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Abstract

This paper explores the cohomological consequences of the existence of moduli spaces for flat bundles with bounded rank and irregularity at infinity and gives unconditional proofs. Namely, we prove the existence of a universal bound for the dimension of de Rham cohomology of flat bundles with bounded rank and irregularity on surfaces. In any dimension, we prove a Lefschetz recognition principle stating the existence of hyperplane sections distinguishing flat bundles with bounded rank and irregularity after restriction. We obtain in any dimension a universal bound for the degrees of the turning loci of flat bundles with bounded rank and irregularity. Along the way, we introduce a new operation on the group of b-divisors on a smooth surface (the partial discrepancy) and prove a closed formula for the characteristic cycles of flat bundles on surfaces in terms of the partial discrepancy of the irregularity b-divisor attached to any flat bundle by Kedlaya.

The goal of this paper is to explore the cohomological consequences of the existence of moduli spaces for flat bundles with bounded rank and irregularity at infinity and to prove them unconditionally in the surface case.

Let U be a smooth complex projective variety. From Simpson's work [Sim94], flat bundles on U with given rank form a complex variety, the *de Rham space of* U. If U is quasi-projective, flat bundles acquire singularities at infinity making the situation more involved. However, *regular singular* flat bundles again give rise to moduli spaces at the cost of rigidifying the situation by a choice of logarithmic lattice [Nit93]. Let X be a smooth compactification of U such that $D := X \setminus U$ has simple normal crossings and let $j : U \to X$ be the inclusion. A flat bundle $\mathcal{E} = (E, \nabla)$ on U is regular singular [Del70] if E extends to a vector bundle F on X with $\nabla(F) \subset F \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$. Regular singularity is independent of a choice of compactification. When an extra condition on the residues is imposed, F is called a *Deligne lattice*. A natural question to ask is what lies beyond the regular singular case. In general, the connection ∇ no longer has simple poles at infinity, but Sabbah observed in the 1990s [Sab00] that ∇ still has a simple expression, the so-called *good formal decomposition* at the cost of working formally along D away from a codimension-1 subset of D. The locus of D where this simple expression does not hold is the *turning locus* of $\mathcal{E}$. Away from the turning locus, Deligne's lattices admit

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straightforward generalizations and Malgrange proved in [Mal96] that they extend canonically to X. Armed with Deligne–Malgrange lattices and conjectural bounds for their Chern classes, it is an expectation of Esnault and Langer dating back from 2014 that there should exist a moduli of finite type for flat bundles on U with bounded rank and irregularity at infinity. See [Esn14] and [Ked21, 4.3.4]. Bounded irregularity means very roughly that we bound the poles of the connection restricted to a Deligne–Malgrange lattice. See Definition 2.1 for a rigorous definition. The bound on irregularity is thus embodied by an effective divisor R supported on D. Let us denote by $\mathcal{M}_r(X, D, R)$ Esnault and Langer's expected moduli of rank-r flat bundles on U with irregularity bounded by R along D. As pointed out to the authors by Langer, there is a mismatch between Nitsure's construction and $\mathcal{M}_r(X, D, R)$ for $R = 0$. Indeed, rigidifying the moduli problem with choices of logarithmic lattices makes Nitsure's construction depend on a choice of compactification for X. On the other hand, having regular singularity at infinity is independent of a choice of compactification. Furthermore, different points of Nitsure's construction may underlie the same flat bundle.

The existence of $\mathcal{M}_r(X, D, R)$ has deep consequences for flat bundles which can be stated and studied independently. The first consequence is *cohomological*. For a flat bundle $\mathcal{E}$ on U, let $\mathrm{DR}\,\mathcal{E}$ be the algebraic de Rham complex of $\mathcal{E}$. Following the construction of the jumping loci for character varieties, the following subsets

$$\mathcal{V}^j := \{\mathcal{E} \in \mathcal{M}_r(X, D, R) \text{ such that } \dim H^*(U, \mathrm{DR}\,\mathcal{E}) \geq j\}$$

should form a decreasing sequence of closed subsets of $\mathcal{M}_r(X, D, R)$ when j increases. Since $\mathcal{M}_r(X, D, R)$ is expected to be of finite type, its underlying topological space should be noetherian, so the sequence of $\mathcal{V}^j$ should stabilize for j big enough. On the other hand, de Rham cohomology is always finite dimensional. Hence, there should exist an integer j_0 such that $\mathcal{V}^{j_0}$ is empty. Put otherwise, there should exist a universal bound for the algebraic de Rham cohomology of rank r flat bundles on U with bounded irregularity at infinity. Furthermore, this bound should depend only on X, D, R and r. See Conjecture 2.5 for a statement making precise the dependency in r and R. One of the main results of this paper is an unconditional proof of the existence of this bound in the surface case (Theorem 6.6). To state it, we let $\mathrm{Div}(X, D)$ be the group of divisors of X supported on D and denote by k a field of characteristic 0. For every effective divisor R of X supported on D, for every integer $r \geq 0$, we let $\mathrm{MIC}_r(X, D, R)$ be the category of flat bundles on U with rank smaller than r and irregularity bounded by R along D.

Theorem 0.1. *Let X be a smooth projective surface over k. Let D be a normal crossing divisor of X. Then, there exists a quadratic polynomial $C : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{E}$ of $\mathrm{MIC}_r(X, D, R)$, we have*

$$\dim H^*(U, \mathrm{DR}\,\mathcal{E}) \leq C(R, r).$$

A second consequence of the existence of $\mathcal{M}_r(X, D, R)$ is the *Lefschetz recognition principle*. It relies on the heuristic principle that the formation of $\mathcal{M}_r(X, D, R)$ should come packaged with functorialities. If $f : Y \to X$ is a morphism of smooth projective varieties over k such that f^*R makes sense and such that $f^{-1}(D)$ is a normal crossing divisor, one expects the existence of a morphism of varieties $\mathcal{M}_r(X, D, R) \to \mathcal{M}_r(Y, f^{-1}(D), f^*R)$ induced by the pull-back along $f : Y \to X$. Functorialities would make the study of $\mathcal{M}_r(X, D, R)$ tractable through the curve case, where they should be much easier to construct since flat bundles on curves have no turning points. If $\mathcal{E}_0$ and $\mathcal{E}_1$ are non-isomorphic flat bundles on U, one can find a hyperplane H such that the restrictions of $\mathcal{E}_0$ and $\mathcal{E}_1$ to $X \cap H$ are again non-isomorphic. When viewed as

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points of $\mathcal{M}_r(X, D, R)$, this means geometrically that $\mathcal{E}_0$ and $\mathcal{E}_1$ do not lie in the same fibre of $\mathcal{M}_r(X, D, R) \to \mathcal{M}_r(X \cap H, f_H^{-1}(D), f_H^* R)$ where $f_H : X \cap H \to X$ is the inclusion. One thus expects the induced morphism of schemes

$$\mathcal{M}_r(X, D, R) \longrightarrow \prod_H \mathcal{M}_r(X \cap H, f_H^{-1}(D), f_H^* R)$$

to be a closed immersion. Since $\mathcal{M}_r(X, D, R)$ should be of finite type, this implies the existence of hyperplanes $H_1, \ldots, H_N$ depending only on X, D, R and r such that if we put $f : \bigsqcup X \cap H_i \to X$, then $\mathcal{M}_r(X, D, R) \to \mathcal{M}_r(Y, f^{-1}(D), f^* R)$ is a closed immersion. Put in the language of flat bundles, two objects of $\mathrm{MIC}_r(X, D, R)$ are isomorphic if and only their pull-back to $\bigsqcup X \cap H_i$ are isomorphic. This is the *Lefschetz recognition principle*. We are led to introduce the following.

DEFINITION 0.2. Let X be a smooth projective variety of dimension $n \geq 2$ over a field k of characteristic 0. Let D be a simple normal crossing divisor of X. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Let $\mathcal{C}$ be a class of flat vector bundles on $X - D$. Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{P}$ meeting X transversally. We say that $\mathcal{H}$ *realizes the Lefschetz recognition principle for* $\mathcal{C}$ if for every $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{C}$, the flat bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic if and only if $\mathcal{M}_1|_{X \cap H}$ and $\mathcal{M}_2|_{X \cap H}$ are isomorphic for every $H \in \mathcal{H}$.

Our second main result is a proof of the Lefschetz recognition principle for $\mathcal{C} = \mathrm{MIC}_r(X, D, R)$ in any dimension.

THEOREM 0.3. *Let X be a smooth projective variety of dimension $n \geq 2$ over k. Let D be a simple normal crossing divisor of X. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Then, there exists a polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ of degree 4 such that for every effective divisor R of X supported on D, for every integer $r \geq 0$, there is a dense open subset of $\Omega(R, r) \subset (\mathbb{P}^\vee)^{K(R,r)}$ such that every $(H_1, \ldots, H_{K(R,r)}) \in \Omega(R, r)$ realizes the Lefschetz recognition principle for* $\mathrm{MIC}_r(X, D, R)$.

See Theorem 7.5 for explicit conditions defining the dense open set $\Omega(R, r)$.

Although the statements of cohomological boundedness and the Lefschetz recognition principle were derived separately from the existence of $\mathcal{M}_r(X, D, R)$, it turns out that *for surfaces*, the latter is a consequence of the former. Let us explain how. Let $j : U \to X$ be the open immersion and let $\mathcal{E}$ be a flat bundle on U. The $\mathcal{D}_X$-module $j_* \mathcal{E}$ is coherent and as such, a canonical $\dim U$-cycle $CC(j_* \mathcal{E})$ of the cotangent bundle $T^* X$ was attached to it by Kashiwara and Schapira [KS90]. This is the *characteristic cycle of* $j_* \mathcal{E}$. The cycle $CC(j_* \mathcal{E})$ tells how far the $\mathcal{D}_X$-module $j_* \mathcal{E}$ is from being a flat connection on X and thus provides a geometric measure of the complexity of the differential system underlying $j_* \mathcal{E}$. The cycle $CC(j_* \mathcal{E})$ is Lagrangian. If X is a surface, $CC(j_* \mathcal{E})$ is thus a combination of the zero section of $T^* X$, the conormal bundles of the components of D and the conormal bundles of some points of D. If $\mathcal{E}'$ is another flat bundle on U, then any smooth curve transverse to D and avoiding the points whose conormal bundle contributes to $CC(j_* \mathcal{H}\mathrm{om}(\mathcal{E}_1, \mathcal{E}_2))$ where $\mathcal{E}_1, \mathcal{E}_2 \in \{\mathcal{E}, \mathcal{E}'\}$ distinguishes $\mathcal{E}$ and $\mathcal{E}'$ after restriction. See Corollary 7.4. This observation translates the statement of Theorem 0.3 into the problem of finding a universal bound for the number of points of D whose conormal bundle contributes to $CC(j_* \mathcal{E})$ for $\mathcal{E}$ in $\mathrm{MIC}_r(X, D, R)$. This question is cohomological since by Kashiwara and Dubson's formula [Dub84], every such point contributes to de Rham cohomology. This is how Theorem 0.3 follows from Theorem 0.1 for surfaces.

The main result of [Tey23] implies that the points of the smooth locus of D whose conormal bundle contributes to $CC(j_* \mathcal{E})$ and $CC(j_* \mathcal{E} \,\mathrm{nd}\, \mathcal{E})$ are exactly the turning points of $\mathcal{E}$ along D. From the above discussion, cohomological boundedness thus yields a universal bound on the

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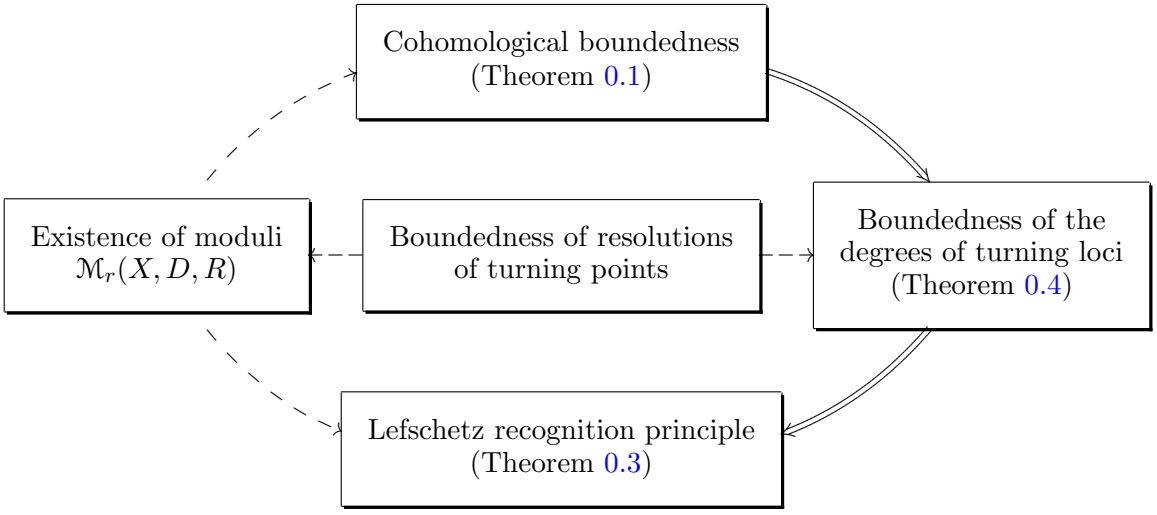
number of turning points of objects in $\mathrm{MIC}_r(X, D, R)$. We prove in any dimension a stronger universal bound (Theorem 6.9).

Theorem 0.4. *Let X be a smooth projective variety over k. Let D be a normal crossing divisor of X. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Then, there exists a quadratic polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{E}$ of $\mathrm{MIC}_r(X, D, R)$, the degree of the turning locus of $\mathcal{E}$ along D is smaller than $K(R, r)$.*

For a flat bundle $\mathcal{E}$, Kedlaya [Ked11] and Mochizuki [Moc11] proved that the turning locus of $\mathcal{E}$ along D can be eliminated by enough blow-up. Theorem 0.4 is thus consistent with Esnault and Langer's expectation that there should exist a universal bound on the number of blow-up needed to achieve good formal decomposition for objects in $\mathrm{MIC}_r(X, D, R)$. Their insight is that controlling the number of blow-ups should give the required bound on the Chern classes of Deligne–Malgrange lattices to construct the moduli $\mathcal{M}_r(X, D, R)$. By purity of turning loci proved by André for D smooth [And07] and by Kedlaya in general [Ked21], we deduce the following (Theorem 6.10).

Theorem 0.5. *Let X be a smooth projective variety over k. Let D be a normal crossing divisor of X. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Then, there exists a quadratic polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{E}$ of $\mathrm{MIC}_r(X, D, R)$, the set of irreducible components of the turning locus of $\mathcal{E}$ along D is smaller than $K(R, r)$.*

We summarize the interplay (at least for surfaces) of the above results and heuristic in the following diagram, to be understood with bounded rank and irregularity.



We finally introduce the main tool of this paper and describe an extra application. Let X be a smooth connected variety over k. Let D be a normal crossing divisor of X. Let $\mathcal{E}$ be a flat bundle on $j : U := X \setminus D \to X$. For every proper birational morphism $p : Y \to X$ where Y is smooth and $E := p^{-1}(D)$ is a normal crossing divisor, Kedlaya attached to $\mathcal{E}$ an effective divisor $\mathrm{Irr}(Y, p^+\mathcal{E})$ supported on E whose coefficient along a component Z is the generic irregularity number of $\mathcal{E}$ along Z or, equivalently, the coefficient of $T^*_Z Y$ in $CC(\jmath_*\mathcal{E})$, where $\jmath : U \to Y$ is the inclusion. This collection of divisors organizes into an element $\mathrm{Irr}\,\mathcal{E}$ of the group $\mathbf{Div}(X) := \varprojlim_{Y \to X} \mathrm{Div}(Y)$ of b-divisors, where $p : Y \to X$ runs over the poset of morphisms as above and where the transition maps are push-forward. In particular, an element of $\mathbf{Div}(X)$ is a $\mathbb{Z}$-valued function on the set

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of divisorial valuations on X. The b-divisor $\operatorname{Irr}\mathcal{E}$ is the *irregularity b-divisor of $\mathcal{E}$ along D*. In the group $\mathbf{Div}(X)$ lies the subgroup of *Cartier b-divisor* defined as $\varinjlim_{Y\to X}\operatorname{Div}(Y)$ where the transition maps are pull-backs. Fundamental to this paper is Kedlaya's theorem [Ked21] that $\operatorname{Irr}\mathcal{E}$ is a nef Cartier b-divisor with the property that $\operatorname{Irr}\mathcal{E}$ and $\operatorname{Irr}\mathcal{E}\mathrm{nd}\,\mathcal{E}$ lie in the subgroup $\operatorname{Div}(X)$ of $\mathbf{Div}(X)$ if and only if $\mathcal{E}$ has good formal decomposition along D. We show in § 5.3 that $\operatorname{Irr}\mathcal{E}$ and $\operatorname{rk}\mathcal{E}$ determine $CC(j_*\mathcal{E})$, and it is an intriguing question to understand how $CC(j_*\mathcal{E})$ relates to $\operatorname{Irr}\mathcal{E}$. If X is a surface, we answer this question using a new operation on b-divisors, the *partial discrepancy* $\delta:\mathbf{Div}(X)\to\mathbf{Div}(X)$. The partial discrepancy measures the failure of a b-divisor Z to lie in the image of $\operatorname{Div}(X)\to\mathbf{Div}(X)$ in the sense that if it does, then $\delta Z=0$. We prove that if Z is a nef Cartier b-divisor, then δZ is an effective b-divisor with finite support when viewed as a function on the set of divisorial valuations on X. Hence, the sum of its values, denoted by $\int_X\delta Z$ is a well-defined positive integer. For a subset A of X, we denote by $\int_A\delta Z$ the sum of the values of δZ over the set of divisorial valuations whose centres on X lie in A. Using the partial discrepancy, we prove the following formula for the characteristic cycle of flat bundles on surfaces (Theorem 5.18).

Theorem 0.6. *Let X be a smooth projective surface over an algebraically closed field of characteristic 0. Let D be a normal crossing divisor of X. Let $\mathcal{E}$ be a flat bundle on $j:U:=X\setminus D\to X$. Then*

$$CC(j_*\mathcal{E})=\operatorname{rk}\mathcal{E}\cdot CC(\mathcal{O}_X(*D))+LC(\operatorname{Irr}(X,\mathcal{E}))+\sum_{P\in D}\left(\int_P\delta\operatorname{Irr}\mathcal{E}\right)\cdot T^*_PX.$$

In the above formula, $LC(\operatorname{Irr}(X,\mathcal{E}))$ is a Lagrangian cycle depending only on the divisor $\operatorname{Irr}(X,\mathcal{E})$ supported on D attached to $\mathcal{E}$. In particular, it depends on $\mathcal{E}$ only via generic data along the components of D. A nice feature of the above formula is to make explicit the means by which the lack of good formal decomposition reflects in the characteristic cycle. By definition of the partial discrepancy, it shows, in particular, that the characteristic cycle is only sensitive to the turning points lying in the smooth locus of the successive inverse images of D by the successive blow-up needed to achieve good formal decomposition. From the Kashiwara and Dubson formula [Dub84], we deduce the following Grothendieck–Ogg Shafarevich-type formula for flat bundles on surfaces (Theorem 5.20).

Theorem 0.7. *Let X be a smooth projective surface over $\mathbb{C}$. Let D be a normal crossing divisor of X. Let $\mathcal{E}$ be a flat bundle on $j:U:=X\setminus D\to X$. Then*

$$\chi(U,\operatorname{DR}\mathcal{E})=\operatorname{rk}\mathcal{E}\cdot\chi(U(\mathbb{C}))+(LC(\operatorname{Irr}(X,\mathcal{E})),T^*_XX)_{T^*X}+\int_D\delta\operatorname{Irr}\mathcal{E}.$$

We now give a linear overview of the paper. Section 1 gathers some general material on $\mathcal{D}$-modules as well as Kedlaya's results on irregularity b-divisors. Section 2 introduces the cohomological boundedness conjecture for flat bundles with bounded rank and irregularity at infinity. The main upshot of § 2 is Corollary 2.17 stating that cohomological boundedness is equivalent to an *a priori* weaker conjecture, the *χ-boundedness conjecture* asking for a universal bound on the global Euler–Poincaré characteristic of the de Rham cohomology. Section 3 is the main technical core of this paper. In a relative situation, it provides a mechanism for deducing χ-boundedness out of χ-boundedness for the generic fibre provided the turning locus is contained in a fibre. See Proposition 3.23. Section 4 is devoted to the construction of the partial discrepancy b-divisor attached to a b-divisor on a smooth surface. Its upshot is Proposition 4.15 ensuring that the partial discrepancy of a nef Cartier b-divisor is a b-divisor with finite support when viewed as a function on the set of divisorial valuations. Section 5 is an application of § 4 to the proof of

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Theorems 0.6 and 0.7. Section 6 gives the proof of cohomological boundedness for surfaces. Section 7 provides the proof of the Lefschetz recognition principle. In § 8, the techniques of this paper are used to obtain a Lefschetz theorem for the differential Galois group of flat bundles under some uncountability assumption of the base field.

1. Geometric and $\mathcal{D}$-module preparations

1.1 Base field

In this paper, k will denote a field of characteristic 0. If X is a variety over k, we denote by X^{sm} the smooth locus of X and by X^{sing} the singular locus of X. If $k \subset K$ is a field extension, we denote by X_K the pull-back of X over K.

1.2 Pair of varieties

A *pair over* k is the data (X, D) of a smooth variety X over k with a reduced divisor D. An *analytic pair* is the data (X, D) of a complex manifold X with a reduced divisor D. In both situations, we denote by $\mathrm{Div}(X, D)$ the group of divisors of X supported on D, that is, $\mathrm{Div}(X, D)$ is the free abelian group over the set of irreducible components of D. We denote by $\mathrm{fdeg} : \mathrm{Div}(X, D) \to \mathbb{Z}$ the formal degree function, that is the group morphism sending each irreducible component of D to 1. Note that if $k \subset K$ is a field extension, then $\mathrm{fdeg}\, R \leq \mathrm{fdeg}\, R_K$, where R_K denotes the pull-back of R to X_K.

If P is a property of algebraic or analytic varieties, a P-pair (X, D) will refer to a pair (X, D) such that X satisfies P. Finally, a P-normal crossing pair (X, D) is a P-pair such that D has normal crossing.

A *morphism of pairs* $f : (Y, E) \to (X, D)$ *over* k is a morphism $f : X \to Y$ of algebraic varieties over k such that $f^{-1}(D) = E$.

1.3 Transversality

We recall the transversality conditions from [Bei16].

Definition 1.1. Let $f : Y \to X$ be a morphism between smooth varieties over k. Let C be a closed conical subset of T^*X. Let $\overline{y}$ be a geometric point above a point y of Y. We say that $f : Y \to X$ is C-transversal at y if

$$\mathrm{Ker}\, df_{\overline{y}} \cap C_{f(\overline{y})} \subset \{0\} \subset T^*_{f(\overline{y})}X,$$

where $df_{\overline{y}} : T^*_{f(\overline{y})}X \to T^*_{\overline{y}}Y$ is the cotangent map of f at $\overline{y}$. We say that $f : Y \to X$ is C-transversal if it is C-transversal at every point of Y.

If $f : Y \to X$ is C-transversal, let $f^\circ C$ be the scheme-theoretic image of $Y \times_X C$ in T^*Y by the canonical map $df : Y \times_X T^*X \to T^*Y$. As proved in [Bei16, Lemma 1.2], the map $df : Y \times_X C \to f^\circ C$ is finite and $f^\circ C$ is a closed conical subset of T^*Y.

In the next transversality criterion, we use an extra operation on closed conical subsets of cotangent bundles. Let $f : X \to Y$ be a proper morphism between smooth varieties over k. Let C be a closed conical subset of T^*X. We denote by $f_\circ C$ the closed conical subset of T^*Y defined as the image of $df^{-1}(C) \subset X \times_Y T^*Y$ by the projection $X \times_Y T^*Y \to T^*Y$, where $df : X \times_Y T^*Y \to T^*X$ is the canonical map.

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Lemma 1.2. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{j} & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & P \end{array} \tag{1.3}$$

be a cartesian diagram of smooth varieties over k whose arrows are closed immersions. Let C be a closed conical subset of T^*X. Let $x \in Y$ and let us abuse notation by viewing x as a point in X, H and P. Then, the following statements hold.

(i) *If g is $i_\circ C$-transversal at x, then f is C-transversal at x.*
(ii) *Assume that X and H are transverse at x. If f is C-transversal at x, then g is $i_\circ C$-transversal at x.*

Proof. Assume that f is not C-transversal at x. This means that there exists a non-zero form ω in C_x vanishing on $T_xY \subset T_xX$. Since (1.3) is cartesian, so is the following induced diagram.

$$\begin{array}{ccc} T_xY & \longrightarrow & T_xH \\ \downarrow & & \downarrow \\ T_xX & \longrightarrow & T_xP \end{array}$$

Via the choice of a supplementary for T_xY in T_xH and a supplementary for $T_xX + T_xH$ in T_xP, we can extend ω to a non-zero form η on T_xP vanishing on T_xH. In particular, η lies in $i_\circ C$ and g is not $i_\circ C$-transversal at x and part (i) is proven.

