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FROBENIUS ACTIONS ON LOCAL COHOMOLOGY MODULES AND DEFORMATION

LINQUAN MA AND PHAM HUNG QUY

Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$. We introduce and study F-full and F-anti-nilpotent singularities, both are defined in terms of the Frobenius actions on the local cohomology modules of R supported at the maximal ideal. We prove that if $R/(x)$ is F-full or F-anti-nilpotent for a nonzero divisor $x \in R$, then so is R. We use these results to obtain new cases on the deformation of F-injectivity.

§1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic $p > 0$. We have the Frobenius endomorphism $F : R \to R$, $x \mapsto x^p$. The *F-singularities* are certain singularities defined via this Frobenius map. They appear in the theory of *tight closure* (cf. [13] for its introduction), which was systematically introduced by Hochster and Huneke [9] and developed by many researchers, including Hara, Schwede, Smith, Takagi, Watanabe, Yoshida and others. A recent active research of F-singularities is centered around the correspondence with the singularities of the minimal model program. We recommend [25] as an excellent survey for recent developments.

In this paper we study the deformation of F-singularities. That is, we consider the problem: if $R/(x)$ has certain property **P** for a regular element $x \in R$, then does R has the property **P**? The classical objects of F-singularities are F-regularity, F-rationality, F-purity and F-injectivity (cf. [13, 25]). It is well known that F-rationality always deforms while F-regularity and F-purity do not deform in general [22, 23]. Whether

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F-injectivity deforms is a long- standing open problem [6] (for recent developments, we refer to [11, 18]). Recall that the Frobenius endomorphism induces a natural Frobenius action on every local cohomology module, F: $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$. The ring R is called *F-injective* if this Frobenius action F is injective for every $i \geqslant 0$. The class of F-injective singularities contains other classes of F-singularities. For an ideal-theoretic characterization of F-injectivity, see [20, Main Theorem D]. We consider this paper as a step toward a solution of the deformation of F-injectivity.

We introduce two conditions: F-full and F-anti-nilpotent singularities, in terms of the Frobenius actions on local cohomology modules of R (we refer to Section 2 for detailed definitions). The first condition is motivated by recent results on Du Bois singularities [18]. The second condition has been studied in [5, 16], and is known to be equivalent to *stably FH-finite*, which means all local cohomology modules of R and $R[[x_1, \ldots, x_n]]$ supported at the maximal ideals have only finitely many Frobenius stable submodules. We prove that F-fullness and F-anti-nilpotency both deform, and we obtain more evidence on deformation of F-injectivity. Our results largely generalize earlier results of [11] in this direction. We list some of our main results here:

THEOREM 1.1. (Theorem 4.2, Corollary 5.16) *$(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ and x a regular element of R. Then we have:*

(1) *if $R/(x)$ is F-anti-nilpotent, then so is R;*
(2) *if $R/(x)$ is F-full, then so is R;*
(3) *if $R/(x)$ is F-full and F-injective, then so is R.*

THEOREM 1.2. (Theorem 5.11) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$. Suppose the residue field $k = R/\mathfrak{m}$ is perfect. Let x be a regular element of R such that* $\operatorname{Coker}(H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R))$ *has finite length for every i. If $R/(x)$ is F-injective, then the map $x^{p-1}F$: $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for every i, in particular R is F-injective.*

§2. Definitions and basic properties

2.1 Modules with Frobenius structure

Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. A Frobenius action on an R-module M, F: $M \to M$, is an additive map such that for all $u \in M$ and $r \in R$, $F(ru) = r^p u$. Such an action induces a natural R-linear map

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$\mathscr{F}_R(M) \to M$,[1] where $\mathscr{F}_R(-)$ denotes the Peskine–Szpiro's Frobenius functor. We say N is an F-*stable* submodule of M if $F(N) \subseteq N$. We say the Frobenius action on M is *nilpotent* if $F^e(M) = 0$ for some e.

We note that having a Frobenius action on M is the same as saying that M is a left module over the ring $R\{F\}$, which may be viewed as a noncommutative ring generated over R by the symbols $1, F, F^2, \ldots$ by requiring that $Fr = r^p F$ for $r \in R$. Moreover, N is an F-stable submodule of M equivalent to requiring that N is an $R\{F\}$-submodule of M. We will not use this viewpoint in this article though.

Let M be an (typically Artinian) R-module with a Frobenius action F. We say the Frobenius action on M is *full* (or simply M is full), if the map $\mathscr{F}_R^e(M) \to M$ is surjective for some (equivalently, every) $e \geqslant 1$. This is the same as saying that the R-span of all the elements of the form $F^e(u)$ is the whole M for some (equivalently, every) $e \geqslant 1$. We say the Frobenius action on M is *anti-nilpotent* (or simply M is anti-nilpotent), if for any F-stable submodule $N \subseteq M$, the induced Frobenius action F on M/N is injective (note that this in particular implies that F acts injectively on M).

LEMMA 2.1. *The Frobenius action on M is anti-nilpotent if and only if every F-stable submodule $N \subseteq M$ is full. In particular, if M anti-nilpotent, then M is full.*

Proof. Suppose M is anti-nilpotent. Let $N \subseteq M$ be an F-stable submodule. Consider the R-span of $F(N)$, call it N'. Clearly, $N' \subseteq N$ is another F-stable submodule of M and $F(N) \subseteq N'$. But since M is anti-nilpotent, F acts injectively on M/N'. Thus we have $N = N'$ and hence N is full.

Conversely, suppose every F-stable submodule of M is full. Suppose there exists an F-stable submodule $N \subseteq M$ such that the Frobenius action on M/N is not injective. Pick $y \notin N$ such that $F(y) \in N$. Let $N'' = N + Ry$. It is clear that N'' is an F-stable submodule of M and the R-span of $F(N'')$ is contained in $N \subsetneq N''$. This shows N'' is not full, a contradiction. ∎

We also mention that whenever M is endowed with a Frobenius action F, then $\widetilde{F} = rF$ defines another Frobenius action on M for every $r \in R$. It is easy to check that if the action $\widetilde{F}$ is full or anti-nilpotent, then so is F.

[1]It is not hard to see that an R-linear map $\mathscr{F}_R(M) \to M$ also determines a Frobenius action on M.

