varepsilon}{3}$.

Then $\beta(b) + z \in P^\infty$ is invertible and

$$\begin{aligned}\|x - (\beta(b) + z)\| &= \|xg + x(1-g) - \beta(b) - z\| \\ &= \|\beta(x) - \beta(b) + (1-g)x(1-g) - z\| \\ &\leq \|\beta(x) - \beta(b)\| + \|(1-g)x(1-g) - z\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

Write $\beta(b) + z = (y_n)$ such that y_n is invertible in P. Therefore, there is a y_n such that $\|x - y_n\| < \varepsilon$, and we conclude that $\mathrm{tsr}(P) = 1$. ∎

Proposition 4.14 *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type and $E: A \to P$ has the tracial Rokhlin property. Suppose that A is simple, stably finite, with real rank zero. Then P has real rank zero.*

Proof Let $x \in P$ be a self-adjoint element and $\varepsilon > 0$. Consider a continuous real valued function f is defined by $f(t) = 1$ for $|t| \leq \frac{\varepsilon}{12}$, $f(t) = 0$ if $|t| \geq \frac{\varepsilon}{6}$, and $f(t)$ is linear if $\frac{\varepsilon}{12} \leq |t| \leq \frac{\varepsilon}{6}$. We may assume that $f(x) \neq 0$. Note that $\|yx\| < \frac{\varepsilon}{6}$ for any $y \in \overline{f(x)Pf(x)}$ with $\|y\| \leq 1$.

Since A is simple and has Property (SP), P has Property (SP) by Proposition 4.7; that is, there is a non-zero projection $e \in \overline{f(x)Pf(x)}$. Moreover, there are orthogonal projections e_1 and e_2 such that $e = e_1 + e_2$ such that $e_2 \sim e_1$ by [22, Lemma 3.5.7]. Then

$$\|x - (1-e_1)x(1-e_1)\| = \|e_1xe_1 + e_1x(1-e_1) + (1-e_1)xe_1)\| < \frac{3\varepsilon}{12} = \frac{\varepsilon}{4}$$

As in the same step in the argument in Proposition 4.13, we have there is an invertible self-adjoint element $z \in P$ such that $\|(1-e_1)x(1-e_1) - z\| < \frac{2\varepsilon}{3}$. Hence, we have $\|x - z\| < \varepsilon$, and we conclude that P has real rank zero. ■

5 The Jiang–Su Absorption

In this section we discuss about the heredity for the Jiang–Su absorption for an inclusion of unital C*-algebras with the tracial Rokhlin property.

Definition 5.1 ([11, Definition 2.1]) A unital C*-algebra A is said to be *tracially* $\mathcal{Z}$*-absorbing* if $A \not\cong \mathbb{C}$ and for any finite set $F \subset A$ and non-zero positive element $a \in A$ and $n \in \mathbb{N}$ there is an order zero contraction $\phi\colon M_n \to A$ such that the following hold:

(i) $1 - \phi(1) \precsim a$;
(ii) for any normalized element $x \in M_n$ and any $y \in F$, we have $\|[\phi(x), (y)]\| < \varepsilon$.

Theorem 5.2 ([11, Theorem 4.1]) *Let A be a unital, separable, simple, nuclear C*-algebra. If A is tracially $\mathcal{Z}$-absorbing, then $A \cong A \otimes \mathcal{Z}$.*

Note that for a simple unital C*-algebra A, if A is $\mathcal{Z}$-absorbing, then A is tracially $\mathcal{Z}$-absorbing ([11, Proposition 2.2]).

Theorem 5.3 *Let $P \subset A$ be an inclusion of unital C*-algebra and let E be a conditional expectation from A onto P with index-finite type. Suppose that A is simple, separable, unital, tracially $\mathcal{Z}$-absorbing, and that E has the tracial Rohklin property. Then P is tracially $\mathcal{Z}$-absorbing.*

Proof Take any finite set $F \subset P$ and non-zero positive element $a \in P$ and $n \in \mathbb{N}$. Since $E\colon A \to P$ has the tracial Rokhlin property, E has the Rokhlin property or A has Property (SP). If E has the Rokhlin property, then P is $\mathcal{Z}$-absorbing ([28, Theorem 3.3]), and we are done. Hence, we can assume that A has Property (SP).