We now prove part (ii). Assume that T_xX and T_xH generate T_xP and that f is C-transversal at x. We argue by contradiction. Let η be a non-zero form of $i_\circ C$ above x such that η vanishes on T_xH. If $i^*\eta = 0$, the fact that T_xX and T_xH generate T_xP yields $\eta = 0$, which is not possible. Thus, $i^*\eta$ is a non-zero form of C above x. By the transversality assumption, $i^*\eta$ does not vanish on T_xY, so η does not vanish on T_xH. Contradiction. This concludes the proof of Lemma 1.2. □

We gather in Lemma 1.4 standard facts on transversality. See [Bei16, 1.2,2.2] and [Sai17, 3.4].

Lemma 1.4. *Let $f : Y \to X$ be a morphism of smooth varieties over k. Let $C, C' \subset T^*X$ be closed conical subsets. Let $x \in X$. Then the following hold.*

(i) *Assume $C_x \subset C'_x$. If f is C'-transversal at x, then f is C-transversal at x.*
(ii) *If f is C'-transversal at x and C-transversal at x, then f is $C \cup C'$-transversal at x.*
(iii) *Let $g : Z \to Y$ be a morphism of smooth varieties over k. The following conditions are equivalent:*
  (a) *f is C-transversal on a neighbourhood of $g(Z)$ and g is $f^\circ(C)$-transversal;*
  (b) *$f \circ g$ is C-transversal.*
(iv) *If $C'' \subset T^*Y$ is a closed conical subset, the set of points $y \in Y$ at which f is C''-transversal is open in Y.*

1.4 Universal hyperplane

Let E be a finite-dimensional vector space over k. Let E^* be its dual. Let $\mathbb{P} = \mathrm{Proj}(\mathrm{Sym}\, E^*)$ and $\mathbb{P}^\vee = \mathrm{Proj}(\mathrm{Sym}\, E)$ be the associated projective spaces. The closed subscheme of $\mathbb{P} \times_k \mathbb{P}^\vee$

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defined by

$$Q := \{(x, H) \in \mathbb{P} \times_k \mathbb{P}^\vee \mid x \in H\}$$

is the universal family of hyperplanes of $\mathbb{P}$. Let (X, D) be a quasi-projective normal crossing pair over k. Let $i : X \to \mathbb{P}$ be an immersion. Put $X_Q = X \times_{\mathbb{P}} Q$. Denote by $p_X^\vee : X_Q \to \mathbb{P}^\vee$ the composition of $X_Q \to Q$ with the canonical projection $p^\vee : Q \to \mathbb{P}^\vee$. Then, Bertini's theorem ensures the existence of a dense open subset $V \subset \mathbb{P}^\vee$ of hyperplanes H transverse to X such that $D \cap H$ has normal crossings. In particular, if η denotes the generic point of $\mathbb{P}^\vee$, we have the following commutative diagram with cartesian squares:

$$\begin{array}{ccccccc}
X_\eta & \longrightarrow & X_V & \longrightarrow & X_Q & \xrightarrow[p_X]{} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q_\eta & \longrightarrow & Q_V & \longrightarrow & Q & \xrightarrow[p]{} & \mathbb{P} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p^\vee} & & \\
\eta & \longrightarrow & V & \longrightarrow & \mathbb{P}^\vee & &
\end{array}$$

where (X_η, D_η) is a quasi-projective pair over η of dimension $\dim X - 1$ and where D_η has normal crossings. The pair (X_η, D_η) is the *generic hyperplane section of* (X, D).

The following lemma provides a slight generalization of [Sai21, 1.3.7 (2)].

Lemma 1.5. *Let $\mathbb{P}$ be a projective space over k. Let X be a smooth subvariety of $\mathbb{P}$. Let $C \subset T^*X$ be a closed conical subset of pure dimension $\dim X$. Then there exists a dense open subset of hyperplanes H transverse to X such that $X \cap H \to X$ is C-transversal.*

Proof. By Bertini's theorem, there exists a dense open subset $V_1 \subset \mathbb{P}^\vee$ of hyperplanes transverse to X. Observe that $i_\circ C$ has pure dimension $\dim \mathbb{P}$. Thus, [Sai21, 1.3.7 (2)] ensures that there exists a dense open subset $V_2 \subset \mathbb{P}^\vee$ of hyperplanes H such that $H \to \mathbb{P}$ is $i_\circ C$-transversal. From Lemma 1.2(i), $V_1 \cap V_2$ thus meets our requirements. □

1.5 Characteristic cycle for coherent $\mathcal{D}$-modules

Let k be a field of characteristic 0. Let X be a smooth variety over k. We endow $\mathcal{D}_X$ with its filtration F by the order of differential operators. In particular, there is a canonical isomorphism $\mathrm{gr}^F \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X}$ where $\pi : T^*X \to X$ is the canonical projection. Every coherent $\mathcal{D}_X$-module $\mathcal{M}$ admits a filtration $F_\mathcal{M}$ compatible with F such that $\mathrm{gr}^{F_\mathcal{M}} \mathcal{M}$ is a coherent $\pi_* \mathcal{O}_{T^*X}$-module. Then,

$$\widetilde{\mathrm{gr}^{F_\mathcal{M}} \mathcal{M}} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_* \mathcal{O}_{T^*X}} \mathrm{gr}^{F_\mathcal{M}} \mathcal{M}$$

is a coherent $\mathcal{O}_{T^*X}$-module. Its characteristic cycle is a cycle of T^*X which does not depend on a choice of filtration as above. It is the *characteristic cycle of* $\mathcal{M}$. We denote it by $CC(\mathcal{M})$. The support of $CC(\mathcal{M})$ is a closed conical subset of T^*X called the *singular support of* $\mathcal{M}$. We denote it by $SS(\mathcal{M})$.

Remark 1.6. The formation of the characteristic cycle commutes with flat base change. In particular, it commutes with base field extension and analytification.

As a consequence of [Gab81], we have the following.

Theorem 1.7. *For any coherent D_X-module $\mathcal{M}$, the irreducible components of $SS(\mathcal{M})$ have dimension $\geq \dim X$.*

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Definition 1.8. We say that a coherent D_X-module $\mathcal{M}$ is *holonomic* if $SS(\mathcal{M})$ has pure dimension $\dim X$.

Holonomy is the correct finiteness condition for D_X-modules, as will be clear from Proposition 1.10 below.

1.6 De Rham cohomology

Let k be a field of characteristic 0. Let X be a smooth algebraic k-variety of dimension d. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. We denote by

$$\mathrm{DR}\,\mathcal{M} : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega^d_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

the algebraic de Rham complex of $\mathcal{M}$, where $\mathcal{M}$ lies in degree 0. The *algebraic de Rham cohomology of* $\mathcal{M}$ is the cohomology of $\mathrm{DR}\,\mathcal{M}$.

If $k = \mathbb{C}$, let $\mathcal{M}^{\mathrm{an}}$ be the analytification of $\mathcal{M}$ and define similarly the analytic de Rham complex of $\mathcal{M}$. The *analytic de Rham cohomology of* $\mathcal{M}$ is the cohomology of $\mathrm{DR}\,\mathcal{M}^{\mathrm{an}}$.

In the proper complex setting, the GAGA theorem for quasi-coherent cohomology identifies algebraic and analytic de Rham cohomology. See [Del70, 6.6.1]. This is the following.

Proposition 1.9. *Let X be a smooth projective variety over $\mathbb{C}$. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{D}_X$-module. Then, the canonical comparison morphism*

$$H^*(X, \mathrm{DR}\,\mathcal{M}) \longrightarrow H^*(X(\mathbb{C}), \mathrm{DR}\,\mathcal{M}^{\mathrm{an}})$$

is an isomorphism.

When *holonomy* is imposed, algebraic de Rham cohomology is finite dimensional even when the ambient variety is not proper.

Proposition 1.10. *Let X be a smooth variety over k. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then, for every integer n, the space $H^n(X, \mathrm{DR}\,\mathcal{M})$ is finite dimensional over k and vanishes if $n \neq 0, \ldots, 2\dim X$.*

Proof. Since the formation of the de Rham complex commutes with push-forward [HTT00, 4.2.5] and since holonomy is preserved under push-forward [HTT00, 3.2.3], we can at the cost of replacing X by a smooth compactification suppose that X is proper over k. From Remark 1.11, we reduce to the case where $k = \mathbb{C}$. From Proposition 1.9, we are left to prove a variant of Proposition 1.10 where now X is a smooth compact complex manifold and where $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. From Kashiwara's perversity theorem [Kas75], we are left to prove that for every perverse complex $\mathcal{F}$ on X, the $\mathbb{C}$-vector space $H^n(X, \mathcal{F})$ is finite dimensional and vanishes if $n \neq -\dim X, \ldots, \dim X$. This is a standard fact from the theory of perverse complexes [BBDG18, 4.2.4]. □

Remark 1.11. As a consequence of the invariance of quasi-coherent cohomology under flat base change, de Rham cohomology is invariant under base field extensions.

1.7 The solution and the irregularity complexes

Let X be a complex manifold of dimension d. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. The *solution complex of* $\mathcal{M}$ is defined as

$$\mathrm{Sol}\,\mathcal{M} := \mathrm{R}\,\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

The following theorem is due to Kashiwara [Kas75].

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Theorem 1.12. *Let X be a smooth complex manifold. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then the complexes* $\operatorname{Sol}\mathcal{M}[\dim X]$ *and* $\operatorname{DR}\mathcal{M}[\dim X]$ *are perverse complexes.*

Let Z be a hypersurface in X. Let $i : Z \to X$ be the inclusion. Let $\mathcal{M}(*Z)$ be the localization of $\mathcal{M}$ along Z. Define the *irregularity complex of* $\mathcal{M}$ *along* Z by

$$\operatorname{Irr}_Z^* \mathcal{M} := i_* i^* \operatorname{Sol}\mathcal{M}(*Z)$$

The following theorem is due to Mebkhout [Meb90].

Theorem 1.13. *Let X be a complex manifold. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let Z be a hypersurface in X. Then* $\operatorname{Irr}_Z^* \mathcal{M}[\dim X]$ *is a perverse complex supported on Z.*

The de Rham and solution complexes are related under sheaf duality; see [Meb82].

Theorem 1.14. *Let X be a complex manifold. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. There is a canonical isomorphism of complexes*

$$\operatorname{DR}\mathcal{M} \longrightarrow \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{C}}(\operatorname{Sol}\mathcal{M}, \mathbb{C})$$

in the derived category of complexes of sheaves.

1.8 Characteristic cycle and the de Rham and solution complexes

For a complex of sheaves $\mathcal{F}$ with bounded and constructible cohomology, Kashiwara and Schapira defined the characteristic cycle $CC(\mathcal{F})$ by means of microlocal analysis [KS90]. The following theorem identifies the characteristic cycle of a holonomic $\mathcal{D}_X$-module with that of its solution complex. See [KS90, 11.3.3] and [Dub84, Theorem 4].

Theorem 1.15. *Let X be a complex manifold. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then*

$$SS(\mathcal{M}) = SS(\operatorname{Sol}\mathcal{M}) \quad \textit{and} \quad CC(\mathcal{M}) = CC(\operatorname{Sol}\mathcal{M}).$$

In particular, the computation of $CC(\mathcal{M})$ can be reduced to a sheaf-theoretic question. If the singular support of $\mathcal{M}$ is already known, the following theorem [Kas75, Theorem 3.5] tells us how to chop off X in order to compute $CC(\operatorname{Sol}\mathcal{M})$.

Theorem 1.16. *Let X be a complex manifold. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let $X_1, \ldots, X_n$ be a Whitney stratification of X such that $SS(\mathcal{M})$ lies in $\bigcup_{i=1}^n T^*_{X_i} X$. Then, the cohomology sheaves of* $(\operatorname{Sol}\mathcal{M})|_{X_i}$ *are local systems on X_i, $i = 1, \ldots, n$.*

From a cohomological perspective, computing the characteristic cycle is useful because of Kashiwara and Dubson's formula [Dub84]. See also [Lau83].

Theorem 1.17. *Let X be a proper complex manifold. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then, we have*

$$\chi(X, \operatorname{DR}\mathcal{M}) = (CC(\mathcal{M}), T^*_X X)_{T^*X}.$$

where $\chi(X, \operatorname{DR}\mathcal{M})$ *denotes the Euler–Poincaré characteristic of* $\operatorname{DR}\mathcal{M}$ *and where* $(-,-)_{T^*X}$ *denotes the intersection number of cycles in T^*X.*

1.9 Characteristic cycle and functorialities

The characteristic cycle for $\mathcal{D}$-modules commutes with proper push-forward. This is the following.

Lemma 1.18. *Let $f : X \to Y$ be a proper morphism between smooth varieties over k. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then, $CC(f_+\mathcal{M}) = f_* CC(\mathcal{M})$.*

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Proof. From Remark 1.6, we can suppose $k = \mathbb{C}$. We have the following chain of equalities:

$$\begin{aligned} CC(f_+\mathcal{M}) &= CC(f_+^{\mathrm{an}}\mathcal{M}^{\mathrm{an}}) = CC(\mathrm{Sol}(f_+^{\mathrm{an}}\mathcal{M}^{\mathrm{an}})) \\ &= CC(Rf_*\,\mathrm{Sol}\,\mathcal{M}^{\mathrm{an}}) = f_*CC(\mathrm{Sol}\,\mathcal{M}^{\mathrm{an}}) = f_*CC(\mathcal{M}). \end{aligned}$$

The first equality follows from Remark 1.6 applied to $f_+\mathcal{M}$. The second equality follows from Theorem 1.15. The third equality follows from the compatibility of the formation of Sol with proper push-forward. The fourth equality follows from [KS90, 9.4.2]. The last equality follows from the above arguments applied to $\mathcal{M}$. □

Definition 1.19. Let $f : Y \to X$ be a morphism between smooth varieties over k or between complex manifolds. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. We say that $f : Y \to X$ is non-characteristic for $\mathcal{M}$ if it is $SS(\mathcal{M})$-transversal.

The following results are due to Kashiwara [Kas95].

Theorem 1.20. *Let $f : Y \to X$ be a morphism between smooth varieties over k. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module such that $f : Y \to X$ is non-characteristic for $\mathcal{M}$. Then the following statements hold.*

(i) *The $\mathcal{D}_Y$-module pull-back $f^+\mathcal{M}$ is concentrated in degree 0. That is, $f^+\mathcal{M} \simeq f^*\mathcal{M}$.*
(ii) *We have $SS(f^+\mathcal{M}) = f^\circ SS(\mathcal{M})$.*
(iii) *If $k = \mathbb{C}$, the comparison morphisms*

$$f^*\,\mathrm{Sol}\,\mathcal{M}^{\mathrm{an}} \longrightarrow \mathrm{Sol}\,f^+\mathcal{M}^{\mathrm{an}} \quad \text{and} \quad \mathrm{DR}\,f^+\mathcal{M}^{\mathrm{an}} \longrightarrow f^*\,\mathrm{DR}\,\mathcal{M}^{\mathrm{an}}$$

are isomorphisms in the derived category of sheaves.

1.10 Meromorphic flat connections

Let X be a smooth algebraic variety over k. A *flat connection on X* or *module with integrable connection on X* is a $\mathcal{D}_X$-module $\mathcal{E} := (E, \nabla)$ whose underlying $\mathcal{O}_X$-module E is a vector bundle of finite rank on X. We denote by $\mathrm{MIC}(X)$ the category of flat connections on X.

If D is a divisor in X, a *meromorphic flat connection on X with poles along D* is a $\mathcal{D}_X$-module $\mathcal{M} := (M, \nabla)$ whose underlying $\mathcal{O}_X$-module M is a locally free sheaf of $\mathcal{O}_X(*D)$-modules of finite rank. We denote by $\mathrm{MIC}(X, D)$ the category of meromorphic flat connections on X with poles along D.

Remark 1.21. The above definitions make sense in the analytic setting, where the same notation will be used.

Remark 1.22. Meromorphic flat connections are holonomic $\mathcal{D}$-modules.

In the algebraic setting, there is no difference between flat connections and meromorphic flat connections. This is expressed by the following proposition [HTT00, 5.3.1].

Proposition 1.23. *Let (X, D) be a pair over k. Let $j : U = X \setminus D \to X$ be the inclusion. Then (j_*, j^*) induce an equivalence of categories between $\mathrm{MIC}(U)$ and $\mathrm{MIC}(X, D)$.*

Since all the action in this paper happens at infinity, we will use the meromorphic viewpoint. The next lemma says that this does not make any difference for cohomology.

Lemma 1.24. *Let (X, D) be a pair over k. Put $U := X \setminus D$ and let $j : U \to X$ be the inclusion. Let $\mathcal{E}$ be an object in $\mathrm{MIC}(U)$. Then the canonical restriction morphism*

$$R\Gamma(X, \mathrm{DR}\,j_*\mathcal{E}) \longrightarrow R\Gamma(U, \mathrm{DR}\,\mathcal{E})$$

is an isomorphism in the derived category of vector spaces over k.

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Proof. We have

$$R\Gamma(U, \mathrm{DR}\,\mathcal{E}) \simeq R\Gamma(X, Rj_*\,\mathrm{DR}\,\mathcal{E}) \simeq R\Gamma(X, j_*\,\mathrm{DR}\,\mathcal{E}) \simeq R\Gamma(X, \mathrm{DR}\,j_*\mathcal{E}),$$

where the second equality comes from the fact that j is an affine morphism, and where the last equality follows from $j_*\,\mathrm{DR}\,\mathcal{E} \simeq \mathrm{DR}\,j_*\mathcal{E}$. □

Proposition 1.25. *Let (X, D) be a pair over k. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be objects of $\mathrm{MIC}(X, D)$. Then, there is a canonical isomorphism*

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_2) \longrightarrow R\Gamma(X, \mathrm{DR}\,\mathcal{H}\mathrm{om}(\mathcal{M}_1, \mathcal{M}_2))$$

in the derived category of k-vector spaces.

Proof. Combining [Bor87, VII 9.8] and [Bor87, VI 5.3.2] yields a canonical isomorphism

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_2) \simeq R\Gamma(X, \mathrm{DR}\,\mathcal{M}_1^\vee \otimes^L_{\mathcal{O}_X} \mathcal{M}_2),$$

where $\mathcal{M}_1^\vee$ denotes the $\mathcal{D}_X$-module dual to $\mathcal{M}_1$. Since $\mathcal{M}_2$ is a locally free $\mathcal{O}_X$-module localized along D, we have

$$\mathcal{M}_1^\vee \otimes^L_{\mathcal{O}_X} \mathcal{M}_2 \simeq \mathcal{M}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{M}_2 \simeq \mathcal{M}_1^\vee(*D) \otimes_{\mathcal{O}_X} \mathcal{M}_2.$$

Observe that $\mathcal{M}_1^\vee(*D)$ is an object of $\mathrm{MIC}(X, D)$. Let $\mathcal{M}_1^*$ be the meromorphic connection dual to $\mathcal{M}_1$. Then [HTT00, 2.6.10] ensures that the restriction of $\mathcal{M}_1^\vee(*D)$ and $\mathcal{M}_1^*$ to $U := X \setminus D$ are canonically isomorphic. From Proposition 1.23, we deduce that $\mathcal{M}_1^\vee(*D)$ and $\mathcal{M}_1^*$ are canonically isomorphic. We thus conclude the proof of Proposition 1.25 using the identification $\mathcal{M}_1^* \otimes_{\mathcal{O}_X} \mathcal{M}_2 \simeq \mathcal{H}\mathrm{om}(\mathcal{M}_1, \mathcal{M}_2)$. □

1.11 Good formal structure for connections

In the next definition, we follow [Ked21, Definition 2.1.3] and fix a field k of characteristic 0.

Definition 1.26. Let (X, D) be a normal crossing pair over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. We say that $\mathcal{M}$ has *good formal structure at $x \in D$* if there is a decomposition

$$\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{S} \cong \bigoplus_{\alpha \in I} \mathcal{E}^{\varphi_\alpha} \otimes_{\mathcal{S}} \mathcal{R}_\alpha, \tag{1.27}$$

where

(1) the ring $\widehat{\mathcal{O}}_{X,x}(*D)$ is the localization along D of the completion of $\mathcal{O}_{X,x}$ along the intersection of the irreducible components of D containing x and $\mathcal{S}$ is a finite étale $\widehat{\mathcal{O}}_{X,x}(*D)$-algebra;
(2) I is a finite set;
(3) each $\mathcal{R}_\alpha$ is a regular differential module over $\mathcal{S}$;
(4) φ_α $(\alpha \in I)$ are elements in $\mathcal{S}$ satisfying that, if φ_α does not lie in the integral closure $\mathcal{S}_0$ of $\mathcal{O}_{X,x}$ in $\mathcal{S}$, then φ_α is a unit of $\mathcal{S}$ and $\varphi_\alpha^{-1} \in \mathcal{S}_0$; furthermore, if $\varphi_\alpha - \varphi_\beta$ does not lie in $\mathcal{S}_0$, then $\varphi_\alpha - \varphi_\beta$ is a unit of $\mathcal{S}$ and $(\varphi_\alpha - \varphi_\beta)^{-1} \in \mathcal{S}_0$.

Definition 1.28. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. We say that a point x of D is a *turning point of* $\mathcal{M}$ if $\mathcal{M}$ does not admit a good formal structure at x. The set of turning points of $\mathcal{M}$ is called the *turning locus of* $\mathcal{M}$. We denote it by $\mathrm{TL}(\mathcal{M})$. If $\mathrm{TL}(\mathcal{M})$ is empty, we say that $\mathcal{M}$ *admits good formal structure along D*.

Remark 1.29. The formation of the turning locus commutes with regular base change. See [Ked21, 4.3.2] for a proof.

The compatibility of good formal structure and pull-back follows from Definition 1.26:

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Proposition 1.30. *Let $f : (Y, E) \to (X, D)$ be a morphism of normal crossing pairs over k. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Let x be a point in E such that $\mathcal{M}$ has good formal structure at $f(x)$. Then, $f^+\mathcal{M}$ has good formal structure at x. In particular, if $\mathcal{M}$ has good formal structure, so does $f^+\mathcal{M}$.*

Purity of the turning locus was proved by André when D is smooth [And07, 3.4.3] and by Kedlaya when D has normal crossing [Ked21, 2.3.1].

Theorem 1.31. *Let (X, D) be a normal crossing pair over k. Then, for any object $\mathcal{M}$ of* $\mathrm{MIC}(X, D)$, *the turning locus of $\mathcal{M}$ is a closed subscheme of D of pure codimension* 1.

The following fundamental theorem was proved by Kedlaya [Ked11, Theorem 8.1.3] and Mochizuki [Moc11, Theorem 19.5] with complement from [Ked21, Corollary 3.2.5].

Theorem 1.32. *Let (X, D) be a pair over k. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Then, there exists a morphism of smooth algebraic varieties $f : Y \to X$ obtained as a composition of blow-up with smooth centres above* $\mathrm{TL}(\mathcal{M})$ *such that $f^{-1}(D)$ is a normal crossing divisor of Y and $f^+\mathcal{M}$ admits good formal structure along $f^{-1}(D)$.*

We thank the anonymous referee for pointing out the following.

Remark 1.33. In Theorem 1.32, it is not obvious from [Ked11, Theorem 8.1.3] and [Moc11, Theorem 19.5] that $f : Y \to X$ can be taken as a composition of blow-up with smooth centres above $\mathrm{TL}(\mathcal{M})$. Such control on f is guaranteed by the subsequent results from [Ked21].

1.12 Irregularity number

Let (X, D) be a normal crossing pair over k of dimension $d + 1 \geq 1$. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. In the setting of Definition 1.26, good formal structure always holds at the generic point η of an irreducible component Z of D. In that case, $\widehat{\mathcal{O}}_{X,\eta}(*D)$ has the form $k[\![x_1, \ldots, x_d, y]\!][y^{-1}]$ and S has the form $K[\![x_1, \ldots, x_d, y^{1/m}]\!][(y^{1/m})^{-1}]$ where $m \geq 1$ and where K/k is a finite extension.

Definition 1.34. For an object $\mathcal{M}$ in $\mathrm{MIC}(X, D)$, the *generic slopes of $\mathcal{M}$ along Z* are the poles orders in the y-variable of the φ_α contributing to (1.27). Let us denote them by $\mathrm{ord}_y \varphi_\alpha$ where $\alpha \in I$ and put

$$r(Z, \mathcal{M}) := \max_{\alpha \in I} \mathrm{ord}_y \varphi_\alpha.$$

The *generic irregularity number of $\mathcal{M}$ along Z* is defined as

$$\mathrm{irr}(Z, \mathcal{M}) := \sum_{\alpha \in I} (\mathrm{ord}_y \varphi_\alpha) \cdot \mathrm{rk}\, \mathcal{R}_\alpha.$$

Remark 1.35. In the setting of Definition 1.34, the generic irregularity number of $\mathcal{M}$ along Z is an integer independent of the choice of the finite étale $\widehat{\mathcal{O}}_{X,\eta}(*D)$-algebra S.