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2.2 F-singularities

We collect some definitions about singularities in positive characteristic. Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ with the Frobenius endomorphism $F : R \to R; x \mapsto x^p$. R is called *F-finite* if R is a finitely generated as an R-module via the homomorphism F. R is called *F-pure* if the Frobenius endomorphism is pure.[2] It is worth to note that if R is either F-finite or complete, then R being F-pure is equivalent to the condition that the Frobenius endomorphism $F : R \to R$ is split [12]. Let $I = (x_1, \ldots, x_t)$ be an ideal of R. Then we denote by $H^i_I(R)$ the ith local cohomology module with support at I (we refer to [3] for the general theory of local cohomology modules). Recall that local cohomology may be computed as the cohomology of the Čech complex

$$0 \to R \to \bigoplus_{i=1}^{t} R_{x_i} \to \cdots \to \bigoplus_{i=1}^{t} R_{x_1 \cdots \widehat{x}_i \cdots x_t} \to R_{x_1 \cdots x_t} \to 0.$$

The Frobenius endomorphism $F : R \to R$ induces a natural Frobenius action $F : H^i_I(R) \to H^i_{I^{[p]}}(R) \cong H^i_I(R)$. A local ring $(R, \mathfrak{m})$ is called *F-injective* if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i \geqslant 0$. This is the case if R is F-pure [12, Lemma 2.2]. One can also characterize F-injectivity using certain ideal closure operations (see [17, 20] for more details).

EXAMPLE 2.2. Let $I = (x_1, \ldots, x_t) \subseteq R$ be an ideal generated by t elements. By the above discussion we have

$$H^t_I(R) \cong R_{x_1 \cdots x_t} \Big/ \mathrm{Im}\left(\bigoplus_{i=1}^{t} R_{x_1 \cdots \widehat{x}_i \cdots x_t} \to R_{x_1 \cdots x_t}\right)$$

and the natural Frobenius action on $H^t_I(R)$ sends $1/(x_1 \cdots x_t)$ to $1/(x_1^p \cdots x_t^p)$. Therefore, it is easy to see the Frobenius action on $H^t_I(R)$ is full (in fact, $\mathscr{F}_R(H^t_I(R)) \to H^t_I(R)$ is an isomorphism). On the other hand, one cannot expect $H^t_I(R)$ is always anti-nilpotent even when R is regular. For example, let $R = k[[x, y]]$ be a formal power series ring in two variables and $I = (x)$. We have

$$H^1_{(x)}(R) \cong k[[y]]x^{-1} \oplus \cdots \oplus k[[y]]x^{-n} \oplus \cdots .$$

[2] A map of R-modules $N \to N'$ is *pure* if for every R-module M the map $N \otimes_R M \to N' \otimes_R M$ is injective for every R-module M.

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Let N be the submodule of $H^1_{(x)}(R)$ generated by $\{y^2x^{-n}\}_{n=1}^{\infty}$, then it is easy to see N is an F-stable submodule of $H^1_{(x)}(R)$. However, $F(yx^{-1}) = y^px^{-p} \in N$ while $yx^{-1} \notin N$. So the Frobenius action on $H^1_{(x)}(R)/N$ is not injective and hence $H^1_{(x)}(R)$ is not anti-nilpotent.

We are mostly interested in the Frobenius actions on local cohomology modules of R supported at the maximal ideal. We introduce two notions of F-singularities.

DEFINITION 2.3.

(1) We say that $(R, \mathfrak{m})$ is F-full, if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is full for every $i \geqslant 0$. This means $\mathscr{F}_R(H^i_{\mathfrak{m}}(R)) \to H^i_{\mathfrak{m}}(R)$ is surjective for every $i \geqslant 0$.
(2) We say that $(R, \mathfrak{m})$ is F-anti-nilpotent, if the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is anti-nilpotent for every $i \geqslant 0$.

The concept of F-anti-nilpotency is not new, it was introduced and studied in [5] and [16] under the name *stably FH-finite*: that is, all local cohomology modules of R and $R[[x_1, \dots, x_n]]$ supported at their maximal ideals have only finitely many F-stable submodules. It is a nontrivial result [5, Theorem 4.15] that this is equivalent to R being F-anti-nilpotent.

REMARK 2.4.

(1) It is clear that F-anti-nilpotent implies F-injective and F-full (see Lemma 2.1). Moreover, F-pure local rings are F-anti-nilpotent [16, Theorem 1.1]. In particular, F-pure local rings are F-full.
(2) We can construct many F-anti-nilpotent (equivalently, stably FH-finite) rings that are not F-pure [20, Sections 5 and 6].
(3) Cohen–Macaulay rings are automatically F-full, since $\mathscr{F}_R(H^d_{\mathfrak{m}}(R)) \to H^d_{\mathfrak{m}}(R)$ is an isomorphism. But even F-injective Cohen–Macaulay rings are not necessarily F-anti-nilpotent [5, Example 2.16].

We give some simple examples of rings that are not F-full, we see a family of such rings in Example 3.6.

EXAMPLE 2.5.

(1) Let $R = k[s^4, s^3t, st^3, t^4]$ where k is a field of characteristic $p > 0$. Then R is a graded ring with s^4, t^4 a homogeneous system of parameters.

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A simple computation shows that the class

$$\left[\frac{(s^3t)^2}{s^4}, -\frac{(st^3)^2}{t^4}\right] \in R_{s^4} \oplus R_{t^4}$$

spans the local cohomology module $H^1_{\mathfrak{m}}(R)$. In particular, $[H^1_{\mathfrak{m}}(R)]$ sits only in degree 2 and thus the natural Frobenius map kills $H^1_{\mathfrak{m}}(R)$. R is not F-full.

(2) Let $R = (k[x, y, z]/(x^3 + y^3 + z^3))\#k[s, t]$ be the Segre product of $A = (k[x, y, z]/(x^3 + y^3 + z^3))$ and $B = k[s, t]$, where k is a field of characteristic $p > 0$ with $p \equiv 2 \bmod 3$. Then R is a normal domain, since it is a direct summand of $A \otimes_k B = A[s, t]$. Moreover, a direct computation (for example see [18, Examples 4.11 and 4.16]) shows that

$$H^2_{\mathfrak{m}_R}(R) = [H^2_{\mathfrak{m}_R}(R)]_0 \cong [H^2_{\mathfrak{m}_A}(A)]_0 = k.$$

Since $p \equiv 2 \bmod 3$, we know the natural Frobenius map kills $[H^2_{\mathfrak{m}_A}(A)]_0$. Hence R is not F-full. On the other hand, if $p \equiv 1 \bmod 3$, then it is well known that R is F-pure (since A is) and hence F-anti-nilpotent [16, Theorem 1.1].

REMARK 2.6.