Since A is simple and has Property (SP), P has Property (SP) by Proposition 4.7. Then there exist orthogonal projections p_1, p_2 in $\overline{aPa}$.

Since A is tracially $\mathcal{Z}$-absorbing, there is an order zero contraction $\phi\colon M_n \to A$ such that the following hold:

(a) $1 - \phi(1) \precsim p_1$.
(b) For any normalized element $x \in M_n$ and any $y \in F$ we have $\|[\phi(x), y)]\| < \varepsilon$.

Since $E\colon A \to P$ has the tracial Rokhlin property, there is a projection $e \in A' \cap A^\infty$ such that $(\mathrm{Index}\, E)E^\infty(e) = g$ is a projection and $1 - g \precsim p_2$. Moreover, by Lemma 4.9 there is an injective *homomorphism β from A into $gP^\infty g$ such that $\beta(1) = g$ and $\beta(a) = ag$ for $a \in P$.

(a) Then the function $\beta \circ \phi (= \psi)\colon M_n \to P^\infty$ is an order zero map such that

$$\begin{aligned} 1 - \psi(1) &= 1 - (\beta \circ \phi)(1) = 1 - g + \beta(1 - \phi(1)) \\ &\precsim p_2 + \beta(p_1) = p_2 + p_1\beta(1) \precsim a, \end{aligned}$$

that is, $1 - \psi(1) \precsim a$ in P^∞.

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(b) For any normalized element $x \in M_n$ and $y \in F$,

$$\begin{aligned} \|[\psi(x), y]\| &= \|[\beta(\phi(x)), y]\| = \|\beta(\phi(x))y - y\beta(\phi(x))\| \\ &= \|\beta(\phi(x))\beta(y) - \beta(y)\beta(\phi(x))\| \\ &= \|\beta(\phi(x)y - y\phi(x))\| \le \|\phi(x)y - y\phi(x)\| < \varepsilon. \end{aligned}$$

Since $C^*(\phi(M_n))$ is semiprojective in the sense of [24, Definition 14.1.3], there is a $k \in \mathbb{N}$ and a *-homomorphism $\widetilde{\beta}: C^*(\phi(M_n))) \to \Pi P/\oplus_{i=1}^k P \to P^\infty$ such that $\pi_k \circ \widetilde{\beta} = \beta$, where π_k is the canonical map from $\Pi P/\oplus_{i=1}^k P$ to P^∞. Write $\widetilde{\beta}(x) = (\widetilde{\beta}_l(x)) + \oplus_{i=1}^k P$ and $g = (g_l)$ for some projections $g_l \in P$ for $l \in \mathbb{N}$. Then we have for sufficiently large l the order zero map $\widetilde{\beta}_l \circ \phi: M_n \to P$ satisfies

(c) $$\begin{aligned} 1 - \widetilde{\beta}_l \circ \phi(1) &= 1 - \widetilde{\beta}_l(\phi(1)) = 1 - g_l + g_l - \widetilde{\beta}_l(\phi(1)) \\ &= 1 - g_l + \widetilde{\beta}_l(1 - \phi(1))\ (\widetilde{\beta}_l(1) = g_l) \\ &\le 1 - g_l + \widetilde{\beta}_l(p_1) \le p_2 + p_1', \end{aligned}$$

where $p_1' \in p_1 P p_1$ is projection such that $\widetilde{\beta}_l(p_1) \sim p_1'$. Note that since $\|\widetilde{\beta}_l(p_1) - p_1 g_l\|$ is very small ($\|\widetilde{\beta}_l(p_1) - p_1 g_l\| < \frac{1}{4}$ is enough), there are projections $p_1' \in p_1 P p_1$ such that $\widetilde{\beta}_l(p_1) \sim p_1'$. Since $p_2 \perp p_1'$, $1 - g_l \le p_2$, and $\widetilde{\beta}_i(p_1) \sim p_1'$, from [6, 1.1 Proposition] we have $1 - g_l + \widetilde{\beta}(p_1) \le p_2 + p_1'$. Hence, we have

$$1 - \widetilde{\beta}_l \circ \phi(1) \le p_2 + p_1' \le a$$

(d) For any normalized element $x \in M_n$ and $y \in F$ $\|[\widetilde{\beta}_n \circ \phi(x), y]\| < 3\varepsilon$.
This implies that P is tracially $\mathcal{Z}$-absorbing. ∎

The following is the main theorem in this paper.