In general, irregularity numbers and irregularity complexes are related through a theorem of Malgrange [Mal71, Theorem 1.4]. We record it in the following form useful for us.

Lemma 1.36. *Let (X, D) be an analytic pair. Let Z be an irreducible component of D. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Then, the generic irregularity number of $\mathcal{M}$ along Z is the generic rank along Z of the perverse complex* $\mathrm{Irr}^*_D \mathcal{M}$.

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1.13 *b*-divisors

Let X be a smooth variety over k. A *modification of* X is the datum of a smooth variety Y over k and a map $Y \to X$ which is proper, dominant and an isomorphism away from a nowhere-dense closed subset of X. Following [Ked21], we introduce the following definition.

Definition 1.37. The group of integral b-divisors $\mathbf{Div}(X)$ of X is the limit

$$\varprojlim_{Y \to X} \mathrm{Div}(Y),$$

where $Y \to X$ runs over the category of modifications of X and where the transition maps are push-forward.

For D in $\mathbf{Div}(X)$, we denote by $D(Y)$ the component of D along a modification $Y \to X$. For every irreducible divisor E of Y, we denote by $m(E, D)$ the multiplicity of $D(Y)$ along E.

Definition 1.38. The group of integral Cartier b-divisors $\mathbf{CDiv}(X)$ on X is the colimit

$$\varinjlim_{Y \to X} \mathrm{CDiv}(Y),$$

where $Y \to X$ runs over the category of modifications of X and where the transition maps are pull-back.

Let $\mathrm{ZR}^{\mathrm{divis}}(X)$ be the subset of the Zariski–Riemann space of X consisting of divisorial valuations centred on X. As explained in [Ked21], the group of integral b-divisors identifies with the set of functions $m : \mathrm{ZR}^{\mathrm{divis}}(X) \to \mathbb{Z}$ such that for every modification $Y \to X$, there is only a finite number of divisorial valuations $v \in \mathrm{ZR}^{\mathrm{divis}}(X)$ centred at an irreducible divisor of Y such that $m(v) \neq 0$. Thus, the order on $\mathbb{Z}$ induces an order $\leq$ on $\mathbf{Div}(X)$. Furthermore, there is a canonical injective map

$$\mathbf{CDiv}(X) \longrightarrow \mathbf{Div}(X).$$

One of the main players of this paper is a Cartier b-divisor with an extra property called *nef* in [Ked21]. In view of Lemmas 1.4.9 and 1.4.10 from [Ked21], we can define nef Cartier b-divisor as follows.

Definition 1.39. Let X be a smooth variety over k. Let D be a Cartier b-divisor on X. We say that D is *nef* if for every modification $Y \to X$, we have $D \leq D(Y)$ in $\mathbf{Div}(X)$ where $D(Y)$ is viewed as a b-divisor via $\mathbf{CDiv}(X) \to \mathbf{Div}(X)$.

1.14 The irregularity *b*-divisor

Let (X, D) be a pair over k. We recall the following definition from [Ked21, 3.1.1].

Definition 1.40. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. We denote by $\mathrm{Irr}\,\mathcal{M}$ the unique b-divisor on X such that for every modification $p : Y \to X$, the multiplicity of $\mathrm{Irr}\,\mathcal{M}$ along an irreducible divisor E of Y is the generic irregularity $\mathrm{irr}(E, p^+\mathcal{M})$ of $p^+\mathcal{M}$ along E. We put

$$\mathrm{Irr}(Y, p^+\mathcal{M}) := (\mathrm{Irr}\,\mathcal{M})(Y) \text{ in } \mathrm{Div}(Y).$$

The following lemma is obvious.

Lemma 1.41. *Let (X, D) be a pair over k. Let $k \subset K$ be a field extension. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Then,* $\mathrm{Irr}(X_K, \mathcal{M}_K) = \mathrm{Irr}(X, \mathcal{M})_K$.

Remark 1.42. In the setting of Definition 1.40, one can also define $R(\mathcal{M})$ the unique rational b-divisor on X such that for every modification $p : Y \to X$, the multiplicity of $R(\mathcal{M})$ along an

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irreducible divisor E of Y is the highest generic slope $r(E, p^+\mathcal{M}) \in \mathbb{Q}_{\geq 0}$ of $p^+\mathcal{M}$ along E, and put $R(Y, p^+\mathcal{M}) := (R(\mathcal{M}))(Y)$ in $\mathrm{Div}(Y)_{\mathbb{Q}}$.

Remark 1.43. Equivalently, the b-divisors $R(\mathcal{M})$ and $\mathrm{Irr}\,\mathcal{M}$ will be viewed as $\mathbb{Q}$-valued functions on the set of divisorial valuations of X.

The following theorem is due to Kedlaya [Ked21, 3.2.3].

Theorem 1.44. *Let (X, D) be a normal crossing pair over k. For any object $\mathcal{M}$ of $\mathrm{MIC}(X, D)$, the b-divisor $\mathrm{Irr}\,\mathcal{M}$ is a nef Cartier b-divisor. In particular, we have*

$$\mathrm{Irr}\,\mathcal{M} \leq \mathrm{Irr}(X, \mathcal{M})$$

in $\boldsymbol{Div}(X)$, where $\mathrm{Irr}(X, \mathcal{M})$ is viewed as a b-divisor via $\boldsymbol{CDiv}(X) \to \boldsymbol{Div}(X)$.

Corollary 1.45. *Let (X, D) be a normal crossing pair over k. Let R be an effective divisor of X supported on D. Let $\mathcal{M}$ be an object in $\mathrm{MIC}(X, D)$. Then the following conditions are equivalent:*

(i) $\mathrm{Irr}(X, \mathcal{M}) \leq R$ *in* $\mathrm{Div}(X)$;
(ii) *for every point* 0 *in* D, *for every locally closed smooth curve $C \to X$ in X meeting D at* 0 *only, we have*

$$\mathrm{irr}(0, \mathcal{M}|_C) \leq (C, R)_0$$

where $(C, R)_0$ denotes the intersection number of C with R at 0.

Proof. If condition (ii) holds, we get condition (i) by using generically enough smooth curves transverse to the irreducible components of D. We now show that condition (i) implies condition (ii). To do this, we can suppose that $R = \mathrm{Irr}(X, \mathcal{M})$. Let 0 be a point in D and let $C \to X$ be a locally closed smooth curve meeting D at 0 only. Let $p : Y \to X$ be a modification of X which is an isomorphism above $X \setminus D$. By valuative criterion for properness, the immersion $C \to X$ factors uniquely through an immersion $C \to Y$ followed by p. In particular, we have

$$\mathrm{irr}(0, \mathcal{M}|_C) = \mathrm{irr}(0, (p^+\mathcal{M})|_C).$$

On the other hand, the projection formula yields

$$(C, \mathrm{Irr}(X, \mathcal{M}))_0 = (C, p^* \mathrm{Irr}(X, \mathcal{M}))_0.$$

From Theorem 1.44, we further have

$$\mathrm{Irr}(Y, p^+\mathcal{M}) \leq p^* \mathrm{Irr}(X, \mathcal{M})$$

in $\mathrm{Div}(Y)$. Hence, the sought-after inequality for $X, \mathcal{M}, C \to X$ and $R = \mathrm{Irr}(X, \mathcal{M})$ follows from the analogous inequality for $Y, p^+\mathcal{M}, C \to Y$ and $R = \mathrm{Irr}(Y, p^+\mathcal{M})$. From Theorem 1.32, we are thus left to suppose that $\mathcal{M}$ has good formal structure and that C is transverse to D at 0. In that case, the sought-after inequality is an equality. □

Remark 1.46. For surfaces, Corollary 1.45 was proved by Sabbah [Sab00, 3.2.3].

The property of having good formal structure can be read from the irregularity b-divisor. This is due to Kedlaya [Ked21, 3.1.2] as a consequence of Theorem 1.44.

Theorem 1.47. *Let (X, D) be a normal crossing pair over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Then, $\mathcal{M}$ has good formal structure if and only if*

$$\mathrm{Irr}\,\mathcal{M} = \mathrm{Irr}(X, \mathcal{M}) \quad \textit{and} \quad \mathrm{Irr}\,\mathrm{End}\,\mathcal{M} = \mathrm{Irr}(X, \mathrm{End}\,\mathcal{M})$$

in $\boldsymbol{Div}(X)$.

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The proof of Corollary 1.45 combined with Theorem 1.47 yields the following curve-like criterion for testing good formal structure.

Corollary 1.48. *Let (X, D) be a normal crossing pair over k. Then, an object $\mathcal{M}$ in $\mathrm{MIC}(X, D)$ has good formal structure if and only if for every point 0 in D, for every locally closed smooth curve $C \to X$ in X meeting D at 0 only, we have*

$$\mathrm{irr}(0, \mathcal{M}|_C) = (C, \mathrm{Irr}(X, \mathcal{M}))_0 \quad \text{and} \quad \mathrm{irr}(0, (\mathrm{End}\,\mathcal{M})|_C) = (C, \mathrm{Irr}(X, \mathrm{End}\,\mathcal{M}))_0.$$

Putting Corollaries 1.45 and 1.48 together yields the following differential analogue of a result [Hu19] of the first named author.

Corollary 1.49. *Let (X, D) be a pair over k where D is smooth and irreducible. Let $\mathcal{M}$ be an object in $\mathrm{MIC}(X, D)$. Then,*

$$\mathrm{irr}(D, \mathcal{M}) = \sup_{(0,C)} \frac{\mathrm{irr}(0, \mathcal{M}|_C)}{(C, D)_0}$$

where 0 runs over the points of D and where C runs over all smooth locally closed curves of X meeting D at 0 only.

2. Cohomological and χ-boundedness conjectures

2.1 Connections with bounded irregularity

Definition 2.1. Let (X, D) be a normal crossing pair over k. Let $\mathcal{M}$ be an object in $\mathrm{MIC}(X, D)$. Let R be an effective divisor of X supported on D. We say that $\mathcal{M}$ has *irregularity bounded by* R if the equivalent conditions of Corollary 1.45 are satisfied.

We denote by $\mathrm{MIC}(X, D, R)$ the full subcategory of $\mathrm{MIC}(X, D)$ consisting in connections with irregularity bounded by R.

For an integer $r \geq 0$, we denote by $\mathrm{MIC}_r(X, D, R)$ the full subcategory of $\mathrm{MIC}(X, D, R)$ spanned by connections with rank smaller than r.

2.2 Bounded irregularity and $\mathcal{H}om$

The $\mathcal{H}om$ construction for connections preserves boundedness of irregularity in the following sense.

Proposition 2.2. *Let (X, D) be a normal crossing pair over k. Let R be an effective divisor of X supported on D. Let $r \geq 0$ be an integer. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be objects of $\mathrm{MIC}_r(X, D, R)$. Then, $\mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2)$ is an object of $\mathrm{MIC}_{r^2}(X, D, 2r^2 \cdot R)$.*

Proof. We can suppose that D is irreducible with generic point η. Let $(\varphi_\alpha, \mathcal{R}_\alpha), \alpha \in I$ and $(\psi_\beta, \mathcal{S}_\beta), \beta \in J$ be the constituents of the good formal structures of $\mathcal{M}_1$ and $\mathcal{M}_2$ at η as in (1.27). With the notation from Definition 1.34, we have

$$\mathrm{irr}(D, \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2)) = \sum_{I \times J} (\mathrm{ord}_y\, \psi_\beta - \varphi_\alpha) \cdot (\mathrm{rk}\,\mathcal{R}_\alpha) \cdot (\mathrm{rk}\,\mathcal{S}_\beta).$$

Since $(\mathrm{ord}_y\, \psi_\beta - \varphi_\alpha) \leq (\mathrm{ord}_y\, \varphi_\alpha) + (\mathrm{ord}_y\, \psi_\beta)$, we deduce

$$\begin{aligned}\mathrm{irr}(D, \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2)) &\leq |J|\,\mathrm{rk}\,\mathcal{M}_2 \sum_I (\mathrm{ord}_y\, \varphi_\alpha) \cdot (\mathrm{rk}\,\mathcal{R}_\alpha) + |I|\,\mathrm{rk}\,\mathcal{M}_1 \sum_J (\mathrm{ord}_y\, \psi_\beta) \cdot (\mathrm{rk}\,\mathcal{S}_\beta) \\ &\leq (\mathrm{rk}\,\mathcal{M}_2)^2 \cdot \mathrm{irr}(D, \mathcal{M}_1) + (\mathrm{rk}\,\mathcal{M}_1)^2 \cdot \mathrm{irr}(D, \mathcal{M}_2).\end{aligned}$$

The proof of Proposition 2.2 thus follows. □

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2.3 Bounded irregularity and pull-back

Bounded irregularity behaves well with respect to inverse image.

Proposition 2.3. *Let $f : (Y, E) \to (X, D)$ be a morphism of normal crossing pairs over k. Let R be an effective divisor of X supported on D. Then, for every object $\mathcal{M}$ in $\mathrm{MIC}(X, D, R)$, the connection $f^+\mathcal{M}$ has irregularity bounded by f^*R.*

Proof. By writing f as a composition of a closed immersion followed by a smooth morphism, we are left to suppose that f is either a closed immersion or a smooth morphism. In both cases, the sought-after bound is local around the maximal points of E, so we can suppose in each case that E is smooth and irreducible.

Let us assume that f is a smooth morphism. Then, the pull-back $E \to D$ is smooth. Since E is smooth, so is the open set $f(E)$ of D. Thus, $f(E)$ lies in a unique irreducible component T of D. Since f is smooth, so is the pull-back $E \to T$. Hence, the generic point η_E of E lies above the generic point η_T of T and a uniformizer of $\mathcal{O}_{X,\eta_T}$ pulls back to a uniformizer of $\mathcal{O}_{Y,\eta_E}$. Thus, the irregularity numbers of $\mathcal{M}|_{\eta_T}$ and $(f^+\mathcal{M})|_{\eta_E}$ are equal. Hence, $\mathrm{Irr}(Y, f^+\mathcal{M}) = f^* \mathrm{Irr}(X, \mathcal{M})$ and Proposition 2.3 is proved if f is smooth.

We now assume that f is a closed immersion with E smooth and irreducible. Let 0 be a point in E and let $C \to Y$ be a locally closed smooth curve of Y meeting E at 0 only. Corollary 1.45 applied to $f(C)$ and $\mathcal{M}$ on X gives

$$\mathrm{irr}(0, (f^+\mathcal{M})|_C) = \mathrm{irr}(f(0), \mathcal{M}|_{f(C)}) \leq (f(C), R)_{f(0)} = (C, f^*R)_0,$$

where the last equality follows from the projection formula. Hence, condition (ii) of Corollary 1.45 is satisfied and Proposition 2.3 follows. □

2.4 Bounded irregularity and change of compactification I

Bounded irregularity behaves well with respect to compactification by normal crossing pairs.

Proposition 2.4. *Let U be a smooth variety over k. Let (X_1, D_1) and (X_2, D_2) be proper normal crossing pairs over k compactifying U, that is $U = X_1 \setminus D_1 = X_2 \setminus D_2$. Let $j_1 : U \to X_1$ and $j_2 : U \to X_2$ be the inclusions. Let R_1 be an effective divisor of X_1 supported on D_1. Then, there exists an effective divisor R_2 of X_2 supported on D_2 depending only on (X_1, D_1), (X_2, D_2) and linearly on R_1 such that for every object $\mathcal{M}_1$ of $\mathrm{MIC}(X_1, D_1, R_1)$, the connection $j_{2*}j_1^*\mathcal{M}_1$ lies in $\mathrm{MIC}(X_2, D_2, R_2)$.*

Proof. Using resolution of singularities, we construct a pair (X_3, D_3) dominating (X_1, D_1) and (X_2, D_2) such that D_3 has normal crossing. That is, there is a commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & \swarrow^{j_1} & \downarrow{\scriptstyle j_3} & \searrow^{j_2} & \\ X_1 & \xleftarrow[p_1]{} & X_3 & \xrightarrow[p_2]{} & X_2 \end{array}$$

with modifications as horizontal arrows where D_3 lies over D_1 and D_2. Let $\mathcal{M}_1 \in \mathrm{MIC}(X_1, D_1, R_1)$. Put $\mathcal{M}_2 = j_{2*}j_1^*\mathcal{M}_1$ and $\mathcal{M}_3 = j_{3*}j_1^*\mathcal{M}_1$. From Proposition 1.23, we have

$$p_1^+\mathcal{M}_1 = \mathcal{M}_3 = p_2^+\mathcal{M}_2.$$

Let E be an irreducible component of D_2. We have to bound $\mathrm{irr}(E, \mathcal{M}_2)$ by means of (X_i, D_i), $i = 1, 2$ and R_1 only. Let E' be the strict transform of E in X_3. Then,

$$\mathrm{irr}(E, \mathcal{M}_2) = \mathrm{irr}(E', p_2^+\mathcal{M}_2) = \mathrm{irr}(E', \mathcal{M}_3) = \mathrm{irr}(E', p_1^+\mathcal{M}_1).$$

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From Theorem 1.44, we further have

$$\operatorname{Irr}(X_3, p_1^+\mathcal{M}_1) \leq p_1^* \operatorname{Irr}(X_1, \mathcal{M}_1) \leq p_1^* R_1.$$

We deduce

$$\operatorname{irr}(E, \mathcal{M}_2) \leq m(E', p_1^* R_1) = m(E, p_{2*} p_1^* R_1).$$

Hence, $R_2 = p_{2*} p_1^* R_1$ depends linearly on R_1 and satisfies the condition of Proposition 2.4. □

2.5 Cohomological boundedness conjecture

Recall that if G is an abelian group and $d \geq 0$ an integer, a *polynomial $C : G \to \mathbb{Z}$ of degree at most d* is an element of $\operatorname{Sym}^\bullet_{\mathbb{Z}}(G^\vee)$ of degree at most d, where $G^\vee$ is the abelian group dual to G.

Conjecture 2.5. *Let (X, D) be a projective normal crossing pair of dimension d over k. There exists a polynomial $C : \operatorname{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ of degree at most d, affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\operatorname{MIC}_r(X, D, R)$, we have*

$$\dim H^*(X, \operatorname{DR}\mathcal{M}) \leq C(R, r).$$

Remark 2.6. If a polynomial $C : \operatorname{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ as above exists, we say that *cohomological boundedness holds for (X, D) with bound C*.

Remark 2.7. Let $d \geq 0$ be an integer. Let k be a field of characteristic 0. We say that *cohomological boundedness holds in dimension d over k* if it holds for every choice of (X, D) as above with (X, D) defined over k and $\dim X = d$.

Remark 2.8. Let $d \geq 0$ be an integer. We say that *cohomological boundedness holds in dimension d* if it holds in dimension d over every field of characteristic 0.

Lemma 2.9. *Cohomological boundedness holds in dimension* 1.

Proof. From Lemma A.1, it is enough to treat the case of a smooth connected projective curve X of genus g over $\mathbb{C}$. Let D be a reduced divisor of X. Let $\mathcal{M}$ be an object of $\operatorname{MIC}(X, D)$. From Proposition 1.9 and Theorem 1.14, we are left to bound the cohomology of $\operatorname{Sol}(\mathcal{M}^{\mathrm{an}})$. Let $j : U = X \setminus D \to X$ be the open immersion and put $\mathcal{L} := \operatorname{Sol}(\mathcal{M}^{\mathrm{an}})|_{U(\mathbb{C})}$. Observe that $\mathcal{L}$ is a local system of rank $\operatorname{rk}\mathcal{M}$ on $U(\mathbb{C})$. From Lemma 1.36, there is the following distinguished triangle.

$$j_!\mathcal{L} \longrightarrow \operatorname{Sol}(\mathcal{M}^{\mathrm{an}}) \longrightarrow \bigoplus_{P \in D} \mathbb{C}_P^{\operatorname{irr}(P,\mathcal{M})}[-1] \xrightarrow{+1}$$

We have $H^0(X(\mathbb{C}), j_!\mathcal{L}) \simeq 0$. If $\mathcal{L}^*$ denotes the dual of $\mathcal{L}$, Poincaré duality further gives

$$\dim H^2(X(\mathbb{C}), j_!\mathcal{L}) = \dim H^0(U(\mathbb{C}), \mathcal{L}^*) \leq \operatorname{rk}\mathcal{M}.$$

On the other hand, $\chi_c(U(\mathbb{C}), \mathcal{L}) = \operatorname{rk}\mathcal{M} \cdot \chi(U(\mathbb{C}), \mathbb{C}) = \operatorname{rk}\mathcal{M} \cdot (2 - 2g - |D|)$. Then, Lemma 2.9 follows from the long exact sequence in cohomology induced by the above distinguished triangle. □

2.6 The χ-boundedness conjecture

We formulate an *a priori* weaker version of the cohomological boundedness conjecture.

Conjecture 2.10. *Let (X, D) be a projective normal crossing pair of dimension d over k. There exists a polynomial $C : \operatorname{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ of degree at most d, affine in the last variable such*

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that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have

$$|\chi(X, \mathrm{DR}\,\mathcal{M})| \leq C(R, r).$$

Remark 2.11. We adopt for the χ-boundedness conjecture the terminology from Remarks 2.6–2.8 for the cohomological boundedness conjecture.

Remark 2.12. Let $d \geq 0$ be an integer. Similarly as in Lemma A.1, the χ-boundedness conjecture holds in dimension d if it holds in dimension d over $\mathbb{C}$.

Cohomological boundedness trivially implies χ-boundedness. The goal of the next subsection is to show that they are equivalent.

2.7 The cohomological and χ-boundedness conjectures are equivalent

Proposition 2.13. *Let $\mathbb{P}$ be a projective space over k. Let X be a smooth subvariety of $\mathbb{P}$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Let $H \in \mathbb{P}^\vee(k)$ be a hyperplane such that $X \cap H$ is smooth and $X \cap H \to X$ is non-characteristic for $\mathcal{M}$. Then the canonical comparison morphism*

$$H^n(X, \mathrm{DR}\,\mathcal{M}) \longrightarrow H^n(X \cap H, \mathrm{DR}(\mathcal{M}|_{X\cap H}))$$

is an isomorphism if $n < \dim X - 1$ and is injective if $n = \dim X - 1$.

Proof. From Remark 1.11, we reduce to the case where $k = \mathbb{C}$. From Proposition 1.9, we are left to prove a variant of Proposition 2.13 where now X is the analytification of a smooth complex algebraic variety and where $\mathcal{M}$ is the analytification of a holonomic $\mathcal{D}$-module. Then Theorem 1.12 ensures that $\mathrm{DR}\,\mathcal{M}[\dim X]$ is a perverse complex. From the Lefschetz hyperplane theorem for perverse complexes [dCat17, 2.4.2], the canonical comparison morphism

$$H^n(X, \mathrm{DR}\,\mathcal{M}) \longrightarrow H^n(X \cap H, (\mathrm{DR}\,\mathcal{M})|_{X\cap H})$$

is an isomorphism if $n < \dim X - 1$ and is injective if $n = \dim X - 1$. Since $X \cap H \to X$ is non-characteristic for $\mathcal{M}$, we conclude the proof of Proposition 2.13 using Theorem 1.20. □

Remark 2.14. When X is a projective complex manifold, Proposition 2.13 is also valid via the same reasoning with either the analytic de Rham complex or the solution complex.

