J. Austral. Math. Soc. (Series A) 35 (1983), 194–196

A CHARACTERIZATION OF SEMI-PRIME IDEALS IN NEAR-RINGS

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(Received 21 June 1982)

Communicated by R. Lidl

Abstract

It is well-known that in any near-ring, any intersection of prime ideals is a semi-prime ideal. The aim of this note is to prove that any ideal is a prime ideal if and only if it is equal to its prime radical. As a consequence of this we have any semi-prime ideal I in a near-ring N is the intersection of minimal prime ideals of I in N and that I is the intersection of all prime ideals containing I.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 16 A 76; secondary 08 A 40.

1. Preliminaries

A near-ring is an algebraic system, $(N, +, \cdot)$ satisfying (i) $(N, +)$ is a group, (ii) $(N, \cdot)$ is a semi-group and (iii) $(x + y)z = xz + yz$ for all x, y, z in N. We abbreviate $(N, +, \cdot)$ by N.

If S and T are subsets of N, we denote the set $\{st\colon s \in S, t \in T\}$ by ST. A normal subgroup I of $(N, +)$ is called an ideal of $N(I \trianglelefteq N)$ if $IN \subset I$ and for all $n, n' \in N$ and all $i \in I$, $n(n' + i) - nn' \in I$. An ideal P of N is called a prime ideal if for any ideal I and J of N, $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$. An ideal I of N is called a semi-prime ideal if for any ideal J of N, $J^2 \subseteq I$ implies that $J \subseteq I$. An ideal minimal in the set of all prime ideals containing some given ideal I is called a minimal prime ideal of I in N.

If x is an element of N, then the principal ideal generated by x is denoted by (x). If S is a subset of N, we write $N - S = \{n \in N/n \notin S\}$. A subset M of a near-ring N is called an m-system if for any $a, b \in M$ there exist $a_1 \in (a)$ and

https://doi.org/10.1017/S1446788700025660 Published online by Cambridge University Press

$b_1 \in (b)$ such that $a_1 b_1 \in M$. A subset S of a near-ring N is called an Sp-system if for any $s \in S$, there exists $s_1 \in (s)$ and $s_1 \in (s)$ such that $s_1 s_2 \in S$. The prime radical $\mathcal{P}(I)$ of the ideal I consists of those elements $n \in N$ with the property that every m-system which contains n, contains an element of I.

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In this section we prove the main theorem. Before proving this, we state the following result.

LEMMA 2.1 (Van der Walt [3]). *If I is an ideal in the near-ring N then the prime radical $\mathcal{P}(I)$ of the ideal I is the intersection of all the minimal prime ideals of I in N.*

THEOREM 2.2. *An ideal I in a near-ring N is a semi-prime ideal in N if and only if $\mathcal{P}(I) = I$.*

PROOF. The "if" part is an immediate consequence of Theorem 3 of Van der Walt [3] and the fact that the intersection of prime ideals is a semi-prime ideal. To prove the "only if" part, suppose that I is a semi-prime ideal in N. Certainly, $I \subseteq \mathcal{P}(I)$, so let us assume $I \subset \mathcal{P}(I)$ and seek a contradiction. Suppose $a \in \mathcal{P}(I)$ with $a \notin I$. Hence $N - I$ is an Sp-system. So by 2.92 of Pilz [1], there exists an m-system M in N such that $a \in M \subseteq N - I$. Now $a \in \mathcal{P}(I)$ and, by the definition of $\mathcal{P}(I)$, every m-system which contains a meets I. But $I \cap (N - I)$ is empty, and therefore $M \cap I$ is empty. This gives the desired contradiction and completes the proof of the theorem.

As corollaries, we get the following results proved by Sambasiva Rao [2].

COROLLARY 2.3. *If I is any semi-prime ideal in the near-ring N, then I is the intersection of all minimal prime ideals of I in N.*

PROOF. Immediate from Lemma 2.1 and Theorem 2.2.

COROLLARY 2.4. *If I is a semi-prime ideal in a near-ring N, then I is the intersection of all prime ideals containing I.*

PROOF. Immediate from Theorem 2.2 and Van der Walt [3], Theorem 3.

References

[1] G. Pilz (1977), *Near-rings* (North-Holland, Amsterdam-New York).
[2] V. Sambasiva Rao, 'A characterization of semi-prime ideals in near-rings', *J. Austral. Math. Soc. Ser. A* **32** (1982), 212–214.
[3] A. P. J. Van der Walt, 'Prime ideals and nil radicals in near-rings', *Arch. Math.* (*Basel*) **15** (1964), 408–414.

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