Theorem 5.4 *Let $P \subset A$ be an inclusion of unital C^*-algebras and let E be a conditional expectation from A onto P with index-finite type. Suppose that A is simple, separable, nuclear, $\mathcal{Z}$-absorbing, and that E has the tracial Rohklin property. Then P is $\mathcal{Z}$-absorbing.*

Proof This follows from Theorems 5.3 and 5.2. ∎

Corollary 5.5 *Let A be an infinite dimensional simple, unital, simple, nuclear C^*-algebra and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Suppose that A is $\mathcal{Z}$-absorbing. Then we have the following:*

(i) *the fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ are $\mathcal{Z}$-absorbing* ([11]);
(ii) *for any subgroup H of G the fixed point algebra A^H is $\mathcal{Z}$-absorbing.*

Proof (i) Since the canonical conditional expectation $E: A \to A^\alpha$ has the tracial Rokhlin property by Proposition 4.8, A^α is $\mathcal{Z}$-absorbing, by Theorem 5.4.

Let $|G| = n$. Then $A \rtimes_\alpha G$ is isomorphic to $pM_n(A^\alpha)p$ for some projection $p \in M_n(A^\alpha)$ by Example 4.1(ii). Since A^α is $\mathcal{Z}$-absorbing, $pM_n(A^\alpha)p$ is $\mathcal{Z}$-absorbing by [37, Corollary 3.1], hence $A \rtimes_\alpha G$ is $\mathcal{Z}$-absorbing.

(ii) Since $\alpha_{|H}: H \to \mathrm{Aut}(A)$ has the tracial Rokhlin property by [8, Lemma 5.6], we know that A^H is $\mathcal{Z}$-absorbing, by (i). ∎

6 Cuntz-equivalence for Inclusions of C*-algebras

In this section we study the heredity for Cuntz equivalence for an inclusion of unital C*-algebras with the tracial Rokhlin property.

Let $M_\infty(A)^+$ denote the disjoint union $\bigcup_{n=1}^\infty M_n(A)^+$. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $a \oplus b = \mathrm{diag}(a, b) \in M_{n+m}(A)^+$, and write $a \precsim b$ if there is a sequence $\{x_k\}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Put $W(A) = M_\infty(A)^+/\sim$, and let $\langle a\rangle \in W(A)$ be the equivalence class containing a. Then $W(A)$ is a positive ordered abelian semigroup equipped with the relations

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle, \quad \langle a\rangle \le \langle b\rangle \Longleftrightarrow a \precsim b, \quad a, b \in M_\infty(A)^+.$$

We call $W(A)$ the Cuntz semigroup.

Lemma 6.1 *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type and let $E\colon A \to P$ have the tracial Rokhlin property. Suppose that positive elements $a, b \in P^\infty$ satisfy $eb = be$ and $a \precsim eb$ in A^∞, where e is a Rokhlin projection for E. Then $a \precsim b$ in P^∞.*