Corollary 2.15. *In the setting of Proposition 2.13, assume the existence of $C \geq 0$ with*

$$\dim H^*(X \cap H, \mathrm{DR}(\mathcal{M}|_{X\cap H})) \leq C.$$

Then, if d denotes the dimension of X, we have

$$\dim H^*(X, \mathrm{DR}\,\mathcal{M}) \leq |\chi(X, \mathrm{DR}\,\mathcal{M})| + 4d \cdot C.$$

Proof. We can suppose that $k = \mathbb{C}$. From Proposition 2.13, we have for $n < d$,

$$\dim H^n(X, \mathrm{DR}\,\mathcal{M}) \leq C.$$

Assume that $n > d$. Then,

$$\begin{aligned}
\dim H^n(X, \mathrm{DR}\,\mathcal{M}) &= \dim H^n(X(\mathbb{C}), \mathrm{DR}\,\mathcal{M}^{\mathrm{an}}) \\
&= \dim H^{2d-n}(X(\mathbb{C}), \mathrm{Sol}\,\mathcal{M}^{\mathrm{an}}) \\
&\leq \dim H^{2d-n}((X \cap H)(\mathbb{C}), \mathrm{Sol}(\mathcal{M}^{\mathrm{an}}|_{(X\cap H)(\mathbb{C})})) \\
&\leq \dim H^{n-2}((X \cap H)(\mathbb{C}), \mathrm{DR}(\mathcal{M}^{\mathrm{an}}|_{(X\cap H)(\mathbb{C})})) \\
&\leq \dim H^{n-2}(X \cap H, \mathrm{DR}(\mathcal{M}|_{X\cap H})) \leq C,
\end{aligned}$$

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where the first and fifth inequality follow from Proposition 1.9, the second follows from Theorem 1.14 and Poincaré–Verdier duality on $X(\mathbb{C})$, the third follows from Remark 2.14 since $2d - n < d$ and the fourth inequality follows from Theorem 1.14 and Poincaré–Verdier duality on $(X \cap H)(\mathbb{C})$. Furthermore, Proposition 1.10 yields

$$\begin{aligned}\dim H^d(X, \mathrm{DR}\,\mathcal{M}) &\leq |\chi(X, \mathrm{DR}\,\mathcal{M})| + \sum_{n \neq d} \dim H^n(X, \mathrm{DR}\,\mathcal{M}) \\ &\leq |\chi(X, \mathrm{DR}\,\mathcal{M})| + 2d \cdot C.\end{aligned}$$

Putting all the above inequalities together finishes the proof of Corollary 2.15. □

Proposition 2.16. *Let (X, D) be a projective normal crossing pair of dimension d over k. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Assume that χ-boundedness holds for (X, D) with bound K and that cohomological boundedness holds with bound C for the generic hyperplane section (X_η, D_η) of (X, D). Then, cohomological boundedness holds for (X, D) with bound $K + 4d \cdot C'$ where C' is the composition of C with the linear map $\mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathrm{Div}(X_\eta, D_\eta) \oplus \mathbb{Z}$ deduced from $X_\eta \to X$.*

Proof. Let R be an effective divisor of X supported on D. Let r be an integer. By Bertini's theorem, there exists a dense open subset V in $\mathbb{P}^\vee$ such that for every hyperplane $H \in V(k)$, the pair $(X \cap H, D \cap H)$ is smooth of dimension $d - 1$ and $D \cap H$ has normal crossing. Let η be the generic point of $\mathbb{P}^\vee$. Following the notation from § 1.4, consider the following commutative diagram with cartesian squares.

$$\begin{array}{ccccccc} X_\eta & \longrightarrow & X_V & \longrightarrow & X_Q & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Q_\eta & \longrightarrow & Q_V & \longrightarrow & Q & \longrightarrow & \mathbb{P} \\ \downarrow & & \downarrow & & \downarrow & & \\ \eta & \longrightarrow & V & \longrightarrow & \mathbb{P}^\vee & & \end{array}$$

Let $\pi : X_Q \to \mathbb{P}^\vee$ be the composition morphism and let $\pi_V : X_V \to V$ and $\pi_\eta : X_\eta \to \eta$ be its pull-back above V and η, respectively. Let D_V and R_V be the pull-back of D and R to X_V and let D_η and R_η be their pull-back to X_η. Observe that (X_η, D_η) is a pair of dimension $d - 1$ over η where D_η has normal crossing. Since cohomological boundedness holds for (X_η, D_η) with bound C, we have for every object $\mathcal{N}_\eta$ in $\mathrm{MIC}_r(X_\eta, D_\eta, R_\eta)$,

$$\dim H^*(X_\eta, \mathrm{DR}\,\mathcal{N}_\eta) \leq C(R_\eta, r).$$

On the other hand, for every object $\mathcal{N}$ in $\mathrm{MIC}_r(X_V, D_V, R_V)$ we have

$$(\pi_{V+}\mathcal{N})_\eta \simeq \pi_{\eta+}\mathcal{N}_\eta \simeq R\Gamma(X_\eta, \mathrm{DR}\,\mathcal{N}_\eta)[d-1].$$

Thus, the space $H^n(X_\eta, \mathrm{DR}\,\mathcal{N}_\eta)$ is the generic fibre of the $\mathcal{D}_V$-module

$$\mathcal{H}^{n-d+1}\pi_{V+}\mathcal{N}$$

for every $n \geq 0$. Let $\mathcal{M}$ be an object in $\mathrm{MIC}_r(X, D, R)$. From Proposition 2.3, the pull-back $\mathcal{M}_V$ of $\mathcal{M}$ to X_V lies in $\mathrm{MIC}_r(X_V, D_V, R_V)$. Since k is infinite, Lemma 1.5 ensures the existence of a hyperplane $H \in V(k)$ such that $X \cap H \to X$ is non-characteristic for $\mathcal{M}$ and such that the cohomology modules of $\pi_{V+}\mathcal{M}_V$ are flat connections in a neighbourhood of $H \in V(k)$. In particular,

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the inclusion $i : \{H\} \to V$ is non-characteristic for the cohomology modules of $\pi_{V+}\mathcal{M}_V$. Hence, Theorem 1.20 combined with base change for $\mathcal{D}$-modules [HTT00, 1.7.3] yields

$$\begin{aligned} i^*\mathcal{H}^{n-d+1}\pi_{V+}\mathcal{M}_V &\simeq \mathcal{H}^{n-d+1}i^+\pi_{V+}\mathcal{M}_V \\ &\simeq \mathcal{H}^n\pi_{H+}(\mathcal{M}_V|_{X\cap H}) \\ &\simeq H^n(X \cap H, \mathrm{DR}\,\mathcal{M}|_{X\cap H}), \end{aligned}$$

where $\pi_H : X \cap H \to \{H\}$ is the pull-back of π above $H \in V(k)$. Since $\mathcal{H}^{n-d+1}\pi_{V+}\mathcal{M}_V$ is a flat connection in a neighbourhood of H, we also have

$$\dim H^n(X_\eta, \mathrm{DR}(\mathcal{M}_V)_\eta) = \dim i^*\mathcal{H}^{n-d+1}\pi_{V+}\mathcal{M}_V.$$

Thus,

$$\dim H^*(X \cap H, \mathrm{DR}\,\mathcal{M}|_{X\cap H}) \leq C(R_\eta, r).$$

Then, Corollary 2.15 yields

$$\dim H^*(X, \mathrm{DR}\,\mathcal{M}) \leq |\chi(X, \mathrm{DR}\,\mathcal{M})| + 4d \cdot C(R_\eta, r) \leq K(R, r) + 4d \cdot C(R_\eta, r),$$

which concludes the proof of Proposition 2.16. □

Corollary 2.17. *Let $d \geq 2$ be an integer. Then cohomological boundedness holds in dimension d if it holds in dimension $d-1$ and if χ-boundedness holds in dimension d.*

From Lemma 2.9 and Corollary 2.17, we deduce the following.

Corollary 2.18. *The cohomological and χ-boundedness conjectures are equivalent.*

3. Nearby slopes and boundedness

3.1 Resolution relative to a normal crossing divisor

Let (X, D) be a normal crossing pair over k. If C is a closed subscheme of X, we say that C *and D have simultaneously only normal crossing* if for every point x of C, there exists a regular system of parameters $(x_1, \ldots, x_n)$ for $\mathcal{O}_{X,x}$ such that $\mathcal{I}_D$ is generated at x by $x_1 \cdots x_k$ and $\mathcal{I}_C$ is generated at x by some monomials in the x_j for $j = 1, \ldots, n$ (therefore, by some x_j if C is smooth at x).

Definition 3.1. Let Z be a closed subscheme of X. We say that a blow-up $p : Y \to X$ with centre C is *admissible with respect to* (Z, D) if the following conditions are satisfied:

(i) C is a smooth nowhere dense subset of Z_{red};
(ii) C and D have simultaneously only normal crossing.

Then, we denote by Z' the strict transform of Z and put $D' = p^{-1}(D) \cup p^{-1}(C)$ endowed with its reduced structure. The pair (Z', D') is the *transform of* (Z, D) *by* $p : Y \to X$.

The form of embedded resolution needed in this paper is the following theorem. For a proof, see Theorems 8.4 and 8.6 in [BM89].

Theorem 3.2. *Let (X, D) be a normal crossing pair over k. Let Z be a closed subscheme of X. Then, there exists a composition $p : Y \to X$ of admissible blow-up with respect to the successive transforms of (Z, D) such that the final transform (Z', D') satisfies that Z'_{red} is smooth and (Z', D') have simultaneously only normal crossing.*

Definition 3.3. A map $p : Y \to X$ as above is a *resolution of Z relative to D*.

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The following lemma is obvious from the definitions.

Lemma 3.4. *Let (X, D) be a normal crossing pair over k. Let Z, T be subschemes of X with $T_{\mathrm{red}} \subset Z_{\mathrm{red}}$. Then a blow-up of X admissible with respect to (T, D) is admissible with respect to (Z, D).*

Lemma 3.5. *Let (X, D) be a normal crossing pair over k. Let Z, T be subschemes of X with $T_{\mathrm{red}} \subset Z_{\mathrm{red}}$ such that T_{red} is nowhere dense in Z_{red}. Then, there exists a resolution $p : Y \to X$ of Z relative to D such that the pull-back scheme $p^{-1}(T)$ is an effective Cartier divisor such that $p^{-1}(T) + D'$ has normal crossings, where (Z', D') is the transform of (Z, D) by $p : Y \to X$.*

Proof. Let $q : X' \to X$ be a blow-up admissible with respect to (T, D). From Lemma 3.4, the map q is admissible with respect to (Z, D). Let (Z', T', D') be the transform of (Z, T, D) by q. Observe that $T'_{\mathrm{red}} \subset Z'_{\mathrm{red}}$ and that T'_{red} is nowhere dense in Z'_{red}. We argue that if Lemma 3.5 holds for (X', D', Z', T'), then it holds for (X, D, Z, T). Indeed, let $\rho : Y' \to X'$ be a resolution of Z' relative to D' such that the pull-back scheme $\rho^{-1}(T')$ is an effective Cartier divisor such that $\rho^{-1}(T') + D''$ has normal crossing, where (Z'', D'') is the transform of (Z', D') by ρ. Then, the composition $\rho \circ q : Y' \to X$ is a resolution of Z relative to D. On the other hand, we have $q^{-1}(T) = T' \cup E$ where E is an effective Cartier divisor. Thus $\rho^{-1}(q^{-1}(T)) = \rho^{-1}(T') \cup \rho^{-1}(E)$ is an effective Cartier divisor. Observe that (Z'', D'') is also the transform of (Z, D) by $\rho \circ q$. Since D'' contains $\rho^{-1}(E)$, the divisors $\rho^{-1}(T') + D''$ and $\rho^{-1}q^{-1}(T) + D''$ have the same support. Hence, $\rho^{-1}q^{-1}(T) + D''$ has normal crossing and Lemma 3.5 is indeed true for (X, D, Z, T).

Using a resolution of T relative to D as given by Theorem 3.2, we are thus left to prove Lemma 3.5 in the case where T_{red} is smooth and T and D have simultaneously only normal crossings. In particular, T_{red} and D have simultaneously only normal crossings. Let $q : X' \to X$ be the blow-up of X at T_{red}. Since $T_{\mathrm{red}} \subset Z_{\mathrm{red}}$ is nowhere dense in Z_{red}, the map $q : X' \to X$ is admissible with respect to (Z, D). Let (Z', D') be the transform of (Z, D) by q. Let $\rho : Y' \to X'$ be a resolution of Z' relative to D'. We claim that the map $p := \rho \circ q : Y' \to X$ does the job. Let (Z'', D'') be the transform of (Z', D') by ρ. Then, $p : Y' \to X$ is a resolution of Z relative to D and (Z'', D'') is the transform of (Z, D) by p. Since $q^{-1}(T_{\mathrm{red}})$ is an effective Cartier divisor of X', so is $q^{-1}(T)$. Hence, $p^{-1}(T)$ is an effective Cartier divisor of Y'. We are thus left to show that $p^{-1}(T) + D''$ has normal crossings. Observe that D'' contains $\rho^{-1}(D')$. Since D' contains $q^{-1}(T)$ by definition, D'' contains $p^{-1}(T)$. We are thus left to showing that D'' has normal crossings, which is obvious by construction. □

3.2 Bounded irregularity and change of compactification II

Proposition 3.7 below provides a generalization of Proposition 2.4 where part of the divisor at infinity is being kept while changing compactification. Before proving it, we need the following.

Lemma 3.6. *Let (X, D) be a normal crossing pair over k and let Z be a divisor of X. Let $p : Y \to X$ be a resolution of Z relative to D and put $F := (p^{-1}(Z \cup D))_{\mathrm{red}}$. Let R be an effective divisor supported on D. Then for every object $\mathcal{M}$ of $\mathrm{MIC}(X, D, R)$, the connection $(p^+\mathcal{M})(*F)$ lies in $\mathrm{MIC}(Y, F, p^*R)$.*

Proof. Put $E = f^{-1}(D)$. By definition of a resolution relative to a normal crossing divisor, the divisors $E \subset F$ have normal crossing. In particular, $p : (Y, E) \to (X, D)$ is a morphism of normal crossing pairs. From Proposition 2.3, we deduce that $p^+\mathcal{M}$ is an object of $\mathrm{MIC}(Y, E, p^*R)$. Along a component of F that is not a component of E, the generic irregularity of $(p^+\mathcal{M})(*F)$ is 0. Thus, $(p^+\mathcal{M})(*F)$ is an object of $\mathrm{MIC}(Y, F, p^*R)$ and Lemma 3.6 is proved. □

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Proposition 3.7. *Let (X, D) be a proper normal crossing pair over k. Let $V \subset X$ be an open subset such that $Z := X \setminus V$ is a divisor and put $D_V := V \cap D$. Let $j : (V, D_V) \to (Y, E)$ be a dense open immersion where (Y, E) is a proper normal crossing pair over k with $Y \setminus E = V \setminus D_V$. Then for every effective divisor R supported on D, there exists an effective divisor S supported on E depending only on V, on $j : (V, D_V) \to (Y, E)$ and linearly on R such that for every object $\mathcal{M}$ of $\mathrm{MIC}(X, D, R)$, the connection $j_*\mathcal{M}|_V$ is an object of $\mathrm{MIC}(Y, E, S)$.*

Proof. Let $p : Y' \to X$ be a resolution of Z relative to D. Put $F := (p^{-1}(Z \cup D))_{\mathrm{red}}$. In particular, $p : Y' \to X$ induces an isomorphism $p^{-1}(V) \to V$. Furthermore, $Y' \setminus F = V \setminus D_V$. Hence, the normal crossing pair (Y', F) is a compactification of $V \setminus D_V$. By assumption, (Y, E) is also a compactification of $V \setminus D_V$. Let $\jmath : V \setminus D_V \to V$ be the inclusion. Let R be an effective divisor of X supported on D. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D, R)$. From Lemma 3.6, the connection $(p^+\mathcal{M})(*F)$ is an object of $\mathrm{MIC}(Y', F, p^*R)$. Proposition 2.4 applied to $V \setminus D_V$, (Y', F), (Y, E) and p^*R yields the existence of an effective divisor S of Y supported on E depending only on (Y', F), (Y, E) and linearly on p^*R such that

$$\mathcal{N} := j_*\jmath_*((p^+\mathcal{M})(*F))|_{V\setminus D_V} \simeq j_*\jmath_*\jmath^+\mathcal{M}|_V \simeq j_*\mathcal{M}|_V$$

is an object of $\mathrm{MIC}(Y, E, S)$. We thus have $\mathcal{N}|_V \simeq \mathcal{M}|_V$ and S only depends on V, on $j : (V, D_V) \to (Y, E)$ and on R. This concludes the proof of Proposition 3.7. □

3.3 Resolution and multiplicity estimate

If X is a smooth connected variety over k, we recall from § 1.13 that any Cartier divisor on X can be seen as a $\mathbb{Z}$-valued function on $\mathrm{ZR}^{\mathrm{divis}}(X)$ via the injection $\mathbf{CDiv}(X) \to \mathbf{Div}(X)$.

Lemma 3.8. *Let (X, D) be a normal crossing pair of dimension d over k. Let C be a smooth subscheme of X which has simultaneously only normal crossing with D. Let v be the divisorial valuation associated to the exceptional divisor of the blow-up $p : Y \to X$ along C. Then, $D(v) \leq d$. If D is smooth, then $D(v)$ is 0 or 1.*

Proof. The question is étale local around the generic point of C. We can thus suppose the existence of a local system of coordinates $(x_1, \ldots, x_d)$ such that $\mathcal{I}_D$ is generated by $\prod_{i\in I} x_i$ where $I \subset \{1, \ldots, d\}$ and $\mathcal{I}_C$ is generated by some x_j for $j \in J \subset \{1, \ldots, d\}$. Then

$$\mathcal{I}_D = \Big(\prod_{i\in I\cap J} x_i \prod_{i\in I\setminus I\cap J} x_i \Big).$$

Since $D(v)$ is simply the multiplicity of p^*D along the exceptional divisor, we deduce

$$D(v) = |I \cap J| \leq d.$$

If D is smooth, then $I \cap J$ is either empty or a singleton depending on whether D contains C, and Lemma 3.8 follows. □

The following lemma appeared as Proposition 4.3.2 in [Tey16] with the tacit assumption that D has *simple* normal crossing. If some components of D are singular, the coefficient $\mathrm{fdeg}\, R$ is not enough and has to be replaced by $d \cdot (\mathrm{fdeg}\, R)$ where d is the dimension of the ambient variety. We spell out the proof here for completeness.

Lemma 3.9. *Let (X, D) be a normal crossing pair of dimension d over k. Let R be an effective divisor supported on D. Let Z be an effective Cartier divisor on X. Let $p : Y \to X$ be a composition of blow-up which are admissible with respect to the successive transforms of (Z, D).*

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Let v be a divisorial valuation centred at a component of $Z(Y) = p^*Z$. Then,

$$R(v) \leq d \cdot (\mathrm{fdeg}\, R) \cdot Z(v), \tag{3.10}$$

where R and Z are viewed as b-divisors via $\boldsymbol{CDiv}(X) \to \boldsymbol{Div}(X)$.

Proof. We argue by induction on the number n of admissible blow-up needed to write p. If $n = 0$, there is nothing to prove. Let us now treat the case of $n+1$ blow-ups. Let

$$p : Y \xrightarrow{\ q\ } X' \xrightarrow{\ r\ } X$$

where r is a composition of n blow-up admissible with respect to the successive transforms of (Z, D), and where q is admissible with respect to the final transform (Z', D') of (Z, D) by r. Let $\mathcal{C}$ be the set of strict transforms in X' of the exceptional divisors that appear in r. For $E \in \mathcal{C}$, we let v_E be the associated divisorial valuation. In what follows, the notation $'$ indicates a strict transform in X' and $''$ indicates a strict transform in Y. Write $R = \alpha_1 D_1 + \cdots + \alpha_m D_m$ with $\alpha_i > 0$ and $Z = \beta_1 Z_1 + \cdots + \beta_k Z_k$ with $\beta_i > 0$. We have

$$Z(X') = \beta_1 Z_1' + \cdots + \beta_k Z_k' + \sum_{E \in \mathcal{C}} Z(v_E) E,$$

with $Z(v_E) > 0$ by admissibility of r with respect to the successive transforms of (Z, D), and

$$R(X') = \alpha_1 D_1' + \cdots + \alpha_k D_k' + \sum_{E \in \mathcal{C}} R(v_E) E.$$

We have to detail the effect of q^* on each component contributing to the above equalities. Let C be the centre of q and let P be its exceptional divisor with associated valuation v_P. Let $E \in \mathcal{C}$. Since C and E have simultaneously only normal crossing and since E is smooth, Lemma 3.8 gives $q^*E = E'' + \epsilon_E P$ where $\epsilon_E \in \{0, 1\}$. Since C and D_i' have simultaneously only normal crossing, Lemma 3.8 gives again

$$q^* D_i' = D_i'' + \epsilon_i P \quad \text{where } \epsilon_i \in \{0, \ldots, d\}.$$

Furthermore, let us write

$$q^* Z_i' = Z_i'' + \mu_i P \quad \text{where } \mu_i \in \mathbb{N}.$$

We have

$$Z(Y) = \sum \beta_i Z_i'' + \sum_{E \in \mathcal{C}} Z(v_E) E'' + \left(\sum \beta_i \mu_i + \sum_{E \in \mathcal{C}} \epsilon_E Z(v_E) \right) P$$

and

$$R(Y) = \sum \alpha_i D_i'' + \sum_{E \in \mathcal{C}} R(v_E) E'' + \left(\sum \alpha_i \epsilon_i + \sum_{E \in \mathcal{C}} \epsilon_E R(v_E) \right) P.$$

By recursion assumption, we have to check the inequality (3.10) for $v = v_P$. By admissibility of q with respect to (Z', D'), the centre C lies in Z'_{red}, that is in one of the Z_i'. Hence, one of the μ_i is strictly positive. Thus,

$$\begin{aligned} d \cdot (\mathrm{fdeg}\, R) \cdot Z(v_P) &= d \cdot (\mathrm{fdeg}\, R) \cdot \left(\sum \beta_i \mu_i + \sum \epsilon_E Z(v_E) \right) \\ &\geq d \cdot (\mathrm{fdeg}\, R) + \sum \epsilon_E \big(d \cdot (\mathrm{fdeg}\, R) \cdot Z(v_E) \big) \\ &\geq \sum \alpha_i \epsilon_i + \sum \epsilon_E R(v_E) = R(v_P) \end{aligned}$$

and the proof of Lemma 3.9 is complete. □

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3.4 Nearby slopes for $\mathcal{D}$-modules

Let X be a smooth algebraic variety over k. Let f be a non-constant function on X. We denote by ψ_f the nearby cycles functor associated to f. For a general reference on nearby cycles, see [MM04]. Inspired by a letter of Deligne to Malgrange [Del07], the following notion of slopes was defined in [Tey16].

Definition 3.11. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. The *nearby slopes of* $\mathcal{M}$ *associated to* f are the rationals $r \geq 0$ such that there exists a germ N of meromorphic connection at 0 in $\mathbb{A}^1$ with slope r verifying

$$\psi_f(\mathcal{M} \otimes f^+N) \neq 0. \tag{3.12}$$

We denote by $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M})$ the set of nearby slopes of $\mathcal{M}$ associated to f.

Remark 3.13. Since N is localized at 0, the connection f^+N is localized along $\operatorname{div} f$. Hence, $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}) = \mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}(*Z))$ where Z denotes the support of $\operatorname{div} f$.

Remark 3.14. The nearby slopes are sensitive to the non reduced structure of $\operatorname{div} f$.

For f as above, Deligne proved that the set $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M})$ is finite. The following theorem is the main result of [Tey16].

Theorem 3.15. *Let X be a smooth algebraic variety over k. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then, there is an integer C depending only on $\mathcal{M}$ such that for every non-constant function f on X, the set $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M})$ is bounded by C.*

In this paper, we will need a boundedness of a different kind where f is fixed but where $\mathcal{M}$ is allowed to vary over the objects of $\mathrm{MIC}(X, D, R)$ where D is a normal crossing divisor and where R is an effective divisor supported on D.

Examples of this kind of bound were obtained in the étale setting in [HT21] and [Hu23] in a semi-stable situation where the horizontal ramification is tame. In our case, the horizontal irregularity will be allowed to be non-trivial. The price to pay for this generality will be to require that the zero locus of f contains the turning locus of $\mathcal{M}$. See Proposition 3.22 below.

Nearby slopes are compatible with proper push-forward. See [Tey16, Theorem 3].

Proposition 3.16. *Let $p : Y \to X$ be a proper morphism of smooth varieties over k. Let f be a non-constant function on X such that $fp = 0$ is a divisor of Y. Then, for every holonomic $\mathcal{D}_Y$-module $\mathcal{M}$, we have*

$$\mathrm{Sl}_f^{\mathrm{nb}}(p_+\mathcal{M}) \subset \mathrm{Sl}_{fp}^{\mathrm{nb}}(\mathcal{M}).$$

The following lemma is useful for dévissages.

Lemma 3.17. *Let $p : (Y, E) \to (X, D)$ be a proper morphism of pairs over k such that the induced morphism $Y \setminus E \to X \setminus D$ is an isomorphism. Let $\mathcal{M}$ be an object in $\mathrm{MIC}(X, D)$. Then, the canonical morphism $p_+p^+\mathcal{M} \to \mathcal{M}$ is an isomorphism.*

Proof. Since p is proper, [Meb04, 3.6-4] ensures that the $\mathcal{D}_X$-module $p_+p^+\mathcal{M}$ is an object of $\mathrm{MIC}(X, D)$. Then, Lemma 3.17 follows from Proposition 1.23 and base change for $\mathcal{D}$-modules [HTT00, 1.7.3]. □

Proposition 3.18. *Let $p : (Y, E) \to (X, D)$ be a proper morphism of pairs over k. Let f be a non-constant function on X such that $fp = 0$ is a divisor of Y. Let Z be the support of $\operatorname{div} f$ and put $T := p^{-1}(Z)$. Assume that $Y \setminus T \to X \setminus Z$ is an isomorphism. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Then,*

$$\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}) \subset \mathrm{Sl}_{fp}^{\mathrm{nb}}((p^+\mathcal{M})(*T)).$$

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Proof. We have

$$\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}) = \mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}(*Z)) = \mathrm{Sl}_f^{\mathrm{nb}}(p_+p^+(\mathcal{M}(*Z))) \subset \mathrm{Sl}_{fp}^{\mathrm{nb}}((p^+\mathcal{M})(*T)).$$

The first equality follows from Remark 3.13. Since $Y \setminus T \to X \setminus Z$ is an isomorphism, the induced morphism $Y \setminus (T \cup E) \to X \setminus (Z \cup D)$ is an isomorphism with $\mathcal{M}(*Z)$ being an object of $\mathrm{MIC}(X, Z \cup D)$. Hence, the second equality follows from Lemma 3.17. The last inclusion follows from Proposition 3.16. Proposition 3.18 is thus proved. □

Over curves, nearby slopes coincide with the traditional notion of slopes recalled in Definition 1.34. This is given by the following lemma proved in [Tey16, 3.3.1].