(1) When R is a homomorphic image of a regular ring A, say $R = A/I$, R is F-full if and only if $H^i_{\mathfrak{m}}(A/J) \to H^i_{\mathfrak{m}}(A/I)$ is surjective for every $J \subseteq I \subseteq \sqrt{J}$. This is because by [15, Lemma 2.2], the R-span of $F^e(H^i_{\mathfrak{m}}(R))$ is the same as the image $H^i_{\mathfrak{m}}(A/I^{[p^e]}) \to H^i_{\mathfrak{m}}(A/I)$, and for every $J \subseteq I \subseteq \sqrt{J}$, $I^{[p^e]} \subseteq J$ for $e \gg 0$. As an application, when $R = A/I$ is F-full, we have $H^i_{\mathfrak{m}}(A/I) = 0$ provided $H^i_{\mathfrak{m}}(A/J) = 0$. Hence depth $A/I \geqslant$ depth A/J for every $J \subseteq I \subseteq \sqrt{J}$.

(2) Suppose R is a local ring essentially of finite type over $\mathbb{C}$ and R is Du Bois (we refer to [21] or [18] for the definition and basic properties of Du Bois singularities). In this case we do have $H^i_{\mathfrak{m}}(A/J) \to H^i_{\mathfrak{m}}(A/I)$ is surjective for every $J \subseteq I = \sqrt{J}$ [18, Lemma 3.3]. This is the main ingredient in proving singularities of dense F-injective type deform [18, Theorem C].

(3) Since F-injective singularity is the conjectured characteristic $p > 0$ analog of Du Bois singularity [1, 21], it is thus quite natural to ask whether F-injective local rings are always F-full. It turns out that this is false in general [18, Example 3.5]. However, constructing such

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examples seems hard. In fact, [5, Example 2.16] (or its variants like [18, Example 3.5]) is the only example we know that is F-injective but not F-anti-nilpotent.

The above remarks motivate us to introduce and study F-fullness and a stronger notion of F-injectivity (see Section 5).

We end this subsection by proving that F-full rings localize. Note that it is proved in [16, Theorem 5.10] that F-anti-nilpotent rings localize.

For convenience, we use $R^{(1)}$ to denote the target ring of the Frobenius map $R \xrightarrow{F} R^{(1)}$. If M is an R-module, then $\mathrm{Hom}_R(R^{(1)}, M)$ has a structure of an $R^{(1)}$-module. We can then identify $R^{(1)}$ with R, and $\mathrm{Hom}_R(R^{(1)}, M)$ corresponds to an R-module which we call $F^\flat(M)$ (we refer to [2, Section 2.3] for more details on this). When R is F-finite, we have $\mathrm{Hom}_R(R^{(1)}, E_R) \cong E_{R^{(1)}}$ and $F^\flat(E) \cong E_R$, where E_R denotes the injective hull of the residue field of $(R, \mathfrak{m})$.

PROPOSITION 2.7. *Let $(R, \mathfrak{m})$ be an F-finite and F-full local ring. Then $R_\mathfrak{p}$ is also F-full for every $\mathfrak{p} \in \mathrm{Spec}\, R$.*

Proof. By a result of Gabber [7, Remark 13.6], R is a homomorphic image of a regular ring A. Let $n = \dim A$. We have

$$\begin{aligned}
&\mathrm{Hom}_{R^{(1)}}(\mathrm{Hom}_R(R^{(1)}, \mathrm{Ext}_A^{n-i}(R, A)), E_{R^{(1)}}) \\
&\quad \cong \mathrm{Hom}_{R^{(1)}}(\mathrm{Hom}_R(R^{(1)}, \mathrm{Ext}_A^{n-i}(R, A)), \mathrm{Hom}_R(R^{(1)}, E_R)) \\
&\quad \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R^{(1)}, \mathrm{Ext}_A^{n-i}(R, A)), E_R) \\
&\quad \cong R^{(1)} \otimes \mathrm{Hom}_R(\mathrm{Ext}_A^{n-i}(R, A), E_R) \\
&\quad \cong R^{(1)} \otimes H^i_\mathfrak{m}(R)
\end{aligned}$$

where the last isomorphism is by local duality. Thus after identifying $R^{(1)}$ with R, we have $\mathscr{F}_R(H^i_\mathfrak{m}(R))$ is the Matlis dual of $F^\flat(\mathrm{Ext}_A^{n-i}(R, A))$. So $\mathscr{F}_R(H^i_\mathfrak{m}(R)) \to H^i_\mathfrak{m}(R)$ is surjective for every i if and only if $\mathrm{Ext}_A^{n-i}(R, A) \to F^\flat(\mathrm{Ext}_A^{n-i}(R, A))$ is injective for every i. The latter condition clearly localizes. So R is F-full implies $R_\mathfrak{p}$ is F-full for every $\mathfrak{p} \in \mathrm{Spec}\, R$. ∎

§3. On surjective elements

The following definition was introduced in [11] and was the key tool in [11].

DEFINITION 3.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and x a regular element of R. x is called a *surjective element* if the natural map on the local

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cohomology module $H^i_{\mathfrak{m}}(R/(x^n)) \to H^i_{\mathfrak{m}}(R/(x))$ induced by $R/(x^n) \to R/(x)$ is surjective for all $n > 0$ and $i \geqslant 0$.

The next proposition is a restatement of [11, Lemma 3.2], so we omit the proof.

PROPOSITION 3.2. *The following are equivalent:*

(i) *x is a surjective element.*
(ii) *For all $0 < h \leqslant k$ the multiplication map*

$$R/(x^h) \stackrel{x^{k-h}}{\to} R/(x^k)$$

induces an injection

$$H^i_{\mathfrak{m}}(R/(x^h)) \to H^i_{\mathfrak{m}}(R/(x^k))$$

for each $i \geqslant 0$.
(iii) *For all $0 < h \leqslant k$ the short exact sequence*

$$0 \to R/(x^h) \stackrel{x^{k-h}}{\to} R/(x^k) \to R/(x^{k-h}) \to 0$$

induces a short exact sequence

$$0 \to H^i_{\mathfrak{m}}(R/(x^h)) \to H^i_{\mathfrak{m}}(R/(x^k)) \to H^i_{\mathfrak{m}}(R/(x^{k-h})) \to 0$$

for each $i \geqslant 0$.