Proof Since $a \precsim eb$ in A^∞, there is a sequence $\{v_n\}_{n\in\mathbb{N}}$ in A^∞ such that

$$\|a - v_n^* ebv_n\| \longrightarrow 0 \ (n \to \infty).$$

Let $E\colon A \to P$ be a conditional expectation of index-finite type. Set

$$w_n = (\mathrm{Index}\, E)^{\frac{1}{2}} E^\infty(ev_n)$$

for each $n \in \mathbb{N}$. Then, since

$$\begin{aligned}
w_n^* bw_n e_P &= (\mathrm{Index}\, E)E^\infty(v_n^* e)bE^\infty(ev_n)e_P \\
&= (\mathrm{Index}\, E)E^\infty(v_n^* eb)E^\infty(ev_n)e_P && (4.1)\\
&= (\mathrm{Index}\, E)e_P v_n^* ebe_P ev_n e_P && (4.2)\\
&= (\mathrm{Index}\, E)e_P v_n^* bee_P ev_n e_P \\
&= e_P v_n^* ebv_n e_P && (4.5)\\
&= E^\infty(v_n^* ebv_n)e_P && (4.2),
\end{aligned}$$

$w_n^* bw_n = E^\infty(v_n^* ebv_n)$ from [38, Lemma 2.1.4]. Therefore,

$$\begin{aligned}
\|a - w_n^* bw_n\| &= \|a - E^\infty(v_n^* ebv_n)\| = \|E^\infty(a - v_n^* ebv_n)\| \\
&\le \|a - v_n^* ebv_n\| \longrightarrow 0(n \longrightarrow \infty).
\end{aligned}$$

This implies that $a \precsim b$ in P^∞. ∎

Proposition 6.2 *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type. Suppose that $E\colon A \to P$ has the tracial Rokhlin property. If two positive elements $a, b \in P$ satisfy $a \precsim b$ in A, then $a \precsim b$ in P.*

Proof Let $a, b \in P$ be positive elements such that $a \precsim b$ in A and $\varepsilon > 0$. Since for any constant $K > 0$ $a \precsim b$ is equivalent to $Ka \precsim Kb$, we can assume that a and b are contractive. If b is invertible, then $a = (a^{1/2}b^{-1/2})b(a^{1/2}b^{-1/2})^*$, and $a \precsim b$ in P. Hence, we may assume that b has 0 in its spectrum.

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Since $a \precsim b$ in A, for every $\varepsilon > 0$ there is $\delta > 0$ and $r \in A$ such that $f_\varepsilon(a) = r f_\delta(b) r^*$ by [33, Proposition 2.4], where $f_\varepsilon\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$f_\varepsilon(t) = \begin{cases} 0, & t \le \varepsilon, \\ \varepsilon^{-1}(t-\varepsilon), & \varepsilon \le t \le 2\varepsilon, \\ 1, & t \ge 2\varepsilon. \end{cases}$$

Set $a_0 = f_\delta(b)^{1/2} r^* r f_\delta(b)^{1/2}$. Then $f_\varepsilon(a) \sim a_0$ by [5, 1.5]. Set a continuous function $g_\delta(t)$ on $[0,1]$ by

$$g_\delta(t) = \begin{cases} \delta^{-1}(\delta - t) & 0 \le t \le \delta, \\ 0 & \delta \le t \le 1. \end{cases}$$

Since b has 0 in its spectrum, $g_\delta(b) \ne 0$ and $g_\delta(b) f_\delta(b) = 0$, which implies that $a_0 g_\delta(b) = 0$. Note that $g_\delta(b) (= c)$ and a_0 belong to $\overline{bPb}$. Therefore, the positive elements a_0 in $\overline{bAb}$ and c in $\overline{bPb}$ satisfy $(a-\varepsilon)_+ \precsim a_0 + c$ in A. Indeed,

$$(a-\varepsilon)_+ \precsim f_\varepsilon(a) \sim a_0 \precsim a_0 + c \quad ([6, \text{Proposition 1.1}]).$$

Take a Rokhlin projection $e \in A' \cap A^\infty$ for E. Then there is a projection $g \in P' \cap P^\infty$ such that $(1-g) \precsim ec$. Hence, $(a-\varepsilon)_+(1-g) \precsim ec$ in A^∞. Note that since $c \in P$, we have $ec = ce$. By Lemma 6.1, $(a-\varepsilon)_+(1-g) \precsim c$ in P^∞.