Lemma 3.19. *Let C be a smooth curve over k. Let 0 be a point in C and let t be a local uniformizer of C at 0. Let $\mathcal{M}$ be an object in $\mathrm{MIC}(C, 0)$. Then the nearby slopes of $\mathcal{M}$ associated to t are the slopes of $\mathcal{M}$ at 0 in the sense of Definition 1.34.*

3.5 Boundedness of nearby slopes

In this paragraph, we will need to resolve the turning locus of a connection $\mathcal{M}$ while resolving a fixed subscheme relatively to the pole locus. Combined with Kedlaya's Theorem 1.44, this resolution gives some control on the nearby slopes of $\mathcal{M}$. The existence of this resolution is achieved by the following.

Lemma 3.20. *Let (X, D) be a normal crossing pair over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Let Z be a divisor of X such that $\mathrm{TL}(\mathcal{M}) \subset Z_{\mathrm{red}}$. Then, there is a morphism of normal crossing pairs $p : (Y, E) \to (X, D)$ such that $p : Y \to X$ is a resolution of Z relative to D and $p^+\mathcal{M}$ has good formal structure.*

Proof. From the Kedlaya–Mochizuki Theorem 1.32, there exists a morphism of smooth algebraic varieties $q : Y' \to X$ obtained as a composition of blow-up above $\mathrm{TL}(\mathcal{M})$ such that $E' = q^{-1}(D)$ is a normal crossing divisor of Y' and $q^+\mathcal{M}$ admits good formal structure along E'. From [Sta-a, 080B], there is a closed subscheme T of X with $T_{\mathrm{red}} \subset \mathrm{TL}(\mathcal{M})$ such that $q : Y' \to X$ identifies with the blow-up of X along T. By the universal property of the blow-up, $q : Y' \to X$ is a final object in the full subcategory of schemes over X consisting in morphisms $Y'' \to X$ such that the inverse image of T is an effective Cartier divisor on Y''. By assumption, we have $T_{\mathrm{red}} \subset \mathrm{TL}(\mathcal{M}) \subset Z_{\mathrm{red}}$. Since Z is a divisor and since $\mathrm{TL}(\mathcal{M})$ has pure codimension 2 in X in virtue of Theorem 1.31, $\mathrm{TL}(\mathcal{M})$ is nowhere dense in Z_{red}. Hence, T_{red} is nowhere dense in Z_{red}. Thus, Lemma 3.5 applies. That is, there exists a resolution $p : Y \to X$ of Z relative to D such that the pull-back scheme $p^{-1}(T)$ is an effective Cartier divisor on Y. Put $E = p^{-1}(D)$. Then, there is the following canonical commutative diagram of normal crossing pairs.

$$\begin{array}{ccc} (Y, E) & \xrightarrow{\quad h \quad} & (Y', E') \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & (X, D) & \end{array}$$

From Proposition 1.30, the connection $h^+(q^+\mathcal{M}) \simeq p^+\mathcal{M}$ has good formal structure and the proof of Lemma 3.20 is complete. □

We recall the following boundedness result for nearby slopes, formulated in terms of the highest slope b-divisor from Remark 1.42. See [Tey16, 3.4.1] for a proof.

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Lemma 3.21. *Let (X,D) be a normal crossing pair over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X,D)$ with good formal structure. Let f be a non-constant function on X such that $\operatorname{div} f$ is supported on D. Let $r \in \mathbb{Q}_{\geq 0}$ such that for every divisorial valuation v centred at an irreducible component of $\operatorname{div} f$, we have*

$$R(\mathcal{M})(v) \leq r \cdot (\operatorname{div} f)(v),$$

where $R(\mathcal{M})$ is the highest slope b-divisor from Remark 1.42. Then the nearby slopes of $\mathcal{M}$ associated to f are smaller than r.

Proposition 3.22. *Let (X,D) be a normal crossing pair of dimension d over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X,D)$. Let f be a non-constant function on X. Suppose that the turning locus of $\mathcal{M}$ lies in the support of $\operatorname{div} f$. Then, the nearby slopes of $\mathcal{M}$ associated to f are bounded by $d \cdot \operatorname{fdeg} \operatorname{Irr}(X,\mathcal{M})$.*

Proof. Put $Z := \operatorname{div} f$. Since $\mathrm{TL}(\mathcal{M}) \subset Z_{\mathrm{red}}$, Lemma 3.20 ensures the existence of a morphism of normal crossing pairs $p : (Y,E) \to (X,D)$ such that $p : Y \to X$ is a resolution of Z relative to D and $p^+\mathcal{M}$ has good formal structure. Let $T := p^{-1}(Z) = p^* \operatorname{div} f = \operatorname{div} fp$ be the pull-back scheme. Let (Z',D') be the transform of (Z,D) by p. Since Z is a divisor, so is Z'. By assumption, Z' and D' have only simultaneously normal crossing. Thus, $Z' \cup D'$ is a normal crossing divisor of Y. Since $Z' \cup D'$ and $T \cup E$ have the same support, we deduce that $T \cup E$ is a normal crossing divisor of Y. We are going to apply Lemma 3.21 on $(Y, T \cup E)$ to $(p^+\mathcal{M})(*T)$ and fp. Since $p^+\mathcal{M} \in \mathrm{MIC}(Y,E)$ has good formal structure, so does $(p^+\mathcal{M})(*T) \in \mathrm{MIC}(Y, T \cup E)$. By assumption, $\operatorname{div} fp$ is supported on $T \cup E$. Let v be a divisorial valuation centred at an irreducible component of $T = \operatorname{div} fp$. If the centre of v is not a component of E, we have

$$R((p^+\mathcal{M})(*T))(v) = 0 \leq d \cdot (\operatorname{fdeg} \operatorname{Irr}(X,\mathcal{M})) \cdot (\operatorname{div} f)(v).$$

Otherwise, $R((p^+\mathcal{M})(*T))(v) = R(\mathcal{M})(v)$. On the other hand, we have

$$\begin{aligned} R(\mathcal{M})(v) &\leq (\operatorname{Irr} \mathcal{M})(v) \\ &\leq (\operatorname{Irr}(X,\mathcal{M}))(v) \\ &\leq d \cdot (\operatorname{fdeg} \operatorname{Irr}(X,\mathcal{M})) \cdot (\operatorname{div} f)(v), \end{aligned}$$

where the first inequality is trivial, the second follows from Kedlaya's Theorem 1.44 and where the last inequality comes from Lemma 3.9. From Lemma 3.21, we deduce that the nearby slopes of $(p^+\mathcal{M})(*T)$ associated to fp are bounded by $d \cdot (\operatorname{fdeg} \operatorname{Irr}(X,\mathcal{M}))$. Finally, Proposition 3.18 applied to the proper morphism of pairs $p : (Y,E) \to (X,D)$ yields

$$\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}) \subset \mathrm{Sl}_{fp}^{\mathrm{nb}}((p^+\mathcal{M})(*T)),$$

which concludes the proof of Proposition 3.22. □

Proposition 3.23. *Let (X,D) be a normal crossing pair of dimension d over $\mathbb{C}$. Let C be a smooth connected curve over $\mathbb{C}$. Let $f : X \to C$ be a dominant proper morphism. Let 0 be a closed point in C. Suppose that the reduced fibre Z of f over 0 is not empty and is contained in D. Suppose that cohomological boundedness holds with bound K for the generic fibre of $f : X \to C$. Then for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X,D,R)$ with $\mathrm{TL}(\mathcal{M}) \subset Z$, we have*

$$|\chi(Z(\mathbb{C}), \operatorname{Sol} \mathcal{M}^{\mathrm{an}})| \leq d \cdot K(R_\eta, r) \cdot \operatorname{fdeg} R,$$

where R_η is the pull-back of R to the generic fibre of $f : X \to C$.

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Proof. By the generic smoothness theorem, the generic fibre of X over C is smooth and the generic fibre of (X, D) over C is again a normal crossing pair. Hence, the statement of cohomological boundedness makes sense over the generic fibre of (X, D) over C. We reproduce here a reasoning carried out in [Tey23, Theorem 2] in a particular case. Let t be a local uniformizer of C around 0. Since $Z \subset D$, we have

$$(\operatorname{Sol} \mathcal{M}^{\mathrm{an}})|_{Z(\mathbb{C})} \simeq \operatorname{Irr}^*_{Z(\mathbb{C})} \mathcal{M}^{\mathrm{an}}.$$

Since the irregularity complex commutes with proper push-forward [Meb90], we deduce

$$\chi(Z(\mathbb{C}), \operatorname{Sol} \mathcal{M}^{\mathrm{an}}) = \chi(Z(\mathbb{C}), \operatorname{Irr}^*_{Z(\mathbb{C})} \mathcal{M}^{\mathrm{an}}) = (-1)^{d-1} \chi(0, \operatorname{Irr}^*_0 f_+^{\mathrm{an}} \mathcal{M}^{\mathrm{an}}).$$

From Theorem 1.13, the irregularity complex of a holonomic $\mathcal{D}_C$-module is a skyscraper sheaf concentrated in degree 1. Hence, Lemma 1.36 gives

$$\begin{aligned} \chi(0, \operatorname{Irr}^*_0 f_+^{\mathrm{an}} \mathcal{M}^{\mathrm{an}}) &= \sum (-1)^i \dim \mathcal{H}^1 \operatorname{Irr}^*_0 \mathcal{H}^{i-1} f_+^{\mathrm{an}} \mathcal{M}^{\mathrm{an}} \\ &= \sum (-1)^i \operatorname{irr}(0, \mathcal{H}^{i-1} f_+^{\mathrm{an}} \mathcal{M}^{\mathrm{an}}). \end{aligned}$$

On the other hand, the properness of f yields from [HTT00, 4.7.2] a canonical identification $f_+^{\mathrm{an}} \mathcal{M}^{\mathrm{an}} \simeq (f_+ \mathcal{M})^{\mathrm{an}}$. Since the irregularity number at 0 is a formal invariant, we deduce

$$\chi(0, \operatorname{Irr}^*_0 f_+^{\mathrm{an}} \mathcal{M}^{\mathrm{an}}) = \sum (-1)^i \operatorname{irr}(0, \mathcal{H}^{i-1} f_+ \mathcal{M}).$$

For every integer i, Lemma 3.19 gives

$$\operatorname{irr}(0, \mathcal{H}^i f_+ \mathcal{M}) \leq \operatorname{rk} \mathcal{H}^i f_+ \mathcal{M} \cdot \operatorname{Max} \operatorname{Sl}_t^{\mathrm{nb}}(\mathcal{H}^i f_+ \mathcal{M}).$$

From Proposition 3.16 and the exactness of nearby cycles, we have for every i

$$\operatorname{Sl}_t^{\mathrm{nb}}(\mathcal{H}^i f_+ \mathcal{M}) \subset \operatorname{Sl}_t^{\mathrm{nb}}(f_+ \mathcal{M}) \subset \operatorname{Sl}_f^{\mathrm{nb}}(\mathcal{M}).$$

We deduce

$$|\chi(Z(\mathbb{C}), \operatorname{Sol} \mathcal{M}^{\mathrm{an}})| \leq \operatorname{Max} \operatorname{Sl}_f^{\mathrm{nb}}(\mathcal{M}) \sum \operatorname{rk} \mathcal{H}^i f_+ \mathcal{M}.$$

Since Z contains the turning locus of $\mathcal{M}$, Proposition 3.22 implies that the nearby slopes of $\mathcal{M}$ associated to f are bounded by $d \cdot \operatorname{fdeg} R$. Since cohomological boundedness holds with bound K for the generic fibre of $f : X \to C$, Proposition 3.23 follows. □

4. Partial discrepancy *b*-divisors

Before proving cohomological boundedness for surfaces, we make a détour through the theory of b-divisor to introduce the partial discrepancy of a b-divisor. As usual, k denotes a field of characteristic 0.

4.1 Chain of blow-up

Let X be a smooth surface over k. Let $n \geq 0$. Borrowing the terminology of [CPR02], a *chain of blow-up of X of length n* is a sequence $\mathbf{p}$ of maps

$$\mathbf{p} : X' := X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_0} X_0 = X \tag{4.1}$$

where p_i is a blow-up at a single closed point P_i of X_i with $p_i(P_{i+1}) = P_i$. We let $E_{\mathbf{p}}$ be the exceptional divisor of p_{n-1}. We denote by $p : X' \to X$ the composition of the p_i, $0 \leq i \leq n-1$ and by $\pi : X_{n-1} \to X$ the composition of the p_i, $0 \leq i \leq n-2$.

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The following theorem was proven by Zariski [Zar39] when k is algebraically closed. See [Cos97] for arbitrary k of characteristic 0.

Theorem 4.2. *Let X be a smooth connected surface over k. Let $k(X)$ be the function field of X. The map associating to any chain of blow-up $\mathbf{p}$ of X the valuation on $k(X)$ induced by $E_{\mathbf{p}}$ is a bijection between the set of chains of blow-up of X and $\mathrm{ZR}^{\mathrm{divis}}(X)$.*

The map $p : X' \to X$ corresponding to a divisorial valuation v via Theorem 4.2 is the smallest modification of X on which the centre of v is a divisor. This is the following Proposition 4.4. To prove it, we need the following.

Lemma 4.3. *Let $f : Y \to X$ be a modification between smooth connected surfaces over k. Let P be a closed point of X. Then either the scheme $f^{-1}(P)$ has pure dimension 1 or $f^{-1}(P)$ is a single point Q. In the latter case, there is a neighbourhood U of P in X such that the induced morphism $f|_U : f^{-1}(U) \to U$ is an isomorphism.*

Proof. This follows from the fact [Sta-b, 0C5R] that a proper birational morphism between smooth connected surfaces is a sequence of blow-up in closed points. □

Proposition 4.4. *Let X be a smooth connected surface over k. Let v be a divisorial valuation of X and let $f : Y \to X$ be a modification such that the centre of v on Y is a divisor. Let $\mathbf{p}$ be the chain of blow-up (4.1) corresponding to v via Theorem 4.2. Then, f admits a unique factorization through $p : X' \to X$.*

Proof. We argue by induction on the length $n \geq 0$ of $\mathbf{p}$. If $n = 0$, there is nothing to do. Assume that $n > 0$ and that $\mathbf{p}$ has length n as in (4.1). Let E be the centre of v on Y. Since the centre of v on X is the point P_0, the map f sends E on P_0. In particular, the scheme $f^{-1}(P_0)$ contains E. From Lemma 4.3, the scheme $f^{-1}(P_0)$ has pure dimension one. It is thus an effective Cartier divisor on Y. From the universal property of the blow-up, we deduce that f factors uniquely through $p_0 : X_1 \to X$ via a modification $g : Y \to X_1$. Observe that the chain

$$\mathbf{q} : X' := X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_1} X_1$$

is the chain corresponding to v via Theorem 4.2 applied to the smooth connected surface X_1. Since $\mathbf{q}$ has length $n-1$, the recursion assumption applies to $g : Y \to X_1$ and v and provides a unique factorization of g through $q : X' \to X_1$. We thus get the desired factorization of f through p. □

4.2 Partial discrepancy divisor

Armed with Zariski's Theorem 4.2, we are now in a position to define the partial discrepancy of a b-divisor on a smooth connected surface endowed with a reduced divisor. Although defined for every $\mathbb{Z}$-valued function on the set of divisorial valuations, the partial discrepancy is better behaved on a subgroup of the group of b-divisors that we now introduce.

Lemma 4.5. *Let (X, D) be a connected pair of dimension 2 over k. Let Z be a b-divisor on X and let us view D as a b-divisor via $\mathrm{Cart}(X) \to \boldsymbol{Div}(X)$. The following conditions are equivalent.*

(i) *The support of $Z : \mathrm{ZR}^{\mathrm{divis}}(X) \to \mathbb{Z}$ lies in the support of $D : \mathrm{ZR}^{\mathrm{divis}}(X) \to \mathbb{Z}$.*
(ii) *For every modification $f : Y \to X$, the support of $Z(Y)$ is contained in the support of f^*D.*

Proof. We assume that condition (i) holds. Let $f : Y \to X$ be a modification and let v be a divisorial valuation centred at a divisor in the support of $Z(Y)$. Then $Z(v) = (Z(Y))(v) \neq 0$. Hence, $D(v) = (f^*D)(v) \neq 0$ and condition (ii) holds. On the other hand, assume that

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condition (ii) holds. Let v be a divisorial valuation on X such that $Z(v) \neq 0$. Then, there exists a modification $f : Y \to X$ such that the centre E of v on Y is a divisor. Since $Z(v) = m(E, Z(Y)) \neq 0$, the divisor E is a component of $Z(Y)$. By assumption, we deduce that E is a component of f^*D. Thus, $D(v) = m(E, f^*D) \neq 0$. □

Definition 4.6. Let (X, D) be a connected pair of dimension 2 over k. Let Z be a b-divisor on X. If the equivalent conditions of Lemma 4.5 are satisfied, we say that Z *is a b-divisor of* (X, D). We denote by $\mathbf{Div}(X, D)$ the subgroup of $\mathbf{Div}(X)$ formed by b-divisors of (X, D).

Let Z be a b-divisor on (X, D) as in Definition 4.6. We define a new function

$$\delta Z : \mathrm{ZR}^{\mathrm{divis}}(X) \longrightarrow \mathbb{Z}$$

as follows. Let $v \in \mathrm{ZR}^{\mathrm{divis}}(X)$. Let $\mathbf{p}$ be the chain of blow-up of X as in (4.1) corresponding to v via Theorem 4.2. If $n = 0$ or if $n \geq 1$ and P_{n-1} is a singular point of the support of π^*D, we put $(\delta Z)(v) = 0$. Otherwise, we put

$$(\delta Z)(v) := (Z(X_{n-1}))(v) - Z(v),$$

where $Z(X_{n-1})$ is viewed as a b-divisor via $\mathrm{Cart}(X_{n-1}) \to \mathbf{Div}(X)$.

Remark 4.7. Assume that $n \geq 1$ and that P_{n-1} is not a singular point of the support of π^*D. Then, if P_{n-1} does not lie in the support of $Z(X_{n-1})$, we have $(Z(X_{n-1}))(v) = 0$. Otherwise, P_{n-1} lies in the support of $Z(X_{n-1})$. Since the support of π^*D contains that of $Z(X_{n-1})$, P_{n-1} is not a singular point of the support of $Z(X_{n-1})$. Thus, $(Z(X_{n-1}))(v)$ is simply the multiplicity of the unique irreducible component of $Z(X_{n-1})$ passing through P_{n-1}.

Example 4.8. If Z lies in the image of $\mathrm{Cart}(X) \to \mathbf{Div}(X)$, then $\delta Z = 0$.

Proposition 4.9. *Let (X, D) be a connected pair of dimension 2 over k. Let Z be a b-divisor of (X, D). Then δZ is a b-divisor of (X, D).*

Proof. Let $f : Y \to X$ be a modification of X. Let v be a divisorial valuation centred at a divisor of Y. Let $\mathbf{p}$ be the chain of blow-up (4.1) corresponding to v via Theorem 4.2. Assume that $n \geq 1$ and that P_{n-1} is not a singular point of the support of π^*D. From Remark 4.7, there are two cases. If P_{n-1} does not lie in support of $Z(X_{n-1})$, then

$$(\delta Z)(v) = (Z(X_{n-1}))(v) - Z(v) = -Z(v)$$

and since Z is a b-divisor, there is only a finite number of v as above for which $(\delta Z)(v) \neq 0$. Otherwise P_{n-1} is a smooth point of the support of $Z(X_{n-1})$. If w is the divisorial valuation corresponding to the unique irreducible component of $Z(X_{n-1})$ passing through P_{n-1}, we thus have $(Z(X_{n-1}))(v) = Z(w)$, so that

$$(\delta Z)(v) = Z(w) - Z(v).$$

On the other hand, Proposition 4.4 implies the existence of the following commutative triangle.

$$\begin{array}{ccc} Y & \longrightarrow & X' \\ & \searrow^{f} & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

Since the centre of w on X' is a divisor, so is its centre on Y. Since Z is a b-divisor, we again conclude that there is only a finite number of v as above for which $(\delta Z)(v) \neq 0$. This proves that δZ is a b-divisor of X. We are left to prove that δZ is a b-divisor of (X, D). Let v be

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a divisorial valuation with $D(v) = 0$. Since Z is a b-divisor of (X, D), we have $Z(v) = 0$ and $(Z(X_{n-1}))(v) = 0$. This concludes the proof of Proposition 4.9. □

DEFINITION 4.10. Let (X, D) be a connected pair of dimension 2 over k. Let Z be a b-divisor of (X, D). Then δZ is called the *partial discrepancy of* Z.

The following proposition expresses the partial discrepancy without any reference to the chains of blow-up provided by Zariski's Theorem 4.2.

PROPOSITION 4.11. *Let (X, D) be a connected pair of dimension 2 over k. Let Z be a b-divisor of (X, D). Let $f : Y \to X$ be a modification of X. Let $Y' \to Y$ be the blow-up of Y at a closed point P and let E be its exceptional divisor. Let v be a divisorial valuation on X such that the centre of v on Y is P and the centre of v on Y' is E. If P is a singular point of the support of f^*D, then $(\delta Z)(v) = 0$. Otherwise, we have $(\delta Z)(v) = (Z(Y))(v) - Z(v)$ where $Z(Y)$ is viewed as a b-divisor via* $\mathrm{Cart}(Y) \to \boldsymbol{Div}(X)$.

Proof. Let $\mathbf{p}$ be the chain of blow-up (4.1) corresponding to v via Theorem 4.2. We use the notation from § 4.1. Since the centre of v on Y is a point, so is the centre of v on X. Thus, we have $n \geq 1$. From Proposition 4.4 applied to the composition $Y' \to Y \to X$ and v, there is the following commutative diagram.

$$\begin{array}{ccccc} E & \longrightarrow & Y' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ P & \longrightarrow & Y & & X_{n-1} \\ & & & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & & & X \end{array}$$

Let w be the divisorial valuation on X corresponding to the penultimate exceptional divisor E_{n-1} of the chain $\mathbf{p}$. In particular, the centres of v and w on X_{n-1} are P_{n-1} and E_{n-1}, respectively. They are, in particular, distinct with that of w being a divisor. Thus, E and the centre of w on Y' are distinct divisors. Since all what $Y' \to Y$ does is to contract E to P, we deduce that the centre of w on Y is again a divisor. A second application of Proposition 4.4 to $f : Y \to X$ and w yields the following commutative diagram.

$$\begin{array}{ccccc} E & \longrightarrow & Y' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ P & \longrightarrow & Y & \xrightarrow{h} & X_{n-1} \\ & & & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & & & X \end{array}$$

Observe that the scheme $h^{-1}(P_{n-1})$ contains P. If $h^{-1}(P_{n-1})$ was an effective Cartier divisor, then the universal property of blow-up would provide the following commutative diagram.

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & \nearrow_{h} & \downarrow{\scriptstyle p_{n-1}} \\ Y & \longrightarrow & X_{n-1} \end{array}$$

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In particular, the centre of v on Y would be a divisor. Contradiction. Hence, $h^{-1}(P_{n-1})$ is not an effective Cartier divisor. From Lemma 4.3, we deduce the existence of a neighbourhood U of P_{n-1} in X_{n-1} such that h induces an isomorphism $h|_U : h^{-1}(U) \to U$ over X. Proposition 4.11 then follows. □

Remark 4.12. The terminology *partial* discrepancy divisor comes from the fact that we are looking at the failure of Z to lie in the image of $\mathrm{Cart}(Y) \to \mathbf{Div}(X)$ with $Y \to X$ a modification by only contemplating *the smooth points* of $Z(Y)$. The justification for this comes from Theorem 5.20, showing that de Rham cohomology only sees the partial discrepancy of the irregularity b-divisors.

Remark 4.13. Let (X, D) be a connected pair of dimension 2 over k. Let $f : Y \to X$ be a modification and put $E := (f^*D)_{red}$. Then, the canonical isomorphism $\mathbf{Div}(X) \to \mathbf{Div}(Y)$ induces an isomorphism $\mathbf{Div}(X, D) \to \mathbf{Div}(Y, E)$. We thus have the following square.

$$\begin{array}{ccc} \mathbf{Div}(X, D) & \xrightarrow{\sim} & \mathbf{Div}(Y, E) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathbf{Div}(X, D) & \xrightarrow{\sim} & \mathbf{Div}(Y, E) \end{array}$$

We stress the fact that the above square does not commute. The lemma below asserts, however, that the only obstruction lies at valuations centred at divisors of Y.

