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DEFORMATION METHODS AND THE STRONG UNBOUNDED REPRESENTATION TYPE OF p-GROUPS

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Introduction.

A basic problem in the representation theory of a finite group G is the determination of all indecomposable G-modules. Thus, for $G = C(n)$ $=$ a cyclic group of order n over an arbitrary field, the indecomposable representations, finite in number, are known from the theory of a single linear transformation. In 1954 Higman [9] showed that, in sharp contrast to the classical case of characteristic zero, an arbitrary finite group G has indecomposables of arbitrarily high dimension over any field of prime characteristic p iff the p-Sylow subgroup of G is non-cyclic (cf. unbounded representation type [3, p. 431]). Examples published by Heller and Reiner [8] in 1961 indicated that this phenomenon is even more extensive; reinterpreting a result of Dieudonné [4] as classifying the indecomposable modules for a square zero algebra on two generators, they showed that $G = C(p) \times C(p)$ (and therefore many other groups) has infinitely many non-isomorphic indecomposables in every even dimension over an infinite field of characteristic p (cf. strong unbounded representation type). At present, all $C(p) \times C(p)$ indecomposables are known only in the case $p = 2$, the result also being given (essentially) in [4] (cf. also [1], [2], [12]). In particular, the four-group $C(2) \times C(2)$ affords only two (dual) indecomposable representations in each *odd* dimension ≥ 3.

This paper contributes, by way of examples and a suggested technique, to a fuller description of this plethora of G-modules. Our study of the deformation of algebra representations [5], [6], [7], when brought to bear on the Heller-Reiner modules for a non-cyclic abelian p-group G, has led us to these observations:

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(1) If G is not the four-group, then it has infinitely many indecomposables in each *odd* dimension ≥ 3 also, provided of course that the field k is infinite (Section 2).

(2) For $p = 2$, the odd-dimensional Heller-Reiner indecomposables for the four-group G must decompose in every nontrivial generic deformation and are actually rigid as modules for the square zero algebra on two generators. This phenomenon may be viewed as the deformation-theoretic counterpart of the fact (see above) that there are no other indecomposables but these two. We partially re-obtain this finiteness result in the course of straightening-out some deformations (Section 4).

(3) A result recently announced by Janusz ([10], [11]) that almost all non-cyclic abelian p-groups have infinitely many *faithful* indecomposables in every sufficiently large dimension is obtained by a slight extension of our methods (Section 3).

Actually our results grew out of a deceptive intuition. We speculated that if an indecomposable module were not rigid (as are the principal indecomposables and the irreducibles), then a generic deformation of that module would provide a parametrized family of non-isomorphic indecomposables of the same dimension. Thus we began by rediscovering the Heller-Reiner modules referred to above and trying to deform them. The even-dimensional family is in fact obtained as a deformation of a particularly simple member of it. Secondly, we found a common deformation of the two odd-dimensional types (no longer factoring through an algebra with square zero radical) valid for $G = C(p^e) \times C(p^f)$ except $C(2) \times C(2)$. It is easy to prove the indecomposability of all these modules directly (see (1) above). Subsequently we extended our methods slightly to include Janusz' faithfulness result ((3) above).

These examples lent support to our potentially very useful deformation principle, but then we discovered a counterexample (cf. (5.2)) a generic deformation of an indecomposable need not be indecomposable. Nonetheless, we suspect that every indecomposable over an algebraically closed field arises by deforming one defined over the prime field. As an illustration of the usefulness of a deformation-theoretic approach, we mention that the rigidity and straightening-out theorems (cf. [6]) imply that certain modules *cannot* belong to a larger parametrized family!

Section 1 briefly describes the Heller-Reiner modules. Then we begin to deform. Section 2 gives the common deformation of the two odd-dimensional classes. Section 3 describes a simple way of extending these results to yield faithful representations of groups with exponent $>p$. Section 4 contains a rigidity result for $C(2) \times C(2)$-modules and Section 5 contains some examples of deformations. It is only in these last two sections that any technical deformation-theoretic language appears.