Then we have in P^∞

$$\begin{aligned} (a-\varepsilon)_+ &= (a-\varepsilon)_+ g + (a-\varepsilon)_+(1-g) \\ &= \beta((a-\varepsilon)_+) + (a-\varepsilon)_+(1-g) \\ &\precsim \beta(a_0) + (a-\varepsilon)_+(1-g) && ([6, \text{Proposition 1.1}]) \\ &\precsim \beta(a_0) + c \in \overline{bP^\infty b}, && ([6, \text{Proposition 1.1}]) \end{aligned}$$

where $\beta\colon A \to gP^\infty g$ is defined as in Lemma 4.9. Hence, $(a-\varepsilon)_+ \precsim b$ in P^∞.

Since $\varepsilon > 0$, we have $a \precsim b$ in P^∞, and $a \precsim b$ in P. ∎

The following result implies that the canonical inclusion from $K_0(P)$ into $K_0(A)$ is injective.

Corollary 6.3 *Under the same assumption in Proposition 6.2, if two projections $p, q \in P$ satisfy $p \precsim q$ in A, then $p \precsim q$ in P.*

7 The Strict Comparison Property

In this section we study the strict comparison property for a Cuntz semigroup and show that for an inclusion $P \subset A$ of exact, unital C*-algebras with the tracial Rokhlin property if A has strict comparison, then so does P. When $E\colon A \to P$ has the Rokhlin property, the statement is proved in [29].

A *dimension function* on a C*-algebra A is a function $d\colon M_\infty(A)^+ \to \mathbb{R}^+$ that satisfies $d(a \oplus b) = d(a) + d(b)$, and $d(a) \le d(b)$ if $a \precsim b$ for all $a, b \in M_\infty(A)^+$. If τ is a positive trace on A, then

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \lim_{\varepsilon \to 0+} \tau\big(f_\varepsilon(a)\big), \quad a \in M_\infty(A)^+$$

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defines a dimension function on A. Every lower semicontinuous dimension function on an exact C*-algebra arises in this way ([3, Theorem II.2.2], [10], [17]). For the Cuntz semigroup $W(A)$ an additive order preserving mapping $\widetilde{d}\colon W(A) \to \mathbb{R}^+$ is given by $\widetilde{d}(\langle a \rangle) = d(a)$ from a dimension function d on A. We use the same symbol to denote the dimension function on A and the corresponding state on $W(A)$.

Recall that an C*-algebra A has strict comparison if, whenever $x, y \in W(A)$ are such that $d(x) < d(y)$ for every dimension function d on A, we have $x \le y$. If A is simple, exact and unital , then the strict comparison property is equivalent to the strict comparison property by traces; that is, for all $x, y \in W(A)$ one has that $x \le y$ if $d_\tau(x) < d_\tau(y)$ for all tracial states τ on A ([34, Corollary 4.6]).

Let $\mathrm{T}(A)$ be the set of all traces on a C*-algebra A.

Theorem 7.1 *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type. Suppose that A is simple and exact and A has strict comparison and $E\colon A \to P$ has the tracial Rokhlin property. Then P has strict comparison.*

Proof Since $E\colon A \to P$ is of index-finite type and A is simple and exact, P is exact and simple by Proposition 4.7 (i). Note that the strict comparison property is equivalent to the the strict comparison property given by traces, *i.e.*, for all $x, y \in W(A)$ one has that $x \le y$ if $d_\tau(x) < d_\tau(y)$ for all tracial states τ on A (see [35, Remark 6.2] and [34, Corollary 4.6]).

Since $E \otimes \mathrm{id}\colon A \otimes M_n \to P \otimes M_n$ is of index-finite type and has the tracial Rokhlin property, it suffices to verify the condition that whenever $a, b \in P$ are positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T(P)}$, then $a \le b$.