PROPOSITION 3.3. *The following are equivalent:*

(i) *x is a surjective element.*
(ii) *The multiplication map $H^i_{\mathfrak{m}}(R) \stackrel{x}{\to} H^i_{\mathfrak{m}}(R)$ is surjective for all $i \geqslant 0$.*

Proof. By Proposition 3.2, x is a surjective element if and only if all maps in the direct limit system $\{H^i_{\mathfrak{m}}(R/(x^h))\}_{h\geqslant 1}$ are injective. This is equivalent to the condition

$$\phi_h : H^i_{\mathfrak{m}}(R/(x^h)) \to \varinjlim_h H^i_{\mathfrak{m}}(R/(x^h)) \cong H^i_{\mathfrak{m}}(H^1_{(x)}(R)) \cong H^{i+1}_{\mathfrak{m}}(R)$$

is injective for all $h \geqslant 1$ and all $i \geqslant 0$ (the last isomorphism comes from an easy computation using local cohomology spectral sequences and noting that x is a nonzero divisor on R, see also [11, Lemma 2.2]).

Claim 3.4. ϕ_h is exactly the connection maps in the long exact sequence of local cohomology induced by $0 \to R \xrightarrow{\cdot x^h} R \to R/(x^h) \to 0$:

$$\cdots \to H^i_{\mathfrak{m}}(R/(x^h)) \xrightarrow{\phi_h} H^{i+1}_{\mathfrak{m}}(R) \xrightarrow{x^h} H^{i+1}_{\mathfrak{m}}(R) \to \cdots .$$

Proof of claim. Observe that by definition, ϕ_h is the natural map in the long exact sequence of local cohomology

$$\cdots \to H^i_{\mathfrak{m}}(R/(x^h)) \xrightarrow{\phi_h} H^i_{\mathfrak{m}}(R_x/R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R_x/R) \to \cdots$$

which is induced by $0 \to R/(x^h) \to R_x/R \xrightarrow{\cdot x^h} R_x/R \to 0$ (note that x^h is a nonzero divisor on R and $H^1_x(R) \cong R_x/R$). However, it is easy to see that the multiplication by x^h map $H^i_{\mathfrak{m}}(R_x/R) \xrightarrow{\cdot x^h} H^i_{\mathfrak{m}}(R_x/R)$ can be identified with the multiplication by x^h map $H^{i+1}_{\mathfrak{m}}(R) \xrightarrow{\cdot x^h} H^{i+1}_{\mathfrak{m}}(R)$ because we have a natural identification $H^i_{\mathfrak{m}}(R_x/R) \cong H^i_{\mathfrak{m}}(H^1_x(R)) \cong H^{i+1}_{\mathfrak{m}}(R)$ (see for example [11, Lemma 2.2]). This finishes the proof of the claim. ∎

From the claim it is immediate that x is a surjective element if and only if the long exact sequence splits into short exact sequences:

$$0 \to H^i_{\mathfrak{m}}(R/(x^h)) \to H^{i+1}_{\mathfrak{m}}(R) \xrightarrow{x^h} H^{i+1}_{\mathfrak{m}}(R) \to 0.$$

But this is equivalent to saying that the multiplication map $H^i_{\mathfrak{m}}(R) \xrightarrow{x^h} H^i_{\mathfrak{m}}(R)$ is surjective for all $h \geqslant 1$ and $i \geqslant 0$, and also equivalent to $H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R)$ is surjective for all $i \geqslant 0$. ∎

We next link the notion of surjective element with F-fullness. This is inspired by [18, 24].

PROPOSITION 3.5. *Let x be a regular element of $(R, \mathfrak{m})$. If $R/(x)$ is F-full, then x is a surjective element. In particular, if $R/(x)$ is F-anti-nilpotent, then x is a surjective element.*

Proof. We have natural maps:

$$\mathscr{F}^e_R(H^i_{\mathfrak{m}}(R/(x))) \xrightarrow{\alpha_e} R/(x) \otimes_R \mathscr{F}^e_R(H^i_{\mathfrak{m}}(R/(x))) \cong \mathscr{F}^e_{R/(x)}(H^i_{\mathfrak{m}}(R/(x))) \xrightarrow{\beta_e} H^i_{\mathfrak{m}}(R/(x)).$$

If $R/(x)$ is F-full, then β_e is surjective for every e. Since α_e is always surjective, the natural map $\mathscr{F}^e_R(H^i_{\mathfrak{m}}(R/(x))) \to H^i_{\mathfrak{m}}(R/(x))$ is surjective for

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every e. Now simply notice that for every $e > 0$, the map $\mathscr{F}_R^e(H^i_{\mathfrak{m}}(R/(x))) \to H^i_{\mathfrak{m}}(R/(x))$ factors through $H^i_{\mathfrak{m}}(R/(x^{p^e})) \to H^i_{\mathfrak{m}}(R/(x))$, so $H^i_{\mathfrak{m}}(R/(x^{p^e})) \to H^i_{\mathfrak{m}}(R/(x))$ is surjective for every $e > 0$. This clearly implies that x is a surjective element. ∎

The above propositions allow us to construct a family of non F-full local rings:

EXAMPLE 3.6. Let $(R, \mathfrak{m})$ be a local ring with finite length cohomology, that is, $H^i_{\mathfrak{m}}(R)$ has finite length for every $i < \dim R$ (under mild conditions, this is equivalent to saying that R is Cohen–Macaulay on the punctured spectrum). Let x be an arbitrary regular element in R. If R is not Cohen–Macaulay, then we claim that $R/(x)$ is not F-full (and hence not F-anti-nilpotent). For suppose it is, then x is a surjective element by Proposition 3.5, hence $H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R)$ is surjective for every i by Proposition 3.3. But since R has finite length cohomology, we also know that a power of x annihilates $H^i_{\mathfrak{m}}(R)$ for every $i < \dim R$. This implies $H^i_{\mathfrak{m}}(R) = 0$ for every $i < \dim R$. So R is Cohen–Macaulay, a contradiction.

We learned the following argument from [11, Lemma A.1]. Since it is a crucial technique of this paper, we provide a detailed proof.

PROPOSITION 3.7. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and x a regular element of R. Let s be a positive integer such that the map $H^{s-1}_{\mathfrak{m}}(R) \xrightarrow{x} H^{s-1}_{\mathfrak{m}}(R)$ is surjective and the Frobenius action on $H^{s-1}_{\mathfrak{m}}(R/(x))$ is injective, then the map*

$$H^s_{\mathfrak{m}}(R) \xrightarrow{x^{p-1}F} H^s_{\mathfrak{m}}(R)$$

is injective.