Lemma 4.14. *Let (X, D) be a connected pair of dimension 2 over k. Let $f : Y \to X$ be a modification over k and put $E := (f^*D)_{red}$. Let Z be a b-divisor on (X, D) and let T be the image of Z under $\mathbf{Div}(X, D) \to \mathbf{Div}(Y, E)$. Let v be a divisorial valuation on X centred at a closed point of Y. Then, δZ and δT coincide at v.*

Proof. We unveil the definition of $(\delta T)(v)$. Let

$$\mathbf{q} : Y' := Y_n \xrightarrow{q_{n-1}} Y_{n-1} \xrightarrow{q_{n-2}} \cdots \xrightarrow{q_0} Y_0 = Y$$

be the chain of blow-up obtained by applying Theorem 4.2 to Y and v. By assumption on v, there is at least one blow-up, that is $n \geq 1$. Let Q be the last blow-up point. Let $\rho : Y_{n-1} \to Y$ be the composition of the q_i, $0 \leq i \leq n-2$. If Q is a singular point of $(\rho^*E)_{\mathrm{red}} = (\rho^*f^*D)_{\mathrm{red}}$, then by definition $(\delta T)(v) = 0$. On the other hand, Proposition 4.11 applied to Z, v and $f\rho : Y_{n-1} \to X$ yields $(\delta Z)(v) = 0$ so that δZ and δT coincide at v in that case. Assume that Q is not a singular point of $(\rho^*E)_{\mathrm{red}} = (\rho^*f^*D)_{\mathrm{red}}$. Then, by definition, $(\delta T)(v) = (T(Y_{n-1}))(v) - T(v)$. On the other hand, Proposition 4.11 yields again $(\delta Z)(v) = (Z(Y_{n-1}))(v) - Z(v)$. Since $Z(Y_{n-1}) = T(Y_{n-1})$ and $Z(v) = T(v)$, we conclude that δZ and δT coincide at v in that case. This concludes the proof of Lemma 4.14. □

4.3 Integral of finitely supported b-divisors

Let X be a smooth connected surface over k. Let $f : Y \to X$ be a modification and let $A \subset Y$ be a subset. Let $\mathrm{ZR}^{\mathrm{divis}}(X, A) \subset \mathrm{ZR}^{\mathrm{divis}}(X)$ be the subset of divisorial valuations of X whose centre on Y is a point of A. For a b-divisor Z of X with finite support as a function on $\mathrm{ZR}^{\mathrm{divis}}(X)$, we put

$$\int_A Z := \sum_{v \in \mathrm{ZR}^{\mathrm{divis}}(X,A)} Z(v) \in \mathbb{Z}.$$

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The goal of what follows is to prove that the partial discrepancy of a nef Cartier b-divisor is effective and finitely supported.

Proposition 4.15. *Let (X, D) be a connected pair of dimension 2 over k. Let Z be a Cartier b-divisor of (X, D). Let $f : Y \to X$ be a modification of X such that Z lies in the image of $\mathrm{Cart}(Y) \to \boldsymbol{Div}(X)$. Then, δZ is supported on the set of divisorial valuations whose centre on Y is an irreducible component of f^*D.*

Proof. Put $E := (f^*D)_{red}$. Let T be the image of Z under $\mathbf{Div}(X, D) \to \mathbf{Div}(Y, E)$. By assumption, T is a Cartier b-divisor. Example 4.8 thus gives $\delta T = 0$. From Lemma 4.14, we deduce that δZ vanishes at every divisorial valuation centred at a closed point of Y. This concludes the proof of Proposition 4.15. □

Proposition 4.16. *Let (X, D) be a connected pair of dimension 2 over k. Let Z be a Cartier b-divisor of (X, D). Then, δZ is a b-divisor with finite support. In particular, the integral $\int_X \delta Z$ is a well-defined integer. If Z is furthermore nef, δZ is effective. Hence, $\int_X \delta Z$ is a positive integer.*

Proof. That δZ is a b-divisor with finite support follows from Propositions 4.9 and 4.15. We are thus left to show that δZ is effective if Z is nef. Let v be a divisorial valuation on X. Let

$$\mathbf{p} : X' := X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_0} X_0 = X$$

be the chain of blow-up corresponding to v via Theorem 4.2. Since Z is a nef Cartier divisor, we have $Z \leq Z(X_{n-1})$ in $\mathbf{CDiv}(X)$ and Proposition 4.16 follows. □

5. Formula for the characteristic cycle of connections on surfaces

5.1 Local Euler–Poincaré characteristic and characteristic cycle

Let X be a smooth variety over $\mathbb{C}$. Let $D^b_c(X(\mathbb{C}), \mathbb{C})$ be the derived category of sheaves of $\mathbb{C}$-vector spaces on $X(\mathbb{C})$ with bounded and constructible cohomology. For an object K of $D^b_c(X(\mathbb{C}), \mathbb{C})$ and a point x of $X(\mathbb{C})$, we put

$$\chi(x, K) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rk}(\mathcal{H}^i K)_x.$$

We let $CC(K)$ be the characteristic cycle of K. Let $\mathbb{F}(X)$ be the group of $\mathbb{Z}$-valued constructible functions on $X(\mathbb{C})$. Let L_X be the group of Lagrangian cycles of T^*X and let $\mathrm{Eu} : L_X \to \mathbb{F}(X)$ be the *Euler morphism*. The following theorem can be found in [KS90, Th.9.7.11].

Theorem 5.1. *The diagram*

$$\begin{array}{ccc} D^b_c(X(\mathbb{C}), \mathbb{C}) & \xrightarrow{\chi} & \mathbb{F}(X) \\ & {\scriptstyle CC} \searrow \quad \nearrow {\scriptstyle \mathrm{Eu}} & \\ & L_X & \end{array}$$

is commutative and its arrows are isomorphisms.

If S is a constructible subset of X, we denote by $\mathbb{1}_S$ the function on $X(\mathbb{C})$ which sends $x \in S$ to 1 and $x \in X(\mathbb{C}) \setminus S(\mathbb{C})$ to 0. By construction, if S is a smooth closed subvariety of X, then $\mathrm{Eu}(T^*_S X) = (-1)^{\operatorname{codim} S} \mathbb{1}_S$.

We now define a cycle that will show up frequently in this paper.

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Definition 5.2. Let (X, D) be a simple normal crossing pair over k. Let $D_1, \dots, D_n$ be the irreducible components of D. Let $R = \sum a_i D_i$ be an effective divisor supported on D. For every $I \subset \{1, \dots, n\}$, set

$$D_I := \bigcap_{i \in I} D_i \quad \text{and} \quad D_I^\circ := D_I \setminus \bigcup_{i \notin I} D_i.$$

Define

$$LC(R) := \sum_{I \subset \{1,\dots,n\}} \left(\sum_{i \in I} a_i \right) [T^*_{D_I} X].$$

Remark 5.3. When $k = \mathbb{C}$ in the above definition, $LC(R)$ is the unique Lagrangian cycle in T^*X such that

$$\mathrm{Eu}(LC(R)) = \begin{cases} -a_i & \text{if } x \in D_i^\circ,\ i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

5.2 Characteristic cycle of a connection with good formal structure

The easy direction of the main theorem of [Tey23] gives the following.

Lemma 5.4. *Let (X, D) be an analytic pair where D is smooth and connected. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Assume that $\mathcal{M}$ has good formal structure. Then* $\mathrm{Irr}^*_D \mathcal{M}$ *is a local system of rank* $\mathrm{irr}(D, \mathcal{M})$ *concentrated in degree* 1.

Proposition 5.5. *Let (X, D) be a normal crossing pair over $\mathbb{C}$. Let $D_1, \dots, D_n$ be the irreducible components of D. Let $\mathcal{M}$ be an objet of* $\mathrm{MIC}(X, D)$. *Assume that $\mathcal{M}$ has good formal structure. Then,*

$$\chi(x, \mathrm{Sol}\, \mathcal{M}^{\mathrm{an}}) = \begin{cases} \mathrm{rk}\, \mathcal{M} & \text{if } x \in X \setminus D, \\ -\mathrm{irr}(D_i, \mathcal{M}) & \text{if } x \in D_i^\circ,\ i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The restriction of $\mathrm{Sol}\, \mathcal{M}^{\mathrm{an}}$ to the complement of D is a local system concentrated in degree 0. This gives the expected formula in that case. The case where x lies in one of the D_i° follows from Lemma 5.4. Let $I \subset \{1, \dots, n\}$ and suppose that I contains at least two elements. Let x be a point of D_I°. Let $i \in I$ such that the number of irregular values of $\mathcal{M}$ at x having no pole along D_i is maximal. Such an integer exists by goodness assumption on the irregular values. If $i_I : D_I \to X$ and $j_i : D_i^\circ \to X$ are the inclusions, Corollary 3.4 of [Sab17] implies that the natural morphism

$$i_I^* \mathrm{Irr}^*_D \mathcal{M}^{\mathrm{an}} \to i_I^* R j_{i*} j_i^* \mathrm{Irr}^*_D \mathcal{M}^{\mathrm{an}}$$

is an isomorphism. Hence, the germ of $\mathrm{Sol}\, \mathcal{M}^{\mathrm{an}}$ at x is the cohomology complex of a local system of rank $\mathrm{irr}(D_i, \mathcal{M})$ on a torus of dimension $|I| - 1 > 0$. Thus, we have $\chi(x, \mathrm{Sol}\, \mathcal{M}^{\mathrm{an}}) = 0$ and this finishes the proof of Proposition 5.5. □

Proposition 5.6. *Let (X, D) be a simple normal crossing pair over k. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Assume that $\mathcal{M}$ has good formal structure. Then,*

$$CC(\mathcal{M}) = \mathrm{rk}\, \mathcal{M} \cdot CC(\mathcal{O}_X(*D)) + LC(\mathrm{Irr}(X, \mathcal{M}))$$

with $CC(\mathcal{O}_X(*D)) = \sum_{I \subset \{1,\dots,n\}} [T^*_{D_I} X]$.

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Proof. From Remark 1.6, we can suppose $k = \mathbb{C}$. Then, Proposition 5.6 follows from Proposition 5.5 after passing to the associated constructible functions. □

Remark 5.7. As pointed out by Budur, Xiao obtained in [Xia15, Theorem 1.4.1] a variant of Proposition 5.6 where the characteristic cycle is replaced by the *log-characteristic cycle* and where the good formal structure assumption is weakened into a *cleanness* assumption. For a proof of why good formal structure implies cleanness, see Theorems 3.2.11 and 3.3.12 in [Xia15, Theorem 1.4.1].

5.3 An application

Lemma 5.8. *Let (X, D) be a normal crossing pair over k. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be objects of* $\mathrm{MIC}(X, D)$. *Then, the following statements are equivalent:*

(i) $\mathrm{Irr}\,\mathcal{M}_1 = \mathrm{Irr}\,\mathcal{M}_2$ *in* $\boldsymbol{Div}(X)$;
(ii) *for every point* 0 *of* D, *for every locally closed smooth curve* $C \to X$ *in* X *meeting* D *at* 0 *only, we have* $\mathrm{irr}(0, \mathcal{M}_1|_C) = \mathrm{irr}(0, \mathcal{M}_2|_C)$.

Proof. Suppose that statement (i) holds. Let $C \to X$ be a locally closed smooth curve meeting D at 0 only. Let $p : Y \to X$ be a resolution of the turning points of $\mathcal{M}_1$ and $\mathcal{M}_2$ as given by the Kedlaya–Mochizuki Theorem 1.32. By valuative criterion for properness, the immersion $C \to X$ factors uniquely through an immersion $C \to Y$ followed by p. At the cost of blowing-up further, we can suppose that $C \to Y$ intersects $E := f^{-1}(D)$ transversely at a point $0'$ of E^{sm}. Let Z be the component of E containing $0'$. Then

$$\mathrm{irr}(0, \mathcal{M}_1|_C) = \mathrm{irr}(0', (p^+\mathcal{M}_1)|_C) = \mathrm{irr}(Z, p^+\mathcal{M}_1) = \mathrm{irr}(Z, p^+\mathcal{M}_2) = \mathrm{irr}(0, \mathcal{M}_2|_C).$$

Suppose that statement (ii) holds. Let $p : Y \to X$ be a modification and let Z be an irreducible component of $p^{-1}(D)$. Let $0'$ be a point of Z^{sm} such that $p^+\mathcal{M}_1$ and $p^+\mathcal{M}_2$ have good formal decomposition at $0'$. Put $0 := p(0')$. Let $C \to Y$ be a locally closed smooth curve meeting $p^{-1}(D)$ transversely at $0'$ only. Let $C \to X$ be the induced locally closed smooth curve of X. Then

$$\mathrm{irr}(Z, p^+\mathcal{M}_1) = \mathrm{irr}(0', (p^+\mathcal{M}_1)|_C) = \mathrm{irr}(0, \mathcal{M}_1|_C) = \mathrm{irr}(0, \mathcal{M}_2|_C) = \mathrm{irr}(Z, p^+\mathcal{M}_2)$$

and the proof of Lemma 5.8 is complete. □

Proposition 5.9. *Let (X, D) be a normal crossing pair over k. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be objects of* $\mathrm{MIC}(X, D)$ *with the same rank. Suppose that for every point* 0 *of* D, *for every locally closed smooth curve* $C \to X$ *in* X *meeting* D *at* 0 *only, we have*

$$\mathrm{irr}(0, \mathcal{M}_1|_C) = \mathrm{irr}(0, \mathcal{M}_2|_C).$$

Then $CC(\mathcal{M}_1) = CC(\mathcal{M}_2)$.

Proof. From Remark 1.6, we can suppose that $k = \mathbb{C}$. From the Kedlaya–Mochizuki Theorem 1.32 combined with Lemmas 3.17 and 1.18, we can suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ have good formal structure and that D has simple normal crossings. From Lemma 5.8, we have $\mathrm{Irr}(X, \mathcal{M}_1) = \mathrm{Irr}(X, \mathcal{M}_2)$. Hence, Proposition 5.9 follows from Proposition 5.6. □

Remark 5.10. In the étale setting, Proposition 5.9 is a theorem of Kato [Kat21].

5.4 Partial discrepancy of the irregularity divisor

Applying the partial discrepancy construction to the irregularity b-divisor, § 4 can be summarized by the following proposition.

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Proposition 5.11. *Let (X, D) be a normal crossing connected surface over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Then, $\delta \operatorname{Irr} \mathcal{M}$ is an effective b-divisor of (X, D) with finite support. In particular, $\int_X \delta \operatorname{Irr} \mathcal{M}$ is a well-defined positive integer.*

Proof. From Kedlaya's Theorem 1.44, the b-divisor $\operatorname{Irr} \mathcal{M}$ is nef and Cartier. Then, Proposition 5.11 follows from Proposition 4.16. □

The non-trivial contributions to the above integral are located above the turning points. This is the content of the following.

Lemma 5.12. *Let (X, D) be a connected normal crossing surface over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$ and let $V(\mathcal{M})$ be the complement of $\mathrm{TL}(\mathcal{M})$. Then, the restriction of $\delta \operatorname{Irr} \mathcal{M}$ to $\mathrm{ZR}^{\mathrm{divis}}(V(\mathcal{M}))$ is 0. In particular, for every subset $A \subset X$, we have*

$$\int_A \delta \operatorname{Irr} \mathcal{M} = \int_{A \cap \mathrm{TL}(\mathcal{M})} \delta \operatorname{Irr} \mathcal{M}.$$

Proof. We know that $\mathcal{M}|_{V(\mathcal{M})}$ has good formal decomposition. Thus, $(\operatorname{Irr} \mathcal{M})|_{\mathrm{ZR}^{\mathrm{divis}}(V(\mathcal{M}))} = \operatorname{Irr} \mathcal{M}|_{V(\mathcal{M})}$ lies in the image of $\mathrm{Cart}(X) \to \mathbf{Div}(X)$. Lemma 5.12 then follows from Example 4.8. □

5.5 Local Euler–Poincaré characteristic and partial discrepancy divisor

Lemma 5.13. *Let (X, D) be a normal crossing connected surface over k. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Let $q : Y \to X$ be a modification. Let Q be a closed point of Y. Then, $\delta \operatorname{Irr} \mathcal{M}$ and $\delta \operatorname{Irr} q^+\mathcal{M}$ coincide at every valuation of X centred at Q. In particular,*

$$\int_Q \delta \operatorname{Irr} \mathcal{M} = \int_Q \delta \operatorname{Irr} q^+\mathcal{M}.$$

Proof. Put $E := (q^*D)_{red}$. Then, $\operatorname{Irr} q^+\mathcal{M}$ is the image of $\operatorname{Irr} \mathcal{M}$ under $\mathbf{Div}(X, D) \to \mathbf{Div}(Y, E)$. Thus, Lemma 5.13 follows from Lemma 4.14. □

Theorem 5.14. *Let (X, D) be a normal crossing surface over $\mathbb{C}$. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Let P be a closed point of D. If P is a singular point of D, we have*

$$\chi(P, \operatorname{Sol} \mathcal{M}^{\mathrm{an}}) = \int_P \delta \operatorname{Irr} \mathcal{M}. \tag{5.15}$$

Otherwise, we have

$$\chi(P, \operatorname{Sol} \mathcal{M}^{\mathrm{an}}) = -\operatorname{irr}(Z, \mathcal{M}) + \int_P \delta \operatorname{Irr} \mathcal{M}, \tag{5.16}$$

where Z is the component of D containing P.

Proof. We argue by recursion on the number of blow-ups needed to resolve the turning points of $\mathcal{M}$ lying above P. If no blow-up is needed, this means that $\mathcal{M}$ has good formal structure in a neighbourhood of P. Then, Theorem 1.47 ensures that $\operatorname{Irr} \mathcal{M}$ lies in the image of $\mathrm{Cart}(X) \to \mathbf{Div}(X)$. Hence, Example 4.8 yields $\delta \operatorname{Irr} \mathcal{M} = 0$. Theorem 5.14 then follows from Proposition 5.5.

Let $n > 0$. Let $p : Y \to X$ be a sequence of blow-up of length n above P such that $p^+\mathcal{M}$ has good formal structure in a neighbourhood of $p^{-1}(P)$. Then, $p = f \circ q$ where $q : X_1 \to X$ is the blow-up at P and where $f : Y \to X_1$ is a sequence of $(n-1)$-blow-up above P. Let E be the exceptional divisor of q. Let D' be the strict transform of D. Let S be the set of points of E which are either turning points for $q^+\mathcal{M}$ or points of $E \cap D'$. Put $U = E \setminus S$ and let $j : U \to E$

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and $i : S \to E$ be the inclusions. The localization triangle for $(\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_E$ reads as follows:

$$j_!j^*(\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_E \longrightarrow (\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_E \longrightarrow i_*i^*(\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_E.$$

From Lemma 3.17 and the compatibility of Sol with proper push-forward, we deduce

$$\chi(P, \operatorname{Sol}\mathcal{M}^{\mathrm{an}}) = \chi_c(U, (\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_U) + \sum_{Q\in S}\chi(Q, \operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}}).$$

Since $(\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_U$ is a local system of rank $\operatorname{irr}(E, q^+\mathcal{M})$ on U concentrated in degree 1, we have

$$\chi_c(U, (\operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}})|_U) = (|S| - 2)\cdot \operatorname{irr}(E, q^+\mathcal{M}).$$

For a turning point $Q \in E$ for $q^+\mathcal{M}$ not in D', the recursion hypothesis gives

$$\begin{aligned}\chi(Q, \operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}}) &= -\operatorname{irr}(E, q^+\mathcal{M}) + \int_Q \delta \operatorname{Irr} q^+\mathcal{M}\\ &= -\operatorname{irr}(E, q^+\mathcal{M}) + \int_Q \delta \operatorname{Irr}\mathcal{M},\end{aligned}$$

where the second equality follows from Lemma 5.13. For a point Q in $E \cap D'$, the recursion hypothesis gives

$$\chi(Q, \operatorname{Sol} q^+\mathcal{M}^{\mathrm{an}}) = \int_Q \delta \operatorname{Irr} q^+\mathcal{M} = \int_Q \delta \operatorname{Irr}\mathcal{M},$$

where the second equality again follows from Lemma 5.13. Let $\operatorname{nb}_D(P)$ be the number of local branches of D around P passing through P. Note that $\operatorname{nb}_D(P)$ is equal to the cardinality of $E \cap D'$. Putting the above equalities together yields

$$\chi(P, \operatorname{Sol}\mathcal{M}^{\mathrm{an}}) = (\operatorname{nb}_D(P) - 2)\cdot \operatorname{irr}(E, q^+\mathcal{M}) + \int_E \delta \operatorname{Irr}\mathcal{M}.$$

Let v be the valuation of X corresponding to E, so that

$$\chi(P, \operatorname{Sol}\mathcal{M}^{\mathrm{an}}) = (\operatorname{nb}_D(P) - 2)\cdot \operatorname{irr}(E, q^+\mathcal{M}) - (\delta \operatorname{Irr}\mathcal{M})(v) + \int_P \delta \operatorname{Irr}\mathcal{M}.$$

If P is a singular point of D, then $\operatorname{nb}_D(P) = 2$ and $(\delta \operatorname{Irr}\mathcal{M})(v) = 0$. Thus, Theorem 5.14 is true in that case. Otherwise, let Z be the unique irreducible component of D passing through P. Then, $\operatorname{nb}_D(P) = 1$ and $(\delta \operatorname{Irr}\mathcal{M})(v) = \operatorname{irr}(Z, \mathcal{M}) - \operatorname{irr}(E, q^+\mathcal{M})$. Hence, Theorem 5.14 is again true in that case. This finishes the proof of Theorem 5.14. □

Corollary 5.17. *Let (X, D) be a normal crossing surface over an algebraically closed field of characteristic 0. Let $\mathcal{M}$ be an object of $\operatorname{MIC}(X, D)$. Let P be a closed point of D. If P lies in the smooth locus of D, the multiplicity of T^*_PX in $CC(\mathcal{M})$ is*

$$\int_P \delta \operatorname{Irr}\mathcal{M}.$$

If P lies in two irreducible components Z_1 and Z_2 of D, the multiplicity of T^*_PX in $CC(\mathcal{M})$ is

$$\operatorname{rk}\mathcal{M} + \operatorname{irr}(Z_1, \mathcal{M}) + \operatorname{irr}(Z_2, \mathcal{M}) + \int_P \delta \operatorname{Irr}\mathcal{M}.$$

Proof. From Remark 1.6 and Lemma A.6, we can suppose that the base field is $\mathbb{C}$. Suppose that P lies in D^{sm}. Then, at the cost of shrinking X, we can suppose that D is smooth and that T^*_QX

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does not contribute to $CC(\mathcal{M})$ for $Q \neq P$. Thus, we have

$$CC(\mathcal{M}) = \operatorname{rk}\mathcal{M} \cdot T^*_X X + (\operatorname{rk}\mathcal{M} + \operatorname{irr}(D, \mathcal{M})) \cdot T^*_D X + m \cdot T^*_P X,$$

where m is the sought-after multiplicity. Applying Theorem 1.15 and passing to the associated constructible functions yields

$$\chi(-, \operatorname{Sol}\mathcal{M}^{\mathrm{an}}) = \operatorname{rk}\mathcal{M} \cdot \mathbb{1}_X - (\operatorname{rk}\mathcal{M} + \operatorname{irr}(D, \mathcal{M})) \cdot \mathbb{1}_D + m \cdot \mathbb{1}_P.$$

From Theorem 5.14, evaluating at P then gives $m = \int_P \delta \operatorname{Irr}\mathcal{M}$.

Suppose that P lies in two irreducible components Z_1 and Z_2 of D. At the cost of shrinking X, we have

$$CC(\mathcal{M}) = \operatorname{rk}\mathcal{M} \cdot T^*_X X + (\operatorname{rk}\mathcal{M} + \operatorname{irr}(Z_1, \mathcal{M})) \cdot T^*_{Z_1} X + (\operatorname{rk}\mathcal{M} + \operatorname{irr}(Z_2, \mathcal{M})) \cdot T^*_{Z_2} X + m \cdot T^*_P X,$$

where m is the sought-after multiplicity. Applying Theorem 1.15 and passing to the associated constructible functions yields

$$\chi(-, \operatorname{Sol}\mathcal{M}^{\mathrm{an}}) = \operatorname{rk}\mathcal{M} \cdot \mathbb{1}_X - (\operatorname{rk}\mathcal{M} + \operatorname{irr}(Z_1, \mathcal{M})) \cdot \mathbb{1}_{Z_1} - (\operatorname{rk}\mathcal{M} + \operatorname{irr}(Z_2, \mathcal{M})) \cdot \mathbb{1}_{Z_2} + m \cdot \mathbb{1}_P.$$

From Theorem 5.14, evaluating at P gives the expected formula. □

Putting together Proposition 5.6 and Corollary 5.17 yields the following.

Theorem 5.18. *Let (X, D) be a simple normal crossing surface over an algebraically closed field of characteristic 0. Let $\mathcal{M}$ be an object of $\operatorname{MIC}(X, D)$. Then*

$$CC(\mathcal{M}) = \operatorname{rk}\mathcal{M} \cdot CC(\mathcal{O}_X(*D)) + LC(\operatorname{Irr}(X, \mathcal{M})) + \sum_{P \in D} \left(\int_P \delta \operatorname{Irr}\mathcal{M} \right) \cdot T^*_P X.$$

Remark 5.19. In the étale setting, Yatagawa obtained an explicit description of the characteristic cycle for rank-one étale sheaves on surfaces [Yat20]. It seems an interesting question to connect Yatagawa's work to the étale analogue of the irregularity b-divisor.