1. The Heller-Reiner modules.

(1.1) We list here, using a convenient notation, various indecomposable representations for a non-cyclic multiplicative abelian group $G = \prod_{i=1}^{i=m} C(p^{e_i})$ with basis $a_1, \cdots, a_m$, $m \geq 2$, $e_i \geq 1$, over a scalar field k of prime characteristic p. See [8].

The group algebra kG has radical of codimension one. We will define the action of the m generators $a_i - 1$ of the radical of kG so that all products $(a_i - 1)(a_j - 1)$ will act as 0. This is sufficient to define a "height two" representation of kG on a k-space V, that is,

$$V > (\text{rad}\, kG)V > (\text{rad}\, kG)^2 V = (0) .$$

In each case let the k-space $V = X \oplus Y$ with bases $\{x_1, \cdots, x_\beta\}$, $\{x_{\beta+1}, \cdots, x_N\}$ for X, Y. We always put $(a_i - 1)Y = (0)$ for all i, and $(a_i - 1)|_X \in \text{Hom}_k(X, Y)$. With one exception, (1.2), we do not specify the $a_i - 1$ for $i \geq 3$.

(1.2) *The first odd-dimensional class*: Let $\beta = n \geq 1$, $N = 2n + 1$ and, for $1 \leq r \leq n$

$$(a_1 - 1)(x_r) = x_{r+n} , \qquad (a_2 - 1)(x_r) = x_{r+n+1}$$

Note that if $m \geq 3$, then the choices

$$(a_3 - 1)(x_1) = t x_{n+1}, t \in k , \quad (a_3 - 1)(x_r) = 0 , \quad 2 \leq r \leq n ,$$

yield inequivalent representations, since for $\alpha \in k$, $\dim_k \ker [\alpha(a_1 - 1) - (a_3 - 1)] = n + 2$ iff $\alpha = t$. This provides "many" indecomposables in the special case G is an elementary abelian p-group $C(p) \times C(p) \times \cdots \times C(p)$ of dimension $m > 2$. (Contrast the case $p = 2$, $m = 2$.) We use this in (3.2).

(1.3) *The second odd-dimensional class*: These are the duals to (1.2). Let $\beta = n + 1 \geq 2$, $N = 2n + 1$, and

$$(a_1 - 1)(x_1) = 0 , \quad (a_1 - 1)(x_r) = x_{r+n} , \quad 2 \leq r \leq n + 1 ,$$

$$(a_2 - 1)(x_r) = x_{r+n+1}\,, \quad 1 \le r \le n\,, \quad (a_2 - 1)(x_{n+1}) = 0\,.$$

Remark. In [8] it is pointed out that, for $G = C(p^e) \times C(p^f)$ (thus $m = 2$), every height two odd-dimensional kG-indecomposable is either (1.2) or (1.3).

(1.4) *An even-dimensional class*: Let $\beta = n \ge 1$, $N = 2n$, and

$$\begin{aligned} (a_1 - 1)(x_r) &= x_{r+n}\,, && 1 \le r \le n \\ (a_2 - 1)(x_r) &= x_{r+n+1} + tx_{r+n}\,, && 1 \le r \le n-1 \\ (a_2 - 1)(x_n) &= tx_{2n}\,, && t \in k \end{aligned}$$

These representations are inequivalent for distinct $t \in k$ since for $\alpha \in k$, $\dim_k \ker [\alpha(a_1 - 1) - (a_2 - 1)] = n + 1$ iff $\alpha = t$. Consequently an infinite field k affords infinitely many distinct even-dimensional indecomposables for G.

In [8] it is pointed out that, for groups $G = C(p^e) \times C(p^f)$ (thus $m = 2$), every even-dimensional height two kG-indecomposable is of the type (1.4) iff k is algebraically closed.

(1.5) *Faithfulness of these representations when all* $e_i = 1$. Note that each group element a_i^r acts as $r(a_i - 1) + 1$. Thus, if $G = C(p) \times C(p)$ is elementary abelian with $m = 2$, then the above representations are faithful for G.