Let $a, b \in P$ be positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T(P)}$. Then for any tracial state $\tau \in T(A)$ the restriction $\tau_{|P}$ belongs to $T(P)$. Hence, we have $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in \mathrm{T}(A)$. Since A has strict comparison, $a \le b$ in A. Therefore, by Proposition 6.2, $a \le b$ in P, and P has strict comparison. ∎

Corollary 7.2 *Let $P \subset A$ be an inclusion of unital C*-algebras of index-finite type. Suppose that A is simple, the order on projections on A is determined by traces, and $E\colon A \to P$ has the tracial Rokhlin property. Then the order on projections on P is determined by traces.*

Proof Since $E \otimes \mathrm{id}\colon A \otimes M_n \to P \otimes M_n$ is of index-finite type and has the tracial Rokhlin property, it suffices to verify the condition that whenever $p, q \in P$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in \mathrm{T(P)}$, p is Murray–von Neumann equivalent to a subprojection of q in P.

Let $p, q \in P$ be projections such that $\tau(p) < \tau(q)$ for all $\tau \in \mathrm{T(P)}$. Since for any tracial state $\tau \in T(A)$ the restriction $\tau_{|P}$ belongs to $T(P)$, we have $\tau(p) < \tau(q)$ for all tracial states $\tau \in \mathrm{T}(A)$. Since the order on projections on A is determined by traces, p is Murray–von Neumann subequivalent to q in A, and $p \le q$ in A by [33, Proposition 2.1]. Therefore, by Proposition 6.2 $p \le q$ in P, and so p is Murray–von Neumann equivalent to a subprojection of q in P. Hence, the order on projections on P is determined by traces. ∎

The following is well known, but there is no direct proof, so we present it for convenience of the reader.

Lemma 7.3 *Let A be an exact C*-algebra and let p be a projection of A. Suppose that A has strict comparison. Then so does pAp.*

Proof Since for each $n \in \mathbb{N}$ $M_n(A)$ is exact and has strict comparison, we have only to show that whenever $a, b \in pAp$ are positive elements such that $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in \mathrm{T}(pAp)$; then $a \precsim b$.

Let $a, b \in pAp$ be positive elements such that $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in \mathrm{T}(pAp)$. Then for any tracial state $\rho \in \mathrm{T}(A)$ the restriction $\rho_{|pAp}$ belongs to $\mathrm{T}(pAp)$. Since $d_\rho(a) = d_{\rho_{|pAp}}(a) < d_{\rho_{|pAp}}(b) = d_\rho(b)$, we have $a \precsim b$ in A by the assumption. Hence there is a sequence $\{x_n\} \subset A$ such that $\|x_n^* b x_n - a\| \to 0$. Set $y_n = p x_n p \in pAp$ for each $n \in \mathbb{N}$, then

$$\begin{aligned}\|y_n^* b y_n - a\| &= \|p x_n^* p b p x_n p - pap\| = \|p x_n^* b x_n p - pap\| \\ &\leq \|x_n^* b x_n - a\| \to 0,\end{aligned}$$

and $a \precsim b$ in pAp. Therefore, pAp has strict comparison. ∎

Corollary 7.4 *Let A be an infinite-dimensional, simple, separable, unital C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Suppose that A is exact and has strict comparison.*

(i) *The fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ have strict comparison.*
(ii) *For any subgroup H of G the fixed point algebra A^H has strict comparison.*

Proof (i) Since the canonical conditional expectation $E\colon A \to A^\alpha$ has the tracial Rokhlin property by Proposition 4.8, A^α has strict comparison by Theorem 7.1.

Let $|G| = n$. Then $A \rtimes_\alpha G$ is isomorphic to $pM_n(A^\alpha)p$ for some projection $p \in M_n(A^\alpha)$ by Example 4.1 (ii) . Since A^α has strict comparison, $pM_n(A^\alpha)p$ has strict comparison by Lemma 7.3, hence $A \rtimes_\alpha G$ has strict comparison.

(ii) Since $\alpha_{|H}\colon H \to \mathrm{Aut}(A)$ has the tracial Rokhlin property by [8, Lemma 5.6], we know that A^H has strict comparison by (i). ∎

Similarly, we have the following corollary.