Proof. The natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0
\end{array}$$

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induces the following commutative diagram (the left most 0 comes from our hypothesis that the map $H^{s-1}_{\mathfrak{m}}(R) \xrightarrow{x} H^{s-1}_{\mathfrak{m}}(R)$ is surjective):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{s-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^{s}_{\mathfrak{m}}(R) & \xrightarrow{x} & H^{s}_{\mathfrak{m}}(R) & \longrightarrow & \cdots \\
 & & {\scriptstyle F}\big\downarrow & & {\scriptstyle x^{p-1}F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & \\
0 & \longrightarrow & H^{s-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^{s}_{\mathfrak{m}}(R) & \xrightarrow{x} & H^{s}_{\mathfrak{m}}(R) & \longrightarrow & \cdots
\end{array}$$

Suppose $y \in \operatorname{Ker}(x^{p-1}F) \cap \operatorname{Soc}(H^{s}_{\mathfrak{m}}(R))$. Then we have $x \cdot y = 0$ so there exists $z \in H^{s-1}_{\mathfrak{m}}(R/(x))$ such that $\alpha(z) = y$. Following the above commutative diagram we have

$$(\alpha \circ F)(z) = x^{p-1}F(\alpha(z)) = x^{p-1}F(y) = 0.$$

However, since both F and α are injective, we have $z = 0$ and hence $y = 0$. This shows $x^{p-1}F$ is injective and hence completes the proof. □

Proposition 3.7 immediately generalizes the main result of [11]:

Corollary 3.8. (Compare with [11], Main Theorem) *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and x a regular element of R. Suppose $R/(x)$ is F-injective. Then we have*

(i) *The map $H^{t}_{\mathfrak{m}}(R) \xrightarrow{x^{p-1}F} H^{t}_{\mathfrak{m}}(R)$ is injective where $t = \operatorname{depth} R$. In particular, the natural Frobenius action on $H^{t}_{\mathfrak{m}}(R)$ is injective.*

(ii) *Suppose x is a surjective element. Then the map $H^{i}_{\mathfrak{m}}(R) \xrightarrow{x^{p-1}F} H^{i}_{\mathfrak{m}}(R)$ is injective for all $i \geqslant 0$. In particular, R is F-injective.*

(iii) *If $R/(x)$ is F-full (e.g., R is F-anti-nilpotent or R is F-pure), then R is F-injective.*

Proof. (i) Follows from Proposition 3.7 applied to $s = t$, (ii) also follows from Proposition 3.7 (because $H^{i}_{\mathfrak{m}}(R) \xrightarrow{x} H^{i}_{\mathfrak{m}}(R)$ is surjective for every $i \geqslant 0$ by Proposition 3.3), (iii) follows from (ii), because we know x is a surjective element by Proposition 3.5. □

In the next two sections, we show that F-full and F-anti-nilpotent singularities both deform. We also prove new cases of deformation of F-injectivity. These results are generalizations of Proposition 3.7 and Corollary 3.8.

§4. Deformation of F-full and F-anti-nilpotent singularities

In this section we prove that the condition F-full and F-anti-nilpotent both deform. Throughout this section we assume that $(R, \mathfrak{m})$ is a local ring of prime characteristic p. We begin with a crucial lemma.

LEMMA 4.1. *Let x be a surjective element of R. Let $N \subseteq H^i_{\mathfrak{m}}(R)$ be an F-stable submodule. Let $L = \bigcap_t x^t N$. Then L is an F-stable submodule of $H^i_{\mathfrak{m}}(R)$ and we have the following commutative diagram (for every $e \geqslant 1$):*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \end{array}$$

where ϕ is the map $H^{i-1}_{\mathfrak{m}}(R/(x)) \to H^i_{\mathfrak{m}}(R)$.

Proof. Since x is a surjective element, by Proposition 3.3 we know that the map

$$H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R) \text{ is surjective for every } i > 0. \quad (\star)$$

Applying the local cohomology functor to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \end{array}$$

we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R) & \xrightarrow{x} & H^i_{\mathfrak{m}}(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R) & \xrightarrow{x} & H^i_{\mathfrak{m}}(R) & \longrightarrow & 0 \end{array}$$

for all $i \geqslant 1$ and $e \geqslant 1$, where the rows are short exact sequences by $(\star)$.

Therefore, to prove the lemma, it suffices to show that L is F-stable and

$$0 \to H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) \xrightarrow{\phi} H^i_{\mathfrak{m}}(R)/L \xrightarrow{x} H^i_{\mathfrak{m}}(R)/L \to 0$$

is exact. It is clear that L is F-stable since it is an intersection of F-stable submodules of $H^i_{\mathfrak{m}}(R)$. To see the exactness of the above sequence, first note that $\operatorname{Im}(\phi) = 0 :_{H^i_{\mathfrak{m}}(R)} x$, so $L + \operatorname{Im}(\phi) \subseteq L :_{H^i_{\mathfrak{m}}(R)} x$. Thus it is enough to check that $L :_{H^i_{\mathfrak{m}}(R)} x \subseteq L + \operatorname{Im}(\phi)$. Let y be an element such that $xy \in L$.

Since $L = xL$ by the construction of L, there exists $z \in L$ such that $xy = xz$. So $y - z \in \operatorname{Im}(\phi)$ and hence $y \in L + \operatorname{Im}(\phi)$, as desired. ∎

We are ready to prove the main result of this section. This answers [20, Problem 4] for stably FH-finiteness.

THEOREM 4.2. *$(R, \mathfrak{m})$ be a local ring of positive characteristic p and x a regular element of R. Then we have:*

(i) *if $R/(x)$ is F-anti-nilpotent, then so is R;*
(ii) *if $R/(x)$ is F-full, then so is R.*

Proof. We first prove (i). Let N be an F-stable submodule of $H^i_{\mathfrak{m}}(R)$. We want to show that the induced Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is injective. Since $R/(x)$ is F-anti-nilpotent, x is a surjective element by Proposition 3.5. Let $L = \bigcap_t x^t N$. By Lemma 4.1, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \end{array}$$

We first claim that the middle map $x^{p^e-1}F^e : H^i_{\mathfrak{m}}(R)/L \to H^i_{\mathfrak{m}}(R)/L$ is injective. Let $y \in \operatorname{Ker}(x^{p^e-1}F^e) \cap \operatorname{Soc}(H^i_{\mathfrak{m}}(R)/L)$. We have $x \cdot y = 0$, so $y = \phi(z)$ for some $z \in H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$. It is easy to see that $\phi^{-1}(L)$ is an F-stable submodule of $H^{i-1}_{\mathfrak{m}}(R/(x))$ and $F^e(z) = 0$. Since $R/(x)$ is F-anti-nilpotent, we know the Frobenius action F, and hence its iterate F^e, on $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$ is injective. Therefore, $z = 0$ and hence $y = 0$. This proves that $x^{p^e-1}F^e$ and hence F acts injectively on $H^i_{\mathfrak{m}}(R)/L$.