In general, since

$$\dim_{Z/pZ} \operatorname{Hom}_k (X, Y) = \dim_k X \cdot \dim_k Y \cdot [k : Z/pZ]\,,$$

proper choices of the radical generators $a_i - 1 \in \operatorname{Hom}_k (X, Y)$ with $3 \le i \le m$ will yield faithful representations of an elementary abelian G provided the field k is large enough.

See Section 3 for the case where some $e_i > 1$.

2. The common deformation of the two odd-dimensional classes.

(2.1) Now we show that an infinitude of odd-dimensional indecomposable G-modules, which occured at height two in the even-dimensional case (1.4), occurs at height three. Our method is to write down a one-parameter family V_t of deformations of (1.2) and observe that the generic member is indecomposable. The modules (1.2) and (1.3) are the cases $t = 0$ and $t = 1$ of this family.

We require $G = \prod_{i=1}^{i=m} C(p^{e_i})$ with $p^{e_1} \ne 2$, $m \ge 2$, and char $k = p$.

(2.2) *The G-modules* V_t. Let $V = X \oplus Y$ with k-bases $\{x_1, \cdots, x_n\}$ and $\{x_{n+1}, \cdots, x_{2n+1}\}$ for X and Y as in (1.2). We have $a_1^2 \neq 1$ in G. Fix $t \in k$ and define the action of $a_1 - 1$ by

$$(a_1 - 1)(x_1) = (1 - t)x_{n+1}\ , \quad (a_1 - 1)(x_r) = x_{r+n}\ , \quad 2 \leq r \leq n$$
$$(a_1 - 1)(x_{n+1}) = tx_{2n+1}\ , \qquad (a_1 - 1)(x_r) = 0\ , \quad n + 2 \leq r \leq 2n + 1\ .$$

The action of $a_2 - 1$ is defined by

$$(a_2 - 1)(x_r) = x_{r+n+1}\ , \quad 1 \leq r \leq n\ , \quad (a_2 - 1)(Y) = (0)\ .$$

Moreover, for $3 \leq i \leq m$ we require that $(a_i - 1)|_X \in \mathrm{Hom}_k(X, Y)$ and also that $(a_i - 1)(Y) = (0)$.

Now one notes that, except for $(a_1 - 1)^2(x_1) = t(1 - t)x_{2n+1}$, defining all products $(a_i - 1)(a_j - 1)$ and so on to act as zero yields a representation of kG on the space $X \oplus Y$. We denote this module V_t. It is immediate that $t = 0$ and $t = 1$ reduce to (1.2) and (1.3) respectively.

(2.3) V_t *is indecomposable.* We sketch the argument. It is standard, and applies in Section 1 also; see [3, p. 433].

Note first that a nonzero kG-direct summand cannot be contained in Y. Now, if X_0 is any k-space complement to Y in V_t, and if we have a nontrivial k-space decomposition $X_0 = X_1 \oplus X_2$, then one checks that $\dim_k (\mathrm{rad}\, kG)X_j > \dim_k X_j$ for $j = 1, 2$. (This follows from the fact that $a_1 - 1$, $a_2 - 1$ map $X_0 \to Y$ injectively.) Thus, $\dim_k (X_1 \oplus X_2) = n$ and the hypothesis $(\mathrm{rad}\, kG)X_1 \cap (\mathrm{rad}\, kG)X_2 = (0)$ would imply $n + 1 = \dim_k Y \geq n + 2$, a contradiction. Thus X_1 and X_2 cannot generate complementary submodules of V_t. The assertion follows.

(2.4) *Let* $t \neq 0, 1$ *in* k. *Then* V_t *and* V_s *are kG-isomorphic iff* $s = t$ *or* $1 - t$. To prove this, suppose first $s = 1 - t$ and $t \neq 0, 1$. Define a k-linear map $\phi\colon V_t \to V_{1-t}$ on the basis $x_1, \cdots, x_{n+1}, \cdots, x_{2n+1}$ by

$$\phi(x_i) = x_i,\ i \neq n + 1\ , \qquad \phi(x_{n+1}) = t(1 - t)^{-1}x_{n+1}$$

One readily checks that ϕ yields the desired kG-isomorphism.