Corollary 7.5 *Let A be an infinite dimensional simple separable unital C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Suppose that the order on projections on A is determined by traces.*

(i) *The order on projections on the fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ is determined by traces.*
(ii) *For any subgroup H of G the order on projections on the fixed point algebra A^H is determined by traces.*

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References

[1] D. Archey, *Crossed product C*-algebras by finite group actions with the tracial Rokhlin property*. Rocky Mountain J. Math. **41**(2011), no. 6, 1755–1768. http://dx.doi.org/10.1216/RMJ-2011-41-6-1755

[2] B. Blackadar, *The stable rank of full corners in C*-algebras*. Proc. Amer. Math. Soc. **132**(2004), 2945–2950. http://dx.doi.org/10.1090/S0002-9939-04-07148-5

[3] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras*. J. Funct. Anal. **45**(1982), 297–340. http://dx.doi.org/10.1016/0022-1236(82)90009-X

[4] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*. J. Funct. Anal. **99**(1991), 131–149. http://dx.doi.org/10.1016/0022-1236(91)90056-B

[5] J. Cuntz, *The structure of multiplication and addition in simple C*-algebras.*. Math. Scand. **40**(1977), no. 2, 215–233. http://dx.doi.org/10.7146/math.scand.a-11691

[6] ———, *Dimension functions on simple C*-algebras*. Math. Ann. **233**(1978), 145–153. http://dx.doi.org/10.1007/BF01421922

[7] G. A. Elliott and Z. Niu, *On tracial approximation*. J. Funct. Anal. **254**(2008), 396–440. http://dx.doi.org/10.1016/j.jfa.2007.08.005

[8] S. Echterhoff, W. Luck, N. C. Phillips, and S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$*. J. Reine Angew. Math. **639**(2010), 173–221. http://dx.doi.org/10.1515/CRELLE.2010.015

[9] Q. Fan and X. Fang, *Stable rank one and real rank zero for crossed products by finite group actions with the tracial Rokhlin property*. Chin. Anal. Math. Ser. B **30**(2009), 179–186. http://dx.doi.org/10.1007/s11401-007-0563-7

[10] U. Haagerup, *Quasitraces on exact C*-algebras are traces*. C. R. Math. Acad. Sci. Soc. Can. **36**(2014), no. 2–3, 67–92.

[11] I. Hirshberg and J. Orovitz, *Tracially $\mathcal{Z}$-absorbing C*-algebras*. J. Funct. Anal. **265**(2013), 765–785. http://dx.doi.org/10.1016/j.jfa.2013.05.005

[12] M. Izumi, *Inclusions of simple C^*-algebras*. J. Reine Angew. Math. **547**(2002), 97–138. http://dx.doi.org/10.1515/crll.2002.055

[13] ———, *Finite group actions on C^*-algebras with the Rokhlin property. I*. Duke Math. J. **122**(2004), 233–280. http://dx.doi.org/10.1215/S0012-7094-04-12221-3

[14] J. A. Jeong and H. Osaka, *Extremally rich C*-crossed products and the cancellation property*. J. Austral. Math. Soc. Ser. A **64**(1998), 285–301. http://dx.doi.org/10.1017/S1446788700039161

[15] J. A. Jeong and G. H. Park, *Saturated actions by finite dimensional Hopf *-algebras on C^*-algebras*. Intern. J. Math. **19**(2008), 125–144. http://dx.doi.org/10.1142/S0129167X08004583

[16] X. Jiang and H. Su, *On a simple unital projectionless C*-algebras*. Amer. J. Math. **121**(1999), 359–413. http://dx.doi.org/10.1353/ajm.1999.0012

[17] E. Kirchberg, *On the existence of traces on exact stably projectionless simple C*-algebras*. In: Operator Algebras and their Applications (Waterloo, ON, 1884/1995), Fields Institute Communications, 13, American Mathematical Society, Providence, RI, 1997, pp. 171–172.