Note that we have a descending chain $N \supseteq xN \supseteq x^2N \supseteq \cdots$. Since $H^i_{\mathfrak{m}}(R)$ is Artinian, $L = \bigcap_t x^t N = x^n N$ for all $n \gg 0$. We next claim that $L = N$, this will finish the proof because we already showed F acts injectively on $H^i_{\mathfrak{m}}(R)/L$. We have $x^{p^e-1}F^e(N) \subseteq x^{p^e-1}N = L$ for $e \gg 0$, but the map $x^{p^e-1}F^e : H^i_{\mathfrak{m}}(R)/L \to H^i_{\mathfrak{m}}(R)/L$ is injective by the above paragraph. So we must have $N \subseteq L$ and thus $L = N$. This completes the proof of (1).

Next we prove (ii). The method is similar to that of (i). Let N be the R-span of $F(H^i_{\mathfrak{m}}(R))$ in $H^i_{\mathfrak{m}}(R)$, this is the same as the image of $\mathscr{F}_R(H^i_{\mathfrak{m}}(R)) \to H^i_{\mathfrak{m}}(R)$. It is clear that N is an F-stable submodule. We want to show

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$N = H^i_{\mathfrak{m}}(R)$. Since $R/(x)$ is F-full, x is a surjective element by Proposition 3.5. Let $L = \bigcap_t x^t N$. By Lemma 4.1, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle F^e} & & \Big\downarrow{\scriptstyle x^{p^e-1}F^e} & & \Big\downarrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R)/L & \xrightarrow{x} & H^i_{\mathfrak{m}}(R)/L & \longrightarrow & 0 \end{array}$$

The descending chain $N \supseteq xN \supseteq x^2N \supseteq \cdots$ stabilizes because $H^i_{\mathfrak{m}}(R)$ is Artinian. So $L = \bigcap_t x^t N = x^n N$ for $n \gg 0$. The key point is that in the above diagram, the middle Frobenius action $x^{p^e-1}F^e$ is the zero map on $H^i_{\mathfrak{m}}(R)/L$ for $e \gg 0$, because for any $y \in H^i_{\mathfrak{m}}(R)$, $F^e(y) \in N$ and thus $x^{p^e-1}F^e(y) \in L$ for $e \gg 0$. But then since $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$ can be viewed as a submodule of $H^i_{\mathfrak{m}}(R)/L$ by the above commutative diagram, the natural Frobenius action F^e on $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$ is zero, that is, F is nilpotent on $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$.

Since F is nilpotent on $H^{i-1}_{\mathfrak{m}}(R/(x))/\phi^{-1}(L)$, we know that $\phi^{-1}(L)$ must contain all elements $F^e(H^i_{\mathfrak{m}}(R/(x)))$, hence it contains the R-span of $F^e(H^i_{\mathfrak{m}}(R/(x)))$. But $R/(x)$ is F-full, so we must have $\phi^{-1}(L) = H^{i-1}_{\mathfrak{m}}(R/(x))$. But this means the map

$$H^i_{\mathfrak{m}}(R)/L \xrightarrow{x} H^i_{\mathfrak{m}}(R)/L$$

is an isomorphism, which is impossible unless $H^i_{\mathfrak{m}}(R) = L$ (since otherwise any nonzero socle element of $H^i_{\mathfrak{m}}(R)/L$ maps to zero). Therefore, we have $H^i_{\mathfrak{m}}(R) = N = L$. This proves R is F-full and hence finished the proof of (2). ∎

The following is a well-known counter-example of Fedder [6] and Singh [22] for the deformation of F-purity.

Example 4.3. (Compare with [20, Lemma 6.1]) Let K be a perfect field of characteristic $p > 0$ and let

$$R := K[[U, V, Y, Z]]/(UV, UZ, Z(V - Y^2)).$$

Let u, v, y and z denote the image of U, V, Y and Z in R (and its quotients), respectively. Then y is a regular element of R and $R/(y) \cong K[[U, V, Z]]/(UV, UZ, VZ)$ is F-pure by [12, Proposition 5.38]. So $R/(y)$ is F-anti-nilpotent by [16, Theorem 1.1]. By Theorem 4.2 we have R is also F-anti-nilpotent, or equivalently, R is stably FH-finite.

§5. F-injectivity

5.1 F-injectivity and depth

We start with the following definition.

DEFINITION 5.1. (Cf. [3, Definition 9.1.3]) Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$. The *finiteness dimension* $f_{\mathfrak{m}}(M)$ *of* M *with respect to* $\mathfrak{m}$ is defined as follows:

$$f_{\mathfrak{m}}(M) := \inf\{i \mid H^i_{\mathfrak{m}}(M) \text{ is not finitely generated}\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}.$$

REMARK 5.2.

(i) Assume that $\dim M = 0$ or $M = 0$ (recall that a trivial module has dimension -1). In this case, $H^i_{\mathfrak{m}}(M)$ is finitely generated for all i and $f_{\mathfrak{m}}(M)$ is equal to ∞. It will be essential to know when the finiteness dimension is a positive integer. We mention the following result. Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. If $d = \dim M > 0$, then the local cohomology module $H^d_{\mathfrak{m}}(M)$ is not finitely generated. For the proof of this result, see [3, Corollary 7.3.3].

(ii) Suppose $(R, \mathfrak{m})$ is an image of a Cohen–Macaulay local ring. By the Grothendieck finiteness theorem (cf. [3, Theorem 9.5.2]) we have

$$f_{\mathfrak{m}}(M) = \min\{\operatorname{depth} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \,:\, \mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}\}.$$

(iii) M is *generalized Cohen–Macaulay* if and only if $\dim M = f_{\mathfrak{m}}(M)$.

It is clear that $\operatorname{depth} R \leqslant f_{\mathfrak{m}}(R) \leqslant \dim R$. The following result says that if $R/(x)$ is F-injective, then R has 'good' depth.

THEOREM 5.3. *If* $R/(x)$ *is* F*-injective, then* $\operatorname{depth} R = f_{\mathfrak{m}}(R)$.