Conversely, in V_t we have $(\mathrm{rad}\, kG)^2 V_t = k\text{-span}\,\{x_{2n+1}\}$. Moreover, since $a_2 - 1$ gives a k-space isomorphism $X \to k\text{-span}\,\{x_{n+2}, \cdots, x_{2n+1}\}$, we may consider the inverse map here (denoted $(a_2 - 1)^{-1}$) and obtain the equation

$$(a_1 - 1)[(a_1 - 1)(a_2 - 1)^{-1}]^n(x_{2n+1}) = t(1 - t)x_{2n+1}$$

Thus a certain intrinsically defined map on the canonical subspace $(\text{rad } kG)^2 V_t$ is multiplication by $t(1-t)$. It follows that V_t isomorphic to V_s forces $t(1-t) = s(1-s)$, whence $s = t$ or $1-t$. This completes the proof of (2.4).

We note this consequence. See (3.2) also.

(2.5) *If the scalar field k is infinite and $p^{e_1} \neq 2$,* (see (2.1)), *then there are infinitely many odd-dimensional indecomposable G-modules of height three.*

The remarks (1.5) on faithfulness in the situation G is elementary abelian, p odd, apply here as well.

(2.6) *The dual module to V_t.* Now we define a left action of kG on $\text{Hom}_k(V_t, k)$ by $(a * f)(x) = f(ax)$ for $a \in kG$, $f \in \text{Hom}_k(V_t, k)$, and $x \in V_t$. Note that $(ab) * f = a * (b * f)$ because kG is commutative. We denote the module thus obtained V_t^*.

(Note also that a different left action may be obtained by using the natural involution $a \mapsto a^{-1}$ of G, rather than commutativity.)

Now let $\{f_i\}$ be the k-basis for V_t^* dual to $\{x_i\}$, so that $f_i(x_j) = \delta_{ij}$. Then the assignment

$$f_1 \mapsto x_{2n+1}, \cdots, f_{n+1} \mapsto x_{n+1}, \cdots, f_{2n+1} \mapsto x_1$$

extends to a kG-isomorphism $V_t^* \rightleftarrows V_{1-t}$.

We conclude, using (2.4), that V_t is isomorphic with its dual iff $t \neq 0, 1$.

3. Abelian p-groups of higher exponent.

In this section we construct families of indecomposables for abelian p-groups of exponent greater than p.

Let G, k be as in (1.1). Let U denote the p-group of unipotent matrices over k of the form $I + N$, with N strictly lower triangular. Any representation $G \to \text{Aut}_k V$ may, by proper choice of basis, be viewed as a homomorphism $\rho: G \to U$. The group U contains an element of order p^d iff $\dim_k V \geq p^{d-1} + 1$. Furthermore V can be made indecomposable with respect to such an element iff $\dim_k V \leq p^d$. For the time being, assume exponent $G = p^d$ and $p^{d-1} + 1 \leq \dim_k V \leq p^d$, and that, by a further modification of basis if necessary, the representation $\rho: G \to U$ satisfies

$$\rho(a_1) = \begin{bmatrix} 1 & & & 0 \\ 1 & \cdot & & \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{bmatrix} \qquad \text{(in Jordan form)} .$$

The centralizer Γ in U of $\rho(a_1)$ is the subgroup of U of matrices (α_{ij}) with constant subdiagonals: $\alpha_{ij} = \alpha_{i+1,j+1}$ for $i > j$. ($\Gamma = I +$ (commutative algebra generated over k by $\rho(a_1) - I$).) Thus $\rho(G) \subset \Gamma$. By letting the first non-zero subdiagonal be the r-th for appropriate r, one gets a matrix of any order $p^\nu, \nu \leq d$. Further, if a_2 is a second generator of G of order p^ν, choosing distinct constant entries $t \in k$ for the r-th subdiagonal yields inequivalent modules V_t (here $\rho(\alpha(a_1 - 1)^r - (a_2 - 1))$ annihilates $V_t/(a_1 - 1)^{r+1}V_t$ iff $\alpha = t$). Since the appropriate location of the constant subdiagonals is sufficient to determine a representation, we see also that ρ can be made faithful if k is sufficiently large and exponent $G = p^d$.