In the complex setting, Kashiwara and Dubson's formula stated in Theorem 1.17 yields the following Grothendieck–Ogg Shafarevich-type formula for surfaces.

Theorem 5.20. *Let (X, D) be a proper simple normal crossing surface over $\mathbb{C}$. Let $\mathcal{M}$ be an object of $\operatorname{MIC}(X, D)$. Put $U := X \setminus D$. Then*

$$\chi(X, \operatorname{DR}\mathcal{M}) = \operatorname{rk}\mathcal{M} \cdot \chi(U(\mathbb{C})) + (LC(\operatorname{Irr}(X, \mathcal{M})), T^*_X X)_{T^*X} + \int_D \delta \operatorname{Irr}\mathcal{M}.$$

6. Cohomological boundedness

6.1 Cohomological boundedness for surfaces

Lemma 6.1. *Let (X, D) be a proper simple normal crossing surface over k. Let V be an open subset of X such that $D_V := V \cap D$ is the vanishing locus of an algebraic function $f : V \to \mathbb{A}^1$. Then, there exists a commutative diagram of smooth varieties over k*

$$\begin{array}{ccc} V & \xrightarrow{j} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{6.2}$$

satisfying the following conditions.

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(i) *The map $j : V \to Y$ is a dense open immersion.*
(ii) *The map h is proper and dominant.*
(iii) *If $E := ((Y \setminus V) \cup h^{-1}(0))_{\mathrm{red}}$, the pair (Y, E) is a simple normal crossing surface with $Y \setminus E = V \setminus D_V$.*
(iv) *For every effective divisor R supported on D, there exists an effective divisor S supported on E depending only on V, on $j : (V, D_V) \to (Y, E)$ and linearly on R such that for every object $\mathcal{M}$ of $\mathrm{MIC}(X, D, R)$, the connection $j_*\mathcal{M}|_V$ is an object of $\mathrm{MIC}(Y, E, S)$.*

Proof. By Nagata compactification theorem applied to the composition $V \to \mathbb{P}^1$, there exists a commutative diagram (6.2) where $j : V \to Y$ is a dense open immersion with Y a variety over k and where h is proper. Observe that h is dominant because f is dominant. By normalizing and blowing-up enough points above the singular locus $Y^{\mathrm{sing}} \subset Y \setminus V$, resolution of singularities for surfaces ensures that we can suppose Y to be smooth. Put $E := ((Y \setminus V) \cup h^{-1}(0))_{\mathrm{red}}$. Then the equality $Y \setminus E = V \setminus D_V$ is automatic. Furthermore, $E \cap V = h^{-1}(0) \cap V = D_V$ is a simple normal crossing divisor of V. Hence, at the cost of blowing-up enough points above $E \setminus E \cap V$, we can further suppose that E is a simple normal crossing divisor of Y. Conditions (i)–(iii) are thus satisfied. Since h and $\mathbb{P}^1$ are proper, so is Y. Hence, condition (iv) is satisfied as a consequence of Proposition 3.7 and the proof of Lemma 6.1 is complete. □

Lemma 6.3. *Let (X, D) be a simple normal crossing surface over $\mathbb{C}$. Let Z be a reduced divisor of X supported on D. For a smooth point P of Z which is singular in D, let $Z(P)$ be the component of Z containing P. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Let $\mathrm{Irr}(X, Z, \mathcal{M})$ be the effective divisor supported on Z which coincides with $\mathrm{Irr}(X, \mathcal{M})$ on Z. Then,*

$$CC((\mathrm{Sol}\,\mathcal{M}^{\mathrm{an}})|_{Z(\mathbb{C})}) = LC(\mathrm{Irr}(X, Z, \mathcal{M})) + \sum_{P \in Z} \left(\int_P \delta\,\mathrm{Irr}\,\mathcal{M} \right) \cdot T^*_P X + \sum_{P \in Z^{\mathrm{sm}} \cap D^{\mathrm{sing}}} \mathrm{irr}(Z(P), \mathcal{M}) \cdot T^*_P X,$$

where $(\mathrm{Sol}\,\mathcal{M}^{\mathrm{an}})|_{Z(\mathbb{C})}$ is viewed as a complex of sheaves on X supported on $Z(\mathbb{C})$.

Proof. We conclude via Theorem 5.14 by looking at the associated constructible functions. □

Proposition 6.4. *Let (X, D) be a geometrically connected simple normal crossing surface over k. Let C be a smooth connected curve over k. Let $h : X \to C$ be a dominant proper morphism. Let 0 be a closed point in C. Suppose that the reduced fibre Z of h over 0 is not empty and is contained in D. Then, there exists a quadratic polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have*

$$\int_Z \delta\,\mathrm{Irr}\,\mathcal{M} \leq K(R, r).$$

Proof. Let R be an effective divisor of X supported on D. Let r be an integer. Let $\mathcal{M}$ be an object of $\mathrm{MIC}_r(X, D, R)$. From Lemma A.4, we can suppose that k is algebraically closed. From Lemma A.8, we can suppose that $k = \mathbb{C}$. The turning locus of $\mathcal{M}$ consists in a finite set of closed points in D. At the cost of shrinking C, we can thus suppose $\mathrm{TL}(\mathcal{M}) \subset Z$. On the other hand, Lemma 2.9 ensures that cohomological boundedness holds with bound K_1 for the generic fibre of $h : X \to C$. Note that K_1 is an affine map. Hence, Proposition 3.23 implies

$$|\chi(Z(\mathbb{C}), \mathrm{Sol}\,\mathcal{M}^{\mathrm{an}})| \leq 2K_1(R_\eta, r) \cdot \mathrm{fdeg}\,R,$$

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where R_η is the pull-back of R to the generic fibre of $h : X \to C$. Since h is proper, the complex manifold $Z(\mathbb{C})$ is compact. Hence, the index formula for constructible sheaves [Kas85, Theorem 4.3] yields

$$\begin{aligned}\chi(Z(\mathbb{C}), \operatorname{Sol}\mathcal{M}^{\mathrm{an}}) &= (CC((\operatorname{Sol}\mathcal{M}^{\mathrm{an}})|_{Z(\mathbb{C})}), T^*_X X)_{T^*X} \\ &= (LC(\operatorname{Irr}(X, Z, \mathcal{M})), T^*_X X)_{T^*X} + \int_Z \delta \operatorname{Irr}\mathcal{M} \\ &\quad + \sum_{P \in Z^{\mathrm{sm}} \cap D^{\mathrm{sing}}} \operatorname{irr}(Z(P), \mathcal{M}),\end{aligned}$$

where the second equality follows from Lemma 6.3. Hence, we have

$$\int_Z \delta \operatorname{Irr}\mathcal{M} \leq 2K_1(R_\eta, r) \cdot \operatorname{fdeg} R + |(LC(\operatorname{Irr}(X, Z, \mathcal{M})), T^*_X X)_{T^*X}|.$$

Let $Z_1, \ldots, Z_n$ be the irreducible components of Z. Then

$$|(LC(\operatorname{Irr}(X, Z, \mathcal{M})), T^*_X X)_{T^*X}| \leq \sum_{i=1}^{n} m(Z_i, R)|(Z_i, T^*_X X)_{T^*X}|.$$

If $K_2(R)$ denotes the right-hand side of the above inequality, we have

$$\int_Z \delta \operatorname{Irr}\mathcal{M} \leq 2K_1(R_\eta, r) \cdot \operatorname{fdeg} R + K_2(R)$$

and the proof of Proposition 6.4 is complete. □

Putting everything together yields the following absolute version of Proposition 6.4.

Theorem 6.5. *Let (X, D) be a geometrically connected proper simple normal crossing surface over k. Let Z be a subset of D. Then, there exists a quadratic polynomial $K : \operatorname{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have*

$$\int_Z \delta \operatorname{Irr}\mathcal{M} \leq K(R, r).$$

Proof. Let R be an effective divisor supported on D. Let r be an integer. Let $\mathcal{M}$ be an object of $\mathrm{MIC}_r(X, D, R)$. Let $\mathcal{V}$ be a finite cover of X by open subsets such that for every $V \in \mathcal{V}$, the closed subset $V \cap D$ is either empty or is the vanishing locus of an algebraic function $f_V : V \to \mathbb{A}^1$. From Proposition 5.11, the b-divisor $\delta \operatorname{Irr}\mathcal{M}$ is effective. Hence,

$$\int_Z \delta \operatorname{Irr}\mathcal{M} \leq \sum_{V \in \mathcal{V}} \int_{Z \cap V} \delta \operatorname{Irr}\mathcal{M} \leq \sum_{V \in \mathcal{V}} \int_{D \cap V} \delta \operatorname{Irr}\mathcal{M}.$$

Thus, we are left to prove Theorem 6.5 in the case where there exists an open set V in X such that $Z = D \cap V$ and such that Z is the vanishing locus of an algebraic function $f : V \to \mathbb{A}^1$. Observe that the choices of V and f depend on X and D only.

From Lemma 6.1, we can assume the existence of a proper map $h : X \to \mathbb{P}^1$ such that D contains $h^{-1}(0)$ and Z is an open subset of $h^{-1}(0)$. Since $\delta \operatorname{Irr}\mathcal{M}$ is again effective, we can further assume that Z is the reduced fibre of h over 0. This case follows from Proposition 6.4. This concludes the proof of Theorem 6.5. □

We are now in position to prove cohomological boundedness for surfaces.

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Theorem 6.6. *Let (X, D) be a projective normal crossing surface over k. Then, there exists a quadratic polynomial $C : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have*

$$\dim H^*(X, \mathrm{DR}\,\mathcal{M}) \leq C(R, r).$$

Proof. From Lemma 2.9, Corollary 2.17 and Remark 2.12, we are left to prove χ-boundedness for surfaces over $\mathbb{C}$. From Lemma 1.24 and Proposition 2.4, we can suppose that D is a simple normal crossing divisor. We can furthermore suppose that X is connected, and thus geometrically connected. From Theorem 5.20, we are left to find a quadratic polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have we have $\int_D \delta \,\mathrm{Irr}\,\mathcal{M} \leq K(R, r)$. The existence of K is ensured by Theorem 6.5, which finishes the proof of Theorem 6.6. □

6.2 Boundedness and turning locus

To relate the turning locus to de Rham cohomology, we use the following.

Theorem 6.7. *Let (X, D) be a pair over k where D is smooth. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Then, the following conditions are equivalent:*

(i) *$\mathcal{M}$ has good formal structure along D;*
(ii) *the singular support of $\mathcal{M}$ and $\mathrm{End}\,\mathcal{M}$ is contained in $T^*_X X \cup T^*_D X$.*

If, furthermore, $k = \mathbb{C}$, the above conditions are equivalent to the following condition.

(iii) *The complexes $\mathrm{Irr}^*_{D(\mathbb{C})}\,\mathcal{M}^{\mathrm{an}}$ and $\mathrm{Irr}^*_{D(\mathbb{C})}\,\mathrm{End}\,\mathcal{M}^{\mathrm{an}}$ are local systems on $D(\mathbb{C})$ concentrated in degree 1.*

Proof. From Remark 1.6, we can suppose that $k = \mathbb{C}$. The fact that condition (i) implies condition (ii) follows from Proposition 5.6. Suppose that condition (ii) holds. From Theorem 1.16, the cohomology sheaves of $\mathrm{Irr}^*_{D(\mathbb{C})}\,\mathcal{M}$ and $\mathrm{Irr}^*_{D(\mathbb{C})}\,\mathrm{End}\,\mathcal{M}$ are local systems on $D(\mathbb{C})$. From Theorem 1.13, the complexes $\mathrm{Irr}^*_{D(\mathbb{C})}\,\mathcal{M}$ and $\mathrm{Irr}^*_{D(\mathbb{C})}\,\mathrm{End}\,\mathcal{M}$ are thus necessarily concentrated in degree 1 and condition (iii) follows. The fact that condition (iii) implies condition (i) is the main result of [Tey23]. □

Theorem 6.8. *Let (X, D) be a normal crossing surface over an algebraically closed field of characteristic 0. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Let P be a point in the smooth locus of D. Then, P is a turning point of $\mathcal{M}$ if and only if*

$$\int_P \big(\delta\,\mathrm{Irr}(\mathcal{M}) + \delta\,\mathrm{Irr}(\mathrm{End}\,\mathcal{M})\big) > 0.$$

In particular,

$$|\,\mathrm{TL}(\mathcal{M})| \leq |D^{\mathrm{sing}}| + \int_{D^{\mathrm{sm}}} \big(\delta\,\mathrm{Irr}(\mathcal{M}) + \delta\,\mathrm{Irr}(\mathrm{End}\,\mathcal{M})\big).$$

Proof. From Theorem 6.7, P is a turning point of $\mathcal{M}$ if and only if $T^*_P X$ contributes to $CC(\mathcal{M}) + CC(\mathrm{End}\,\mathcal{M})$. Then, Theorem 6.8 follows from Corollary 5.17. □

Theorem 6.9. *Let (X, D) be a projective normal crossing pair over k. Let $X \to \mathbb{P}$ be a closed embedding in some projective space. Then, there exists a quadratic polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have*

$$\deg \mathrm{TL}(\mathcal{M}) \leq K(R, r).$$

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Proof. From Proposition 2.4, we can suppose that D has simple normal crossings. From Remark 1.29, we can suppose that k is algebraically closed. We can thus suppose that X is geometrically connected. If X is a curve, the turning locus is empty and there is nothing to prove. If X is a surface, $\mathrm{TL}(\mathcal{M})$ is a finite set of points. Thus, $\deg \mathrm{TL}(\mathcal{M}) = |\mathrm{TL}(\mathcal{M})|$. On the other hand, Theorems 6.8 and 6.5 ensure the existence of a quadratic polynomial $L : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ affine in the last variable such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, we have

$$|\mathrm{TL}(\mathcal{M})| \leq |D^{\mathrm{sing}}| + L(R, r) + L(R, r^2).$$

Hence, Theorem 6.9 holds in that case. Suppose that X has dimension $d \geq 3$. Let N be the dimension of $\mathbb{P}$. Let $\mathcal{G}$ be the Grassmannian of projective $N - d + 2$-spaces in $\mathbb{P}$. Let η be the generic point of $\mathcal{G}$. Let $\mathcal{Q}$ be the universal family of projective $N - d + 2$-spaces in $\mathbb{P}$. Then, there is the following commutative diagram with cartesian squares.

$$\begin{array}{ccccc} X_\eta & \longrightarrow & X_{\mathcal{Q}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Q}_\eta & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathbb{P} \\ \downarrow & & \downarrow & & \\ \eta & \longrightarrow & \mathcal{G} & & \end{array}$$

Let η as a subscript indicate a pull-back along $X_\eta \to X$. Let R be an effective divisor of X supported on D. Let $r \geq 0$ be an integer. Let $\mathcal{M}$ be an object of $\mathrm{MIC}_r(X, D, R)$. From Kedlaya's purity Theorem 1.31, the closed set $\mathrm{TL}(\mathcal{M})$ has pure dimension $d - 2$. Hence, $\deg \mathrm{TL}(\mathcal{M})$ is the cardinal of $\mathrm{TL}(\mathcal{M}) \cap H$ where H is a generic projective $N - d + 2$-spaces of $\mathbb{P}$. Thus,

$$\deg \mathrm{TL}(\mathcal{M}) = |\mathrm{TL}(\mathcal{M})_\eta| = |\mathrm{TL}(\mathcal{M}_\eta)|,$$

where the last equality follows by Remark 1.29. Since $\mathcal{M}_\eta$ lies in $\mathrm{MIC}_r(X_\eta, D_\eta, R_\eta)$ with (X_η, D_η) a normal crossing surface over η, Theorem 6.9 follows from the surface case. □

As an application of the above theorem, we deduce the following.

Theorem 6.10. *Let (X, D) be a projective normal crossing pair over k. Then, there exists a quadratic polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_r(X, D, R)$, the number of irreducible components of $\mathrm{TL}(\mathcal{M})$ is smaller than $K(R, r)$.*

Proof. Let R be an effective divisor of X supported on D. Let $r \geq 0$ be an integer. Let $\mathcal{M}$ be an object of $\mathrm{MIC}_r(X, D, R)$. Let $Z_1, \ldots, Z_n$ be the irreducible components of $\mathrm{TL}(\mathcal{M})$. From Kedlaya's purity Theorem 1.31, the Z_i have the same dimension $\dim X - 2$. Hence, $\deg \mathrm{TL}(\mathcal{M}) = \deg Z_1 + \cdots + \deg Z_n$. Since each $\deg Z_i$ is strictly positive, we deduce $n \leq \deg \mathrm{TL}(\mathcal{M})$. Thus, Theorem 6.10 follows from Theorem 6.9. □

7. Lefschetz recognition principle

Lemma 7.1. *Let $\mathbb{P}$ be a projective space over k. Let $X \subset \mathbb{P}$ be a smooth subvariety of pure dimension $n \geq 2$. Let $C', C'' \subset T^*X$ be closed conical subsets of pure dimension n where the base of C'' has dimension at most $n - 2$. Put $C = C' \cup C''$. Let $S \subset X$ be a closed subset of dimension $\leq n - 2$ containing the base of C''. Let $Y \subset \mathbb{P}$ be a smooth hypersurface transverse*

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to X such that $X \cap Y \to X$ is C'-transversal and $X \cap Y \cap S$ has dimension $< n-2$. Then, for every sufficiently generic hyperplanes $E_1, \dots, E_{n-2} \in \mathbb{P}^\vee$, the following hold.

(i) *The commutative diagram*

$$\begin{array}{ccccccc} X \cap E_1 \cap \cdots \cap E_{n-2} & \longrightarrow & \cdots & \longrightarrow & X \cap E_1 \cap E_2 & \longrightarrow & X \cap E_1 \\ & \searrow^{i_{n-2}} & & & & \searrow^{i_2} & \downarrow{\scriptstyle i_1} \\ & & & & & & X \end{array}$$

is a diagram of smooth varieties such that for every $j = 1, \dots, n-2$, the map

$$X \cap E_1 \cap \cdots \cap E_j \to X \cap E_1 \cap \cdots \cap E_{j-1}$$

is $i_{j-1}^\circ(C)$-transversal.

(ii) *The scheme $T := X \cap Y \cap E_1 \cap \cdots \cap E_{n-2}$ is a smooth curve of X avoiding S such that $T \to X$ is C-transversal.*

(iii) *The map $T \to X \cap E_1 \cap \cdots \cap E_{n-2}$ is $i_{n-2}^\circ(C)$-transversal.*

Proof. Claim (i) follows from an iterative use of Lemmas 1.5 and 1.4(iii). Let us prove claim (ii). The scheme $X \cap Y$ is a smooth variety of dimension $n-1$ and $X \cap Y \cap S \subset X \cap Y$ is a closed subset of dimension $< n-2$. For $E_1, \dots, E_{n-2} \in \mathbb{P}^\vee$ sufficiently generic, Bertini's theorem ensures that $T := X \cap Y \cap E_1 \cap \cdots \cap E_{n-2} \subset X \cap Y$ is a smooth curve avoiding S. An iterative use of Lemmas 1.5 and 1.4(iii) applied to

$$T \longrightarrow \cdots \longrightarrow X \cap Y \cap E_1 \longrightarrow X \cap Y \longrightarrow X$$

ensures that for $E_1, \dots, E_{n-2} \in \mathbb{P}^\vee$ sufficiently generic, the map $T \to X$ is C'-transversal. Since T avoid S and since S contains the base of C'', the map $T \to X$ is also C''-transversal. From Lemma 1.4(ii), we deduce that $T \to X$ is C-transversal. Claim (ii) is thus proved. Claim (iii) follows from Lemma 1.4(iii) applied to the following.

$$T \longrightarrow X \cap E_1 \cap \cdots \cap E_{n-2} \xrightarrow{i_{n-2}} X$$ □

PROPOSITION 7.2. *Let (X, D) be a projective simple normal crossing pair of dimension $n \geq 2$ over k. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D)$. Let H be a hyperplane such that:*

(i) *H is transverse to X and $X \cap H \to X$ is $SS(\mathcal{O}_X(*D))$-transversal;*

(ii) *$X \cap H$ does not contain any irreducible component of $\mathrm{TL}(\mathcal{M})$.*

Then, for every $E_1, \dots, E_{n-2} \in \mathbb{P}^\vee$ sufficiently generic, the restriction morphism

$$H^0(X, \mathrm{DR}\,\mathcal{M}) \longrightarrow H^0(T, \mathrm{DR}\,\mathcal{M}|_T) \tag{7.3}$$

is an isomorphism, where $T = X \cap H \cap E_1 \cap \cdots \cap E_{n-2}$.

Proof. Let C'' be the union of irreducible components of $SS(\mathcal{M})$ whose bases are subsets of $\mathrm{TL}(\mathcal{M})$. Then, Proposition 5.6 yields

$$SS(\mathcal{M}) \subset SS(\mathcal{O}_X(*D)) \cup C''.$$

Since $\dim \mathrm{TL}(\mathcal{M}) = n-2$, the conditions from Lemma 7.1 are satisfied for $Y = H$, $C' = SS(\mathcal{O}_X(*D))$ and C'', and $S = \mathrm{TL}(\mathcal{M})$. Thus, for every $E_1, \dots, E_{n-2} \in \mathbb{P}^\vee$ sufficiently generic,

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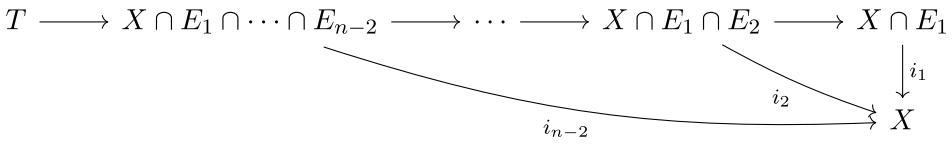
the commutative diagram



is a diagram of smooth varieties such that for every $j = 1, \ldots, n-2$, the map

$$X \cap E_1 \cap \cdots \cap E_j \to X \cap E_1 \cap \cdots \cap E_{j-1}$$

is $i_{j-1}^\circ(SS(\mathcal{M}))$-transversal and $T \to X \cap E_1 \cap \cdots \cap E_{n-2}$ is $i_{n-2}^\circ(SS(\mathcal{M}))$-transversal. From Theorem 1.20, the horizontal and vertical arrows of the above diagram are thus non-characteristic for the successive restrictions of $\mathcal{M}$. From Proposition 2.13, we deduce that each arrow of the diagram

$$H^0(X, \mathrm{DR}\,\mathcal{M}) \longrightarrow H^0(X \cap H_1, \mathrm{DR}\,\mathcal{M}|_{X \cap E_1}) \longrightarrow \cdots \longrightarrow H^0(T, \mathrm{DR}\,\mathcal{M}|_T)$$

induced by the successive restrictions of $\mathcal{M}$ are isomorphisms. This concludes the proof of Proposition 7.2. □

Corollary 7.4. *Let (X, D) be a projective simple normal crossing pair of dimension $n \geq 2$ over k. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Let $\mathcal{M}_1, \mathcal{M}_2$ be objects of* $\mathrm{MIC}(X, D)$. *Let H be a hyperplane such that:*

(i) *H is transverse to X and $X \cap H \to X$ is $SS(\mathcal{O}_X(*D))$-transversal;*
(ii) *$X \cap H$ does not contain any irreducible component of* $\mathrm{TL}(\mathcal{H}\mathrm{om}(\mathcal{M}_1, \mathcal{M}_2))$.

Then, for every $E_1, \ldots, E_{n-2} \in \mathbb{P}^\vee$ sufficiently generic, the restriction morphism

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}_T}(\mathcal{M}_1|_T, \mathcal{M}_2|_T)$$

is an isomorphism, where $T = X \cap H \cap E_1 \cap \cdots \cap E_{n-2}$.

Proof. Combine Proposition 1.25 with Proposition 7.2. □

Theorem 7.5. *Let (X, D) be a projective simple normal crossing pair of dimension $n \geq 2$ over k. Let $X \to \mathbb{P}$ be a closed immersion in some projective space. Then, there exists a polynomial $K : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ of degree 4 such that for every effective divisor R of X supported on D, for every integer $r \geq 0$, every set $\mathcal{H}$ of $K(R, r)$ hyperplanes satisfying the following conditions:*

(i) *H is transverse to X and $X \cap H \to X$ is $SS(\mathcal{O}_X(*D))$-transversal for every $H \in \mathcal{H}$;*
(ii) *$D \cap H, H \in \mathcal{H}$ are closed subsets of D of pure dimension $n-2$ with two by two distinct irreducible components;*

realizes the Lefschetz recognition principle for $\mathrm{MIC}_r(X, D, R)$ *(Definition 0.2). In particular, there is a dense open subset of $K(R, r)$-uples of hyperplanes realizing the Lefschetz recognition principle for* $\mathrm{MIC}_r(X, D, R)$.