Proof. Suppose $t = \operatorname{depth} R < f_{\mathfrak{m}}(R)$. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0
\end{array}$$

induces the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^{t-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^t_{\mathfrak{m}}(R) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \\
0 & \longrightarrow & H^{t-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\alpha} & H^t_{\mathfrak{m}}(R) & \longrightarrow & \cdots
\end{array}$$

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where both α and the left vertical map are injective. But $H^t_{\mathfrak{m}}(R)$ has finite length, $x^{p^e-1}F^e: H^t_{\mathfrak{m}}(R) \to H^t_{\mathfrak{m}}(R)$ vanishes for $e \gg 0$, which is a contradiction. ∎

REMARK 5.4. The assertion of Theorem 5.3 also holds true if $R/(x)$ is F-full. Indeed, by Proposition 3.5 we have x is a surjective element. Hence there is no nonzero $H^i_{\mathfrak{m}}(R)$ of finite length. Thus depth $R = f_{\mathfrak{m}}(R)$.

REMARK 5.5. The above result is closely related to the work of Schwede and Singh in [11, Appendix]. In the proof of [11, Lemma A.2, Theorem A.3], it is claimed that if $R_{\mathfrak{p}}$ satisfies the Serre condition (S_k) for all $\mathfrak{p}$ in $\operatorname{Spec}^\circ(R)$, the punctured spectrum of R, and depth $R = t < k$, then $H^t_{\mathfrak{m}}(R)$ is finitely generated. But this fact may not be true if R is not equidimensional. For instance, let $R = K[[a, b, c, d]]/(a) \cap (b, c, d)$ with K a field. We have depth $R = 1$ and $R_{\mathfrak{p}}$ satisfies (S_2) for all $\mathfrak{p} \in \operatorname{Spec}^\circ(R)$. However, $H^1_{\mathfrak{m}}(R)$ is not finitely generated.

The assertion of [11, Lemma A.2] (and hence [11, Theorem A.3]) is still true. In fact, we can reduce it to the case that R is equidimensional. We fill this gap below.

COROLLARY 5.6. [11, Lemma A.2] *Let $(R, \mathfrak{m})$ be an F-finite local ring. Suppose there exists a regular element x such that $R/(x)$ is F-injective. If $R_{\mathfrak{p}}$ satisfies the Serre condition (S_k) for all $\mathfrak{p} \in \operatorname{Spec}^\circ(R)$, then R is (S_k).*

Proof. We can assume that $k \leqslant d = \dim R$. In fact, we need only to prove that $t :=$ depth $R \geqslant k$. The case $k = 1$ is trivial since R contains a regular element x. For $k \geqslant 2$, since $R/(x)$ is F-injective we have $R/(x)$ is reduced (cf. [21, Proposition 4.3]). Hence $\operatorname{depth}(R/(x)) \geqslant 1$, so depth $R \geqslant 2$. Thus R satisfies the Serre condition (S_2). On the other hand, since R is F-finite, R is a homomorphic image of a regular ring by a result of Gabber [7, Remark 13.6]. In particular, R is universally catenary.[3] But if a universally catenary ring satisfies (S_2), then it is equidimensional (see [10, Remark 2.2(h)]). By Theorem 5.3 and Remark 5.2(ii), there exists a prime ideal $\mathfrak{p} \in \operatorname{Spec}^\circ(R)$ such that depth $R =$ depth $R_{\mathfrak{p}} + \dim R/\mathfrak{p}$. It is then easy to see that depth $R \geqslant \min\{d, k+1\} \geqslant k$. The proof is complete. ∎

[3]Another way to see this is to use the fact that F-finite rings are excellent [14] and hence universally catenary.

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REMARK 5.7. In the above argument, we actually proved that if $k < d$, then depth $R \geqslant k+1$.

5.2 Deformation of F-injectivity

We begin with the following generalization of the notion of surjective elements.

DEFINITION 5.8. (Cf. [4]) A regular element x is called a *strictly filter regular* element if

$$\mathrm{Coker}(H^i_{\mathfrak{m}}(R) \xrightarrow{x} H^i_{\mathfrak{m}}(R))$$

has finite length for all $i \geqslant 0$.

LEMMA 5.9. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. Suppose the residue field $k = R/\mathfrak{m}$ is perfect. Let M be an R-module with an injective Frobenius action F. Suppose L is an F-stable submodule of M of finite length. Then the induced Frobenius action on M/L is injective.*

Proof. First we note that L is killed by $\mathfrak{m}$: suppose $x \in L$, then $F^e(\mathfrak{m} \cdot x) = \mathfrak{m}^{[p^e]} \cdot x = 0$ for $e \gg 0$ since L has finite length. But then $\mathfrak{m} \cdot x = 0$ since F acts injectively. Now we have a Frobenius action F on a k-vector space L. Call the image of $L' \subseteq L$ (which is a k^p-vector subspace of L). Since F is injective, the k^p-vector space dimension of L' is equal to the k-vector space dimension of L. But since $k^p = k$, this implies $L' = L$ and thus F is surjective, hence F is bijective. Now by the injectivity of F again we have $F(x) \notin L$ for all $x \notin L$. Thus $F : M/L \to M/L$ is injective. ∎

EXAMPLE 5.10. The perfectness of the residue field in Lemma 5.9 is necessary. Let $A = \mathbb{F}_p[t]$ and $R = k = \mathbb{F}_p(t)$, where t is an indeterminate. We consider the Frobenius action on the A-module $Ae_1 \oplus Ae_2$ defined by

$$F(f(t), g(t)) = (f(t)^p + tg(t)^p, 0).$$

It is clear that F is injective. Moreover, $Ae_1 \oplus 0$ is an F-stable submodule of $Ae_1 \oplus Ae_2$. Since $F(Ae_1 \oplus Ae_2) \subseteq Ae_1 \oplus 0$, the induced Frobenius action on $(Ae_1 \oplus Ae_2)/(Ae_1 \oplus 0)$ is the zero map. By localizing, we obtain an injective Frobenius action on $M = k \cdot e_1 \oplus k \cdot e_2$ with $L = k \cdot e_1 \oplus 0$ is an F-stable submodule of finite length, but the induced Frobenius action on M/L is not injective.

The following is a generalization of the main result of [11] when $R/\mathfrak{m}$ is perfect.