We combine the representations of Sections 1 and 2 with the above described minimal faithful ones V_t.

(3.1) PROPOSITION. *Let* $\{v_1, \cdots, v_m\}$, $m \geq 3$, *be a basis of* V_t *giving the desired lower triangular form. Let* M *be an indecomposable* G*-module with cyclic submodule* M_0 *isomorphic via* φ *to some submodule* k-span $\{v_\lambda, \cdots, v_m\}$, $\lambda \geq 2$, *of* V_t. *Assume* $(a_1 - 1)^{m-\lambda+1}(M) = (0)$. *Then*

$$W = V_t \oplus M/k\text{-span}\,\{v_\lambda - \varphi^{-1}(v_\lambda), \cdots, v_m - \varphi^{-1}(v_m)\}$$

is indecomposable.

Proof. We denote the images in W of the v_i by $\bar{v}_i$. If $\bar{v}_1 + w \in Z$, a submodule of W, then $(a_1 - 1)^{m-1}(\bar{v}_1 + w) = \bar{v}_n \in Z$. Let $W = Z_1 \oplus Z_2$ and suppose some $\bar{v}_1 + w \in Z_1$. Then as vector space we have a non-trivial decomposition

$$Z_1 = k\text{-span}\,\{(a_1 - 1)^j(\bar{v}_1 + w) : j = 0, \cdots, \lambda - 2\} \oplus (Z_1 \cap \bar{M}) ,$$

where $\bar{M}$ denotes the image of M in W. Since $\bar{v}_j + w' \in Z_2$, $j < \lambda$, implies $\bar{v}_m \in Z_2$, we have $Z_2 \subset \bar{M}$. Thus $\bar{M}$ is decomposed. Since $\bar{M}$ is isomorphic to $M, Z_2 = (0)$.

In this situation, given non-isomorphic V_t or non-isomorphic M's, the resulting modules W are non-isomorphic. In fact, then $\bar{V}_t$ is the only m-dimensional cyclic submodule of W, while $\bar{M} = \ker (a_1 - 1)^{m-\lambda+1}$.

We may apply (3.1) using for M the second odd-dimensional class ($M_0 = k$-span $\{x_{n+1}, x_{2n+1}\}$ with $(a_i - 1)(x_{n+1}) \in k$-span $\{x_{2n+1}\}$, $i > 2$) provided $(a_2 - 1)v_{m-1} = 0$ in V_t. This can be arranged if either a_2 has order $< p^d$ or $n > p^{d-1} + 1$. We may also take M in the even-dimensional class, with $M_0 = k$-span $\{x_n, x_{2n}\}$ and $(a_i - 1)(x_n) \in k$-span $\{x_{2n}\}$, $i > 2$. Note that the choice of $t \in k$ for the action of $a_2 - 1$ is already determined by V_t.

One may also take for M one of the modules of Section 2, with $M_0 = k$-span $\{x_{n+1}, x_{2n+1}\}$ and appropriate choices for action of the $a_i - 1$, $i > 2$, provided again that $(a_2 - 1)v_{m-1} = 0$, and $m > 3$. In this case, however, the hypothesis that $(a_1 - 1)^{m-\lambda+1}(M) = 0$ fails, and a slightly more complicated argument specific to M is needed to prove W indecomposable. For other M, and in the absence of that hypothesis, W may decompose.

A partial summary of the material of Sections 1, 2, and 3 is the following:

(3.2) THEOREM. *With the exception of* $C(2) \times C(2)$, *a non-cyclic abelian p-group has infinitely many indecomposables in every dimension* ≥ 2 *over an infinite field of characteristic p. If* p^d *is the exponent of the group, these representations may be taken faithful in dimensions* $\geq p^{d-1} + 1$.