[18] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C*-algebras*. Commun. Math. Phys. **81**(1981), 429–435. http://dx.doi.org/10.1007/BF01209077

[19] K. Kodaka, H. Osaka, and T. Teruya, *The Rokhlin property for inclusions of C*-algebras with a finite Watatani Index*. In: Operator structures and Dynamical Systems, Contemporary Mathematics, 503, American Mathematical Society, Providence, RI, 2009, pp. 177–195. http://dx.doi.org/10.1090/conm/503/09900

[20] H. Lin, *Tracial AF C*-algebras*. Trans. Amer. Math. Soc. **353**(2001), 693–722. http://dx.doi.org/10.1090/S0002-9947-00-02680-5

[21] ———, *The tracial topological rank of C*-algebras*. Proc. London Math. Soc. (3) **83**(2001), no. 1, 199–234. http://dx.doi.org/10.1112/plms/83.1.199

[22] ———, *An Introduction to the Classification of Amenable C*-algebras*. World Scientific, River Edge NJ, 2001. http://dx.doi.org/10.1142/9789812799883

[23] ———, *Simple nuclear C*-algebras of tracial topological rank one*. J. Funct. Anal. **251**(2007), 601–679. http://dx.doi.org/10.1016/j.jfa.2007.06.016

[24] T. A. Loring, *Lifting Solutions to Perturbing Problems in C^*-algebras*. Fields Institute Monographs, 8, American Mathematical Society, Providence RI, 1997.

[25] H. Osaka, *SP-Property for a pair of C*-algebras*. J. Operator Theory **46**(2001), 159–171.

[26] H. Osaka and N. C. Phillips, *Crossed products by finite group actions with the Rokhlin property*. Math. Z. **270**(2012), 19–42. http://dx.doi.org/10.1007/s00209-010-0784-4

[27] H. Osaka and N. C. Phillips, in preparation.

[28] H. Osaka and T. Teruya, *Strongly self-absorbing property for inclusions of C*-algebras with a finite Watatani index*. Trans. Amer. Math. Soc. **366**(2014), no. 3, 1685–1702. http://dx.doi.org/10.1090/S0002-9947-2013-05907-7

[29] ———, *Permanence of nuclear dimension for inclusions of unital C*-algebras with the Rokhlin property.* Advances in Operator Theory 3(2018), 123–136. http://dx.doi.org/10.22034/aot.1703-1145

[30] N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*. Amer. J. Math. **133**(2011), no. 3, 581–636. http://dx.doi.org/10.1353/ajm.2011.0016

[31] M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*. Proc. London Math. Soc. **46**(1983),301–333. http://dx.doi.org/10.1112/plms/s3-46.2.301

[32] M. Rørdam, *On the structure of simple C*-Algebras tensored with a UHF-algebra*. J. Funct. Anal. **100**(1991), 1–17. http://dx.doi.org/10.1016/0022-1236(91)90098-P

[33] ———, *On the structure of simple C*-Algebras tensored with a UHF-algebra II*. J. Funct. Anal. **107**(1992), 255–269. http://dx.doi.org/10.1016/0022-1236(92)90106-S

[34] ———, *The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras*. Internat. J. Math. **15**(2004), 1065–1084. http://dx.doi.org/10.1142/S0129167X04002661

[35] M. Rørdam and W. Winter, *The Jiang-Su algebra revisited*. J. Reine Angew. Math. **642**(2010), 129–155. http://dx.doi.org/10.1515/CRELLE.2010.039

[36] A. S. Toms, *Characterizing classifiable AH algebras*. C. R. Math. Acad. Sci. Soc. R. Can. **33**(2011), no. 4, 123–126.

[37] A. S. Toms and W. Winter, *Strongly self-absorbing C*-algebras*. Trans. Amer. Math. Soc. **359**(2007), 3999–4029. http://dx.doi.org/10.1090/S0002-9947-07-04173-6

[38] Y. Watatani, *Index for C^*-subalgebras*. Mem. Amer. Math. Soc. **424**(1990). http://dx.doi.org/10.1090/memo/0424

[39] W. Winter and J. Zacharias, *The nuclear dimension of C*-algebras*. Adv. Math. **224**(2010), 461–498. http://dx.doi.org/10.1016/j.aim.2009.12.005

[40] X. Yang and X. Fang, *The tracial class property for crossed products by finite group actions*. Abstr. Appl. Anal., 2012, Art. ID 745369.

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