Theorem 5.11. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring of characteristic* $p>0$. *Suppose the residue field* $k=R/\mathfrak{m}$ *is perfect. Let* x *be a strictly filter regular element. If* $R/(x)$ *is* F*-injective, then the map* $x^{p-1}F\colon H^i_{\mathfrak{m}}(R)\to H^i_{\mathfrak{m}}(R)$ *is injective for every* i*, in particular* R *is* F*-injective.*

Proof. Let $L_i:=\operatorname{Coker}(H^i_{\mathfrak{m}}(R)\xrightarrow{x}H^i_{\mathfrak{m}}(R))$, we have L_i has finite length for all $i\geqslant 0$. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/(x) & \longrightarrow & 0 \end{array}$$

induces the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & L_{i-1} & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R) & \xrightarrow{x} & H^i_{\mathfrak{m}}(R) & \longrightarrow & \cdots\\ & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & L_{i-1} & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x)) & \xrightarrow{\phi} & H^i_{\mathfrak{m}}(R) & \xrightarrow{x} & H^i_{\mathfrak{m}}(R) & \longrightarrow & \cdots \end{array}$$

Therefore, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1} & \xrightarrow{\alpha} & H^i_{\mathfrak{m}}(R) & \xrightarrow{x} & H^i_{\mathfrak{m}}(R) & \longrightarrow & \cdots\\ & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1} & \xrightarrow{\alpha} & H^i_{\mathfrak{m}}(R) & \xrightarrow{x} & H^i_{\mathfrak{m}}(R) & \longrightarrow & \cdots \end{array}$$

with the Frobenius action $F: H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1}\to H^{i-1}_{\mathfrak{m}}(R/(x))/L_{i-1}$ is injective by Lemma 5.9. Now by the same method as in the proof of Proposition 3.7 or Theorem 4.2(i), we conclude that the map $x^{p-1}F: H^i_{\mathfrak{m}}(R)\to H^i_{\mathfrak{m}}(R)$ is injective for all $i\geqslant 0$. ∎

Similarly, we have the following:

Proposition 5.12. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring of characteristic* $p>0$. *Suppose the residue field* $k=R/\mathfrak{m}$ *is perfect. Let* x *be a regular element such that* $R/(x)$ *is* F*-injective. Let* s *be a positive integer such that* $H^{s-1}_{\mathfrak{m}}(R/(x))$ *has finite length. Then the map* $x^{p-1}F: H^{s+1}_{\mathfrak{m}}(R)\to H^{s+1}_{\mathfrak{m}}(R)$ *is injective.*

Proof. The short exact sequence

$$0\to R\xrightarrow{x}R\to R/(x)\to 0$$

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induces the exact sequence

$$\cdots \to H^{s-1}_{\mathfrak{m}}(R/(x)) \to H^s_{\mathfrak{m}}(R) \xrightarrow{x} H^s_{\mathfrak{m}}(R) \to H^s_{\mathfrak{m}}(R/(x)) \to H^{s+1}_{\mathfrak{m}}(R) \to \cdots .$$

Since $H^{s-1}_{\mathfrak{m}}(R/(x))$ has finite length, so is $\mathrm{Ker}(H^s_{\mathfrak{m}}(R) \xrightarrow{x} H^s_{\mathfrak{m}}(R))$. We claim that

$$L_s := \mathrm{Coker}(H^s_{\mathfrak{m}}(R) \xrightarrow{x} H^s_{\mathfrak{m}}(R))$$

also has finite length: to see this we may assume R is complete, since $\mathrm{Ker}(H^s_{\mathfrak{m}}(R) \xrightarrow{x} H^s_{\mathfrak{m}}(R))$ has finite length, this means $H^s_{\mathfrak{m}}(R)^\vee \xrightarrow{x} H^s_{\mathfrak{m}}(R)^\vee$ is surjective when localizing at any $\mathfrak{p} \neq \mathfrak{m}$. But by [19, Theorem 2.4] this implies $H^s_{\mathfrak{m}}(R)^\vee \xrightarrow{x} H^s_{\mathfrak{m}}(R)^\vee$ is an isomorphism when localizing at any $\mathfrak{p} \neq \mathfrak{m}$. Thus $\mathrm{Ker}(H^s_{\mathfrak{m}}(R)^\vee \xrightarrow{x} H^s_{\mathfrak{m}}(R)^\vee)$ has finite length which, after dualizing, shows that $\mathrm{Coker}(H^s_{\mathfrak{m}}(R) \xrightarrow{x} H^s_{\mathfrak{m}}(R))$ has finite length.

We have proved $L_s = \mathrm{Coker}(H^s_{\mathfrak{m}}(R) \xrightarrow{x} H^s_{\mathfrak{m}}(R))$ has finite length. Now the map $x^{p-1}F : H^{s+1}_{\mathfrak{m}}(R) \to H^{s+1}_{\mathfrak{m}}(R)$ is injective by the same argument as in Theorem 5.11. ∎

The following immediate corollary of the above proposition recovers (and in fact generalizes) results in [11].

COROLLARY 5.13. [11, Corollary 4.7] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$. Suppose the residue field $k = R/\mathfrak{m}$ is perfect. Let x be a regular element such that $R/(x)$ is F-injective. Then the map $x^{p-1}F$: $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for all $i \leqslant f_{\mathfrak{m}}(R/(x)) + 1$. In particular, if $R/(x)$ is generalized Cohen–Macaulay, then R is F-injective.*

Because of the deep connections between F-injective and Du Bois singularities [1, 21] and Remark 2.6, we believe that it is rarely the case that an F-injective ring fails to be F-full (again, the only example we know this happens is [18, Example 3.5], which is based on the construction of [5, Example 2.16]). Therefore, we introduce:

DEFINITION 5.14. We say $(R, \mathfrak{m})$ is *strongly F-injective* if R is F-injective and F-full.

REMARK 5.15. In general we have: F-anti-nilpotent $\Rightarrow$ strongly F-injective $\Rightarrow$ F-injective. Moreover, when R is Cohen–Macaulay, strongly F-injective is equivalent to F-injective.

We can prove that strong F-injectivity deform.

COROLLARY 5.16. *Let x be a regular element on $(R, \mathfrak{m})$. If $R/(x)$ is strongly F-injective, then R is strongly F-injective.*

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Proof. We know R is F-injective by Corollary 3.8(iii). But we also know R is F-full by Theorem 4.2(ii). This shows that R is strongly F-injective. □

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Linquan Ma
Department of Mathematics
University of Utah
Salt Lake City
UT 84102
USA
lquanma@math.utah.edu

Pham Hung Quy
Department of Mathematics
FPT University, and
Thang Long Institute of Mathematics and Applied Sciences
Ha Noi
Vietnam
quyph@fe.edu.vn