4. The anomalous case: The four-group in characteristic 2.

In this section we show that the procedure of Section 2 for obtaining an infinite family of odd-dimensional indecomposables fails for $G = C(2) \times C(2)$. In fact we show essentially that in this case neither of the two odd-dimensional Heller-Reiner modules is a member of *any* non-trivial larger parametrized family of modules, indecomposable or decomposable, unless the group algebra kG is a direct summand of the generic members of that family. Our argument involves straightening out a deformation, and the method is reminiscent of Dieudonné's in [4]. Consequently a portion of his characterization of the odd-dimensional indecomposables in this situation emerges from our proof.

A complete statement of results is as follows:

(4.1) THEOREM. *Let G be the four-group, and V be an indecomposable G-module of odd dimension* ≥ 3 *over a field k of characteristic* 2. *Then*

(i) *every non-trivial generic deformation* V_t *of V decomposes directly* *over KG; in fact, $V_t \simeq KG \oplus W$, for some KG-submodule W; and*

(ii) *V is a Heller-Reiner module of the first* (1.2) *or second* (1.3) *class.*

Here K is the power series field $k((t))$. For deformations, we employ only some definitions and two lemmas from [6]. Statement (ii) is of course known. We use some weaker consequences of it, Lemmas (4.2) and (4.3) below, in the proof of (i). This proof shows (ii) to be a consequence of (4.2) and (4.3).

We have $G = \{1, x, y, xy\}$, and write $\alpha = 1 - x$, $\beta = 1 - y$ in the group algebra kG. Thus $\operatorname{rad} kG$ has basis $\alpha, \beta, \alpha\beta$. Suppose $\varphi: kG \to \operatorname{End}_k V$ gives the odd-dimensional indecomposable representation. Then it is well-known that $\varphi(\alpha\beta) = 0$, for otherwise V would contain a submodule isomorphic to the free (and hence injective) left regular module kG, which would therefore split off from V.

Likewise, if $\varphi_t = \varphi + t\Phi_1 + t^2\Phi_2 + \cdots$ affords a deformation V_t such that $\varphi_t(\alpha\beta) \neq 0$, then V_t has a KG-direct summand isomorphic with KG. (Here the underlying space for V_t is V_K, obtained by extending scalars to the power series field K. This procedure is standard; see [6]).

It remains to show, therefore, *that every deformation φ_t with the property $\varphi_t(\alpha\beta) = 0$ is a trivial deformation.* This is true in any characteristic. See (4.4) below.

Since $\alpha\beta$ will act as zero from now on, we may take φ and φ_t as representations of the 3-dimensional quotient algebra $k[a, b] = kG/(\alpha\beta)$, where $a = \alpha + (\alpha\beta)$ and $b = \beta + (\alpha\beta)$. Note $N = \operatorname{rad} k[a, b]$ is square zero on a and b.

Denoting $\varphi(N)V$ by Y, we may write $V = X \oplus Y$, where X is a k-space complement to Y. Since $\varphi(a)Y = \varphi(b)Y = (0)$, we may as well consider the operators $\varphi(a)$ and $\varphi(b)$ on V as mappings $X \to Y$.

(4.2) LEMMA. *If V is indecomposable as a $k[a, b]$-module and if $\dim_k X \leq \dim_k Y$, then both mappings $\varphi(a), \varphi(b): X \to Y$ are injective.*

V must in fact have one of the structures given in (1.2) and (1.4). From (4.1) (ii) we also have

(4.3) LEMMA. *Let V be a $(2n + 1)$-dimensional indecomposable module for $k[a, b]$. Then $\dim_k \varphi(N)V$ equals n or $n + 1$.*

Now all of (4.1) follows from (4.4) using only (4.2) and (4.3).

(4.4) LEMMA. *Let k be any field, and V a $(2n+1)$-dimensional indecomposable module for the algebra $k[a, b]$ afforded by the representation φ. Then*

(i) *V is rigid as a $k[a, b]$-module;*

(ii) *if $\dim_k \varphi(N)V = n+1$, then V has a k-basis $x_1, \cdots, x_n, y_1, \cdots, y_{n+1}$ such that, for $1 \leq i \leq n$,*

$$\varphi(a)x_i = y_i \,, \qquad \varphi(b)x_i = y_{i+1} ;$$

(iii) *otherwise, V is the n-dimensional k-dual of the module in* (ii).

It is straightforward that (iii) follows from (ii), (4.3) and Lemma 4 of [6]. To prove (i) and (ii), suppose we are given a deformation V_t of V afforded by a representation $\varphi_t = \varphi + t\Phi_1 + \cdots$ of $K[a, b]$. We assert first that if $Y = \varphi(N)V$ has dimension $n+1$ over k, then $Z = \varphi_t(N_K)V_K$ has dimension $n+1$ over the power series field K and, moreover, $Z = \ker \varphi_t(a) = \ker \varphi_t(b)$. Here we are considering $\varphi_t(a)$ and $\varphi_t(b)$ as operators on V_t. To see this, note first that Z is contained in both kernels because $ab = ba = 0$. Moreover, each kernel has K-dimension $n+1$, because any square zero operator on V_K has a kernel of dimension $\geq n+1$, while Lemma 8 of [6] says that $\dim_K \ker \varphi_t(a)$ is less than or equal to $\dim_k \ker \varphi(a)$. Thus $\dim_K Z \leq n+1$. On the other hand, Z has K-dimension at least as large as $\dim_k \varphi(N)X = n+1$, again by Lemma 8 of [6].

It follows that the k-space decomposition $V = X \oplus Y$ has a counterpart $V_t = X_K \oplus Z$ over K. We next assert that X_K has a K-space basis $\xi_1, \cdots, \xi_n$ where $\xi_i = \xi_i(t) = x_{i0} + tx_{i1} + \cdots$ (cf. order zero) with the properties that $x_{10}, \cdots, x_{n0}$ is a k-basis for X and also $\varphi_t(b)\xi_i = \varphi_t(a)\xi_{i+1}$ for $i = 1, \cdots, n-1$ when $n \geq 2$. We go by induction on n, noting that the case $n = 1$ is clear.

To begin the induction process, define

$$Z_1 = (\varphi_t(a)X_K) \cap (\varphi_t(b)X_K) \,, \qquad \Xi = \varphi_t(a)|_{X_K}^{-1} Z_1 \,.$$

Note that Z_1 is of codimension 2 in Z, whence Ξ is of codimension 1 in X_K. Now let W be the submodule of V_t generated over $K[a, b]$ by the subspace Ξ,

$$W = \Xi + \varphi_t(a)\Xi + \varphi_t(b)\Xi$$

and, having this, let W_0 be the $k[a, b]$-submodule of V generated by the "constant terms" in W; see Lemma 5 in [6]. It follows that

$$W_0 = \Xi_0 + \varphi(a)\Xi_0 + \varphi(b)\Xi_0$$

where Ξ_0 is the k-space of constant terms in Ξ. Note $\dim_k W_0 \geq 2(n-1)$, since $\dim_k \Xi_0 = n-1$.

We assert that in fact $\dim_k W_0 = 2n-1$ and that W_0 is an indecomposable $k[a,b]$-module. First the assumption $\dim_k W_0 = 2(n-1)$ would imply $\varphi(a)\Xi_0 = \varphi(b)\Xi_0$. Then any $x \in X$, $x \notin \Xi_0$, would generate a 3-dimensional $k[a,b]$-module complement to W_0 in V, a contradiction. Now given a module decomposition $W_0 = T \oplus T'$, assume T' even-dimensional. From previous remarks $\ker \varphi(a)|_{W_0} \cap \ker \varphi(b)|_{W_0}$ is $\varphi(N)W_0$. Thus we have $T' = X' \oplus Y'$ where $X' \subset X$ and $Y' = \varphi(N)T'$. A similar decomposition of T implies $\dim \varphi(N)T > \frac{1}{2}\dim T$, whence $\dim X' = \dim Y'$, and $\varphi(a)|_{X'}$ and $\varphi(b)|_{X'}$ are isomorphisms. Let $X'' \oplus X' = X$. Then $[\varphi(N)X''] \cap Y' = (0)$ by dimensionality. Thus T' has a complement $X'' + \varphi(N)X''$ in V and so is (0).

Now by induction, there exists a K-basis $\xi_2, \cdots, \xi_n$ for Ξ such that each ξ_i has the form $x_{i0} + tx_{i1} + \cdots$ and also $\varphi_t(b)\xi_i = \varphi_t(a)\xi_{i+1}$, $i = 2, \cdots, n-1$. Recalling the definition of Ξ, we note that $\varphi_t(a)\xi_2, \cdots, \varphi_t(a)\xi_n$ is a basis for Z_1. Since $Z_1 \subset \varphi_t(b)X_k$, we may complete our basis of X_k by choosing a series ξ_1 of order 0 in X_K such that $\varphi_t(b)\xi_1 = \varphi_t(a)\xi_2$. This gives us the basis of ξ_i's as asserted.

Now we complete the proof of (i) and (ii) in the statement of (4.4) by defining $x_i = x_{i0}$ and $y_i = \varphi(a)x_i$ for $1 \leq i \leq n$, and also $y_{n+1} = \varphi(b)x_n$. To show that the deformation V_t is trivial we define the mapping $I_t\colon V_K \to V_K$ (a deformation of the identity mapping of the underlying K-space; see [6, Lemma 3]) by

$$I_t(x_i) = \xi_i\ , \quad I_t(y_i) = \varphi_t(a)\xi_i\ , \quad I_t(y_{n+1}) = \varphi_t(b)\xi_n$$

for $1 \leq i \leq n$. One checks readily that I_t affords an equivalence of the representations φ and φ_t, whence the deformation φ_t of φ is trivial. This completes the proof of Lemma (4.4) and thus of Theorem (4.1).

5. Some examples of deformations in the even-dimensional case.

In Section 4 we did not determine whether for $G = C(2) \times C(2)$, the KG-modules of types (1.2) and (1.3) can deform so as to acquire a summand isomorphic to KG. In fact we intend to give a fuller treatment of this problem later in which we show this to be impossible. There the Hochschild cohomology is a useful computational device.

For the present we content ourselves with two examples to illustrate the deformation structure of the even-dimensional Heller-Reiner modules for $G = C(2) \times C(2)$ in characteristic two.

(5.1) *The* 4-*dimensional module of type* (1.4) *with* $t = 0$ *deforms into the left regular module* KG.

We let u represent the variable in the power series field K. Then defining

$$\varphi_u(a_2 - 1)(x_1) = x_4 + ux_2 \,, \qquad \varphi_u(a_2 - 1)(x_3) = ux_4 \,,$$

with other action as before, gives the result. For now x_1 acts as cyclic generator.

Actually here again one may show that for $t \neq 0$, there is no deformation into KG, nor is there even a deformation with a summand isomorphic to KG in the higher dimensional cases of (1.4) with $t \neq 0$. One might say that the *kernel* of these KG-representations is rigid.

(5.2) *All the modules* (1.4) *for* $G = C(2) \times C(2)$ *admit decomposable generic deformations of height two.*

We factor through the algebra $k[a, b]$ of Section 4, letting a act as $a_1 - 1$ and b as $a_2 - 1$. Hence the following is valid in any characteristic. With the given action of a, and an arbitrary map $\varphi(b): X \to Y$, the module $X \oplus Y$ is indecomposable if and only if with respect to the given bases of X and Y the map $\varphi(b): X \to Y$ has an indecomposable matrix. It suffices then to give a deformation preserving the action of a and such that $\varphi_u(b): X \to Y$ has a matrix with distinct eigenvalues in K. For example, we may define

$$\varphi_u(b)(x_r) = \varphi(b)(x_r) + u^r x_{r+n} \,, \qquad 1 \leq r \leq n \,.$$

Note that if $\varphi_u(b)$ has distinct eigenvalues only in the algebraic closure of K then the generic deformation might actually be indecomposable over K in spite of the fact that all specializations of u into k might yield decomposable modules. One may often avoid this difficulty by replacing u by a suitable power u^N before making the deformation.

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