Proof. From Theorem 6.10, there exists a quadratic polynomial $L : \mathrm{Div}(X, D) \oplus \mathbb{Z} \to \mathbb{Z}$ such that for every effective divisor R of X supported on D, for every integer $r \geq 0$ and every object $\mathcal{M}$ of $\mathrm{MIC}_{4r^2}(X, D, 2r^2 \cdot R)$, the number of irreducible components of $\mathrm{TL}(\mathcal{M}) \subset D$ is smaller than $L(2r^2 \cdot R, 4r^2)$. Put

$$K(R, r) = L(2r^2 \cdot R, 4r^2) + 1.$$

Choose a set $\mathcal{H}$ of $K(R, r)$ hyperplanes satisfying conditions (i) and (ii) above. We want to show that $\mathcal{H}$ realizes the Lefschetz recognition principle for $\mathrm{MIC}_r(X, D, R)$.

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Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{MIC}_r(X, D, R)$ such that $\mathcal{M}_1|_{X\cap H}$ and $\mathcal{M}_2|_{X\cap H}$ are isomorphic for $H \in \mathcal{H}$. In particular, for every $H \in \mathcal{H}$ and every smooth subvariety $T \subset X \cap H$, the connections $\mathcal{M}_1|_T$ and $\mathcal{M}_2|_T$ are isomorphic. We now want to show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic. To do this, it is enough to show the existence of $H \in \mathcal{H}$ and a smooth subvariety $T \subset X \cap H$ such that

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_a, \mathcal{M}_b) \longrightarrow \mathrm{Hom}_{\mathcal{D}_T}(\mathcal{M}_a|_T, \mathcal{M}_b|_T)$$

is an isomorphism for every $a, b \in \{1, 2\}$. From Corollary 7.4, it is enough to show the existence of $H \in \mathcal{H}$ such that $X \cap H$ does not contain any irreducible component of $\mathrm{TL}(\mathcal{H}\mathrm{om}(\mathcal{M}_a, \mathcal{M}_b)), a, b \in \{1, 2\}$. If we put

$$\mathcal{M} := \bigoplus_{a,b\in\{1,2\}} \mathcal{H}\mathrm{om}(\mathcal{M}_a, \mathcal{M}_b),$$

we have

$$\bigcup_{a,b\in\{1,2\}} \mathrm{TL}(\mathcal{H}\mathrm{om}(\mathcal{M}_a, \mathcal{M}_b)) \subset \mathrm{TL}(\mathcal{M}).$$

Since turning loci have pure dimension $n-2$ in virtue of Theorem 1.31, we deduce that the irreducible components of the $\mathrm{TL}(\mathcal{H}\mathrm{om}(\mathcal{M}_a, \mathcal{M}_b))$, $a, b \in \{1, 2\}$ are irreducible components of $\mathrm{TL}(\mathcal{M})$. Thus, it is enough to show the existence of $H \in \mathcal{H}$ such that $X \cap H$ does not contain any irreducible component of $\mathrm{TL}(\mathcal{M})$. Note from Proposition 2.2 that $\mathcal{M}$ is an object of $\mathrm{MIC}_{4r^2}(X, D, 2r^2 \cdot R)$. Hence, $\mathrm{TL}(\mathcal{M})$ has strictly less than $K(R, r)$ irreducible components. Thus, condition (ii) above ensures the existence of the sought-after hyperplane. □

8. Tannakian Lefschetz theorem

As usual, k denotes a field of characteristic 0. For an abelian category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{ss}}$ the full subcategory of $\mathcal{C}$ spanned by the semisimple objects. If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories, we let $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ be the full subcategory of $\mathcal{C}$ spanned by the objects sent in $\mathcal{D}^{\mathrm{ss}}$ by F.

As a general reference for Tannakian categories, let us mention [DM82]. If $(\mathcal{C}, \otimes_{\mathcal{C}})$ is a Tannakian category and if M is an object of $\mathcal{C}$, we denote by $\langle M\rangle$ the Tannakian subcategory of $\mathcal{C}$ generated by M. If $\omega : \mathcal{C} \to \mathrm{Vect}_k$ is a neutralization for $(\mathcal{C}, \otimes_{\mathcal{C}})$, we denote by $\pi_1(M, \omega)$ the Tannakian algebraic group of $\langle M\rangle$ at ω.

Example 8.1. Let (X, D) be a pair over k. Then $(\mathrm{MIC}(X, D), \otimes_{\mathcal{O}_X(*D)})$ is an abelian rigid tensor category over k. For a morphism of pairs $f : (Y, E) \to (X, D)$ over k, the pull-back $f^+ : \mathrm{MIC}(X, D) \to \mathrm{MIC}(Y, E)$ is an exact tensor functor. If, furthermore, X is connected and k is algebraically closed, the restriction to any closed point of $X \setminus D$ endows $(\mathrm{MIC}(X, D), \otimes_{\mathcal{O}_X(*D)})$ with a structure of neutral Tannakian category.

LEMMA 8.2. *Let $F : (\mathcal{C}, \otimes_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}})$ be an exact tensor functor between neutral Tannakian categories over k. Then, $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ is a neutral Tannakian subcategory of $(\mathcal{C}, \otimes_{\mathcal{C}})$.*

Proof. Let $f : X \to Y$ be a morphism in $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ and let K be its kernel in $\mathcal{C}$. Since F is exact, $F(K) \simeq \mathrm{Ker}\, F(f)$ is semisimple since $F(X)$ is semisimple [Mil17, 4.14]. Hence, K lies in $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ and similarly for the cokernel of f. Thus, $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ is an abelian subcategory of $\mathcal{C}$. Let X, Y be objects of $F^{-1}(\mathcal{D}^{\mathrm{ss}})$. Since F is a tensor functor between rigid tensor categories, we have

$$F(\underline{\mathrm{Hom}}(X, Y)) \simeq \underline{\mathrm{Hom}}(F(X), F(Y)) \simeq F(X)^\vee \otimes_{\mathcal{D}} F(Y),$$

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where the first equivalence follows from [DM82, 1.9] and the second from [DM82, 1.7]. Since the tensor product of two semisimple finite-dimensional representations of a group is again semisimple [Che54, p.88], we know that $F(X)^\vee \otimes_{\mathcal{D}} F(Y)$ is semisimple. Hence, $\underline{\mathrm{Hom}}(X,Y)$ lies in $F^{-1}(\mathcal{D}^{\mathrm{ss}})$. Furthermore, the identity object of $\mathcal{C}$ lies in $F^{-1}(\mathcal{D}^{\mathrm{ss}})$. Hence, $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ is stable under finite tensor product and dual. Thus, $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ is a rigid abelian tensor subcategory of $(\mathcal{C}, \otimes_{\mathcal{C}})$. A neutralization of $(\mathcal{C}, \otimes_{\mathcal{C}})$ induces a neutralization of $F^{-1}(\mathcal{D}^{\mathrm{ss}})$ and the proof of Lemma 8.2 is complete. □

Remark 8.3. Lemma 8.2 applied to the identity functor implies that $\mathcal{C}^{\mathrm{ss}}$ is a neutral Tannakian subcategory of $(\mathcal{C}, \otimes_{\mathcal{C}})$.

The following ground-breaking theorem is due to Mochizuki [Moc11, § 13.2.3].

Theorem 8.4. *Let (X, D) be a projective simple normal crossing pair over k. Let $\mathcal{L}$ be an ample line bundle on X. Let $\mathcal{M}$ be a semisimple object of* $\mathrm{MIC}(X, D)$. *Then, for every $m_1 > 0$, there is $m > m_1$ such that for every generic hyperplane H of $\mathcal{L}^m$, the restriction $\mathcal{M}|_{X\cap H}$ is a semisimple object of* $\mathrm{MIC}(X \cap H, D \cap H)$.

Remark 8.5. In the statement of Theorem 8.4, the integer m depends on $\mathcal{M}$.

Corollary 8.6. *Let (X, D) be a connected projective simple normal crossing pair over k. Assume that k is algebraically closed. Let $\mathcal{L}$ be an ample line bundle on X. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Then, for every integer $m_1 > 0$, there is an integer $m > m_1$ such that for every generic hyperplane H of $\mathcal{L}^m$, for every semisimple object $\mathcal{N}$ of $\langle\mathcal{M}\rangle$, the restriction $\mathcal{N}|_{X\cap H}$ is a semisimple object of* $\mathrm{MIC}(X \cap H, D \cap H)$.

Proof. Since $\langle\mathcal{M}\rangle$ admits a tensor generator, the general yoga of neutral Tannakian categories [DM82, 2.20 (b)] ensures that so does $\langle\mathcal{M}\rangle^{\mathrm{ss}}$. Let $\mathcal{N}$ be a tensor generator for $\langle\mathcal{M}\rangle^{\mathrm{ss}}$. From Mochizuki's Theorem 8.4 applied to $\mathcal{N}$, there exists an integer $m > m_1$ such that the set of hyperplanes H of $\mathcal{L}^m$ with $\mathcal{N}|_{X\cap H}$ semisimple contains a dense open subset V of $\mathbb{P}(\Gamma(X, \mathcal{L}^m))^\vee$. Let $H \in V(k)$. Let $F : \langle\mathcal{M}\rangle \to \mathrm{MIC}(X \cap H, D \cap H)$ be the exact tensor functor given by the restriction to $X \cap H$. Then, Lemma 8.2 ensures that $F^{-1}(\mathrm{MIC}(X \cap H, D \cap H)^{\mathrm{ss}})$ is a neutral Tannakian subcategory of $\langle\mathcal{M}\rangle$. Since it contains $\mathcal{N}$, it also contains $\langle\mathcal{M}\rangle^{\mathrm{ss}}$. The proof of Corollary 8.6 is thus complete. □

Lemma 8.7. *Let (X, D) be a projective pair of dimension at least 2 over k. Assume that k is uncountable. Let $\mathcal{L}$ be a very ample line bundle on X. Let $\mathcal{C}$ be a full subcategory of* $\mathrm{MIC}(X, D)$ *spanned by a countable number of objects. Then, for every very generic hyperplane H of $\mathcal{L}$, the restriction to $X \cap H$ induces a fully faithful functor on $\mathcal{C}$.*

Proof. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be objects in $\mathcal{C}$. From Lemma 1.5, the set of hyperplanes H of $\mathcal{L}$ transverse to X such that $X \cap H \to X$ is non-characteristic for $\mathcal{H}\mathrm{om}(\mathcal{M}_1, \mathcal{M}_2)$ contains a dense open subset $V(\mathcal{M}_1, \mathcal{M}_2)$. From Corollary 7.4, the set of hyperplanes H of $\mathcal{L}$ transverse to X such that

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}_{X\cap H}}(\mathcal{M}_1|_{X\cap H}, \mathcal{M}_2|_{X\cap H})$$

is an isomorphism contains $V(\mathcal{M}_1, \mathcal{M}_2)$. Then the intersection of the $V(\mathcal{M}_1, \mathcal{M}_2)$ for $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{C}$ gives the sought-after set of very generic hyperplanes. □

Before proving Theorem 8.10, we recall the following abstract lemma [Sta08, 1.1.4].

Lemma 8.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact k-linear fully faithful functor between abelian categories over k. Assume that for every object M of $\mathcal{C}$, the length of M is finite and the endomorphism*

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algebra of M has finite dimension over k. Assume that F sends semisimple objects of $\mathcal{C}$ to semisimple objects of $\mathcal{D}$. Then, the essential image of F is closed under subquotients in $\mathcal{D}$.

Remark 8.9. Let X be a smooth variety over k. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{M}$ has finite length. Furthermore, the space

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \simeq H^0(X, \mathrm{DR}\,\mathcal{M}^\vee \otimes^L_{\mathcal{O}_X} \mathcal{M})$$

is finite dimensional over k as a consequence of Proposition 1.10.

Theorem 8.10. *Let (X,D) be a connected projective simple normal crossing pair of dimension at least 2 over k. Assume that k is uncountable and algebraically closed. Let $\mathcal{L}$ be an ample line bundle on X. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X,D)$. Then, for every integer $m_1 > 0$, there is an integer $m > m_1$such that for every very generic hyperplane H of $\mathcal{L}^m$ and every point x of $(X \setminus D) \cap H$, the induced morphism of differential Galois groups*

$$\pi_1(\mathcal{M}|_{X\cap H}, x) \longrightarrow \pi_1(\mathcal{M}, x)$$

is an isomorphism.

Proof. Let $m_1 > 0$. We have to show the existence of $m > m_1$ such that for every very generic hyperplane H of $\mathcal{L}^m$, the functor $F_H : \langle \mathcal{M} \rangle \to \langle \mathcal{M}|_{X\cap H} \rangle$ of neutralized Tannakian categories induced by the restriction to $X \cap H$ is an equivalence. This amounts to show the existence of $m > m_1$ such that for every very generic H, the functor F_H is fully faithful and its essential image is closed under taking subquotients. From Remark 8.9, Lemma 8.8 applies to F_H and we are left to show the existence of $m > m_1$ such that for every very generic H, the functor F_H is fully faithful and sends semisimple objects to semisimple objects. This follows from Corollary 8.6 and Lemma 8.7. □

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Conflicts of interest

None.

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Appendix A. Base field extension lemmas

The goal of this purely technical appendix is to prove various compatibilities with base field extension of some constructions considered in this paper.

A.1 Cohomological boundedness and base field extension

Lemma A.1. *Let $d \geq 0$ be an integer. Let k be a field of characteristic 0. Then, cohomological boundedness holds in dimension d over k if it holds in dimension d over $\mathbb{C}$.*

Proof. Let (X, D) be a normal crossing pair over k. Let R be an effective divisor of X supported on D. Let κ be a finitely generated extension of $\mathbb{Q}$ such that (X, D) descends to a normal crossing pair (X_κ, D_κ). Put $S := (\mathrm{fdeg}\, R) \cdot D_\kappa$. Then, $R \leq S_k$ and S depends linearly on R. Choose an embedding $\kappa \to \mathbb{C}$. Let $\mathcal{M}$ be an object of $\mathrm{MIC}(X, D, R)$. Then, there exists an intermediate extension $\kappa \subset \kappa_{\mathcal{M}} \subset k$ such that $\kappa_{\mathcal{M}}/\kappa$ is finitely generated and such that $\mathcal{M}$ descends to an object $\mathcal{M}_{\kappa_{\mathcal{M}}}$ of $\mathrm{MIC}(X_{\kappa_{\mathcal{M}}}, D_{\kappa_{\mathcal{M}}})$, where $(X_{\kappa_{\mathcal{M}}}, D_{\kappa_{\mathcal{M}}})$ is the pull-back of (X_κ, D_κ) over $\kappa_{\mathcal{M}}$. From Lemma 1.41, we have

$$\mathrm{Irr}(X_{\kappa_{\mathcal{M}}}, \mathcal{M}_{\kappa_{\mathcal{M}}})_k \leq R \leq S_k = (S_{\kappa_{\mathcal{M}}})_k.$$

Thus,

$$\mathrm{Irr}(X_{\kappa_{\mathcal{M}}}, \mathcal{M}_{\kappa_{\mathcal{M}}}) \leq S_{\kappa_{\mathcal{M}}}.$$

Hence, $\mathcal{M}_{\kappa_{\mathcal{M}}}$ is an object of $\mathrm{MIC}(X_{\kappa_{\mathcal{M}}}, D_{\kappa_{\mathcal{M}}}, S_{\kappa_{\mathcal{M}}})$. Choose an embedding $\kappa_{\mathcal{M}} \to \mathbb{C}$ over κ. Observe that the complex variety

$$\mathbb{C} \times_{\kappa_{\mathcal{M}}} X_{\kappa_{\mathcal{M}}} \simeq \mathbb{C} \times_\kappa X_\kappa$$

does not depend on $\kappa_{\mathcal{M}}$, and thus does not depend on $\mathcal{M}$. Furthermore, $\mathbb{C} \times_\kappa D_\kappa$ is a normal crossing divisor and $\mathbb{C} \times_\kappa X_\kappa$ is projective over $\mathbb{C}$ if X is projective over k. Let us assume that X is projective of dimension d over k. By assumption, let $C : \mathrm{Div}(\mathbb{C} \times_\kappa X_\kappa, \mathbb{C} \times_\kappa D_\kappa) \oplus \mathbb{Z} \to \mathbb{Z}$ be a polynomial of degree at most d affine in the last variable such that cohomological boundedness holds for $(\mathbb{C} \times_\kappa X_\kappa, \mathbb{C} \times_\kappa D_\kappa)$ with bound C. From Lemma 1.41, we know that $(\mathcal{M}_{\kappa_{\mathcal{M}}})_{\mathbb{C}}$ is an object of $\mathrm{MIC}(\mathbb{C} \times_\kappa X_\kappa, \mathbb{C} \times_\kappa D_\kappa, S_{\mathbb{C}})$ with $S_{\mathbb{C}} = (\mathrm{fdeg}\, R) \cdot (\mathbb{C} \times_\kappa D_\kappa)$. From Remark 1.11, we deduce

$$\dim H^*(X, \mathrm{DR}\, \mathcal{M}) \leq C(S_{\mathbb{C}}, r)$$

and the proof of Lemma A.1 is complete. □

A.2 Partial discrepancy and base field extension

The goal of this subsection is to prove Lemma A.6. Lemma A.6 is used to reduce the computation of the characteristic cycle of a connection on a surface to the case where the base field is $\mathbb{C}$.

Lemma A.2. *Let (X, D) be a geometrically connected normal crossing surface over k. Let $k \subset K$ be a field extension. Let w be a divisorial valuation on X and let v be an extension of w on X_K. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Then,*

$$(\mathrm{Irr}\, \mathcal{M}_K)(v) = (\mathrm{Irr}\, \mathcal{M})(w).$$

Proof. Let $f : Y \to X$ be a modification of X such that w is centred at a divisor E of Y. Then, v is centred at an irreducible component F of E_K. Let η_E be the generic point of E and let η_F be the generic point of F. Observe that a uniformizer of $\mathcal{O}_{Y,\eta_E}$ pulls-back along $Y_K \to Y$ to a uniformizer of $\mathcal{O}_{Y_K,\eta_F}$. Hence, $\mathrm{irr}(F, f_K^+ \mathcal{M}_K) = \mathrm{irr}(E, f^+ \mathcal{M})$ and the equality $(\mathrm{Irr}\, \mathcal{M}_K)(v) = (\mathrm{Irr}\, \mathcal{M})(w)$ follows. □

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LEMMA A.3. *Let (X, D) be a geometrically connected normal crossing surface over k. Let $k \subset K$ be a field extension. Let w be a divisorial valuation on X and let v be an extension of w on X_K. Let $\mathcal{M}$ be an object of* $\mathrm{MIC}(X, D)$. *Then,*

$$(\delta \operatorname{Irr} \mathcal{M}_K)(v) = (\delta \operatorname{Irr} \mathcal{M})(w).$$

Proof. If the centre of w on X is a divisor, then the centre of v on X_K is also a divisor. In that case, $(\delta \operatorname{Irr} \mathcal{M}_K)(v) = (\delta \operatorname{Irr} \mathcal{M})(w) = 0$. Hence, we can suppose that w is centred at a closed point of X. Then, there exists a modification $p : Y \to X$ of X such that w is centred at a closed point P of Y and w is centred at the exceptional divisor E of the blow-up $Y' \to Y$ of Y at P. Then, v is centred at a closed point Q of Y_K lying over P. Let $S \subset Y_K$ be the set of closed points of Y_K lying over P. Let Y'_K be the blow-up of Y_K at S. Since blowing-up commutes with flat base change, there is the following canonical cartesian diagram of varieties.

$$\begin{array}{ccc} Y'_K & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Y_K & \longrightarrow & Y \end{array}$$

Hence, v is centred at the exceptional divisor F of Y'_K lying over Q. Thus, Proposition 4.11 applies to v and w. Consider the following cartesian diagram.

$$\begin{array}{ccc} Y_K & \longrightarrow & Y \\ {\scriptstyle f_K}\downarrow & & \downarrow{\scriptstyle f} \\ X_K & \longrightarrow & X \end{array}$$

Since $X_K \to X$ is étale, so is the map $f_K^* D_K \to f^* D$. In particular, the point Q is a singular point of $f_K^* D_K$ if and only if P is a singular point of $f^* D$. Thus, if Q is a singular point of $f_K^* D_K$, we have

$$(\delta \operatorname{Irr} \mathcal{M}_K)(v) = (\delta \operatorname{Irr} \mathcal{M})(w) = 0.$$

Otherwise,

$$\begin{aligned} (\delta \operatorname{Irr} \mathcal{M}_K)(v) &= ((\operatorname{Irr} \mathcal{M}_K)(Y_K))(v) - (\operatorname{Irr} \mathcal{M}_K)(v) \\ &= ((\operatorname{Irr} \mathcal{M})(Y))(w) - (\operatorname{Irr} \mathcal{M})(w) \\ &= (\delta \operatorname{Irr} \mathcal{M})(w), \end{aligned}$$

where the second equality follows from Lemma A.2. Lemma A.3 is thus proved. □

LEMMA A.4. *Let (X, D) be a geometrically connected normal crossing surface over k. Let $k \subset K$ be a field extension. Let P be any point of X. Let Q be a point of X_K lying over P. Then,*

$$\int_P \delta \operatorname{Irr} \mathcal{M} \leq \int_Q \delta \operatorname{Irr} \mathcal{M}_K.$$

Proof. Every divisorial valuation on X centred at P admits an extension to a divisorial valuation on X_K centred at Q. Then, Lemma A.4 follows from Lemma A.3. □

LEMMA A.5. *Let (X, D) be a connected normal crossing surface over k. Assume that k is algebraically closed. Let $k \subset K$ be a field extension. Let P be a rational point of X/k. Let Q be*

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the unique rational point of X_K/K lying over P. Then

$$\int_P \delta \operatorname{Irr} \mathcal{M} = \int_Q \delta \operatorname{Irr} \mathcal{M}_K.$$

Proof. Let $f : Y \to X$ be a modification such that $f^+\mathcal{M}$ has good formal structure. In particular, $f_K^+\mathcal{M}_K$ has good formal structure. Thus, $\operatorname{Irr} \mathcal{M} = \operatorname{Irr}(Y, f^+\mathcal{M})$ in $\mathbf{Div}(X)$ and $\operatorname{Irr} \mathcal{M}_K = \operatorname{Irr}(Y_K, f_K^+\mathcal{M}_K)$ in $\mathbf{Div}(X_K)$. Hence, Proposition 4.15 ensures that $\delta \operatorname{Irr} \mathcal{M}$ is supported on the set A of divisorial valuations associated with the irreducible components of f^*D. Similarly, $\delta \operatorname{Irr} \mathcal{M}_K$ is supported on the set B of divisorial valuations associated with the irreducible components of $f_K^*D_K$. Let F be an irreducible component of f^*D. Since k is algebraically closed, F is geometrically irreducible. Thus, F_K is irreducible. Hence, the map $f_K^*D_K \to f^*D$ induces a bijection on the sets of irreducible components. Thus, the map $B \to A$ induced by restriction is bijective. Let A_P be the subset of A of valuations centred at P on X. Let B_Q be the subset of B of valuations centred at Q on X_K. Then, the bijection $B \to A$ induces a bijection $B_Q \to A_P$. Hence,

$$\int_P \delta \operatorname{Irr} \mathcal{M} = \sum_{w \in A_P} (\delta \operatorname{Irr} \mathcal{M})(w) = \sum_{v \in B_Q} (\delta \operatorname{Irr} \mathcal{M}_K)(v) = \int_Q \delta \operatorname{Irr} \mathcal{M}_K,$$

where the second equality follows from Lemma A.3. Lemma A.6 is thus proved. □

Lemma A.6. *Let (X, D) be a geometrically connected normal crossing surface over k. Let $k \subset K_1$ and $k \subset K_2$ be field extensions where K_1 and K_2 are algebraically closed. Let P be a closed point of X. For $i = 1, 2$, let Q_i be a rational point of X_{K_i}/K_i lying over P. Then,*

$$\int_{Q_1} \delta \operatorname{Irr} \mathcal{M}_{K_1} = \int_{Q_2} \delta \operatorname{Irr} \mathcal{M}_{K_2}.$$

Proof. For $i = 1, 2$, let k_i be the algebraic closure of k in K_i. Since K_1 and K_2 are algebraically closed, k_1 and k_2 are isomorphic over k. Let P_i be the image of Q_i via $X_{K_i} \to X_{k_i}$. Consider the following commutative diagram over k.

$$\begin{array}{ccccccccc}
Q_1 & \longrightarrow & P_1 & \longrightarrow & P & \longleftarrow & P_2 & \longleftarrow & Q_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{K_1} & \longrightarrow & X_{k_1} & \longrightarrow & X & \longleftarrow & X_{k_2} & \longleftarrow & X_{K_2}
\end{array}$$

Choose an isomorphism between k_1 and k_2 over k. Let P_{21} be the pull-back of P_2 along the induced isomorphism $X_{k_1} \to X_{k_2}$. Then,

$$\int_{P_2} \delta \operatorname{Irr} \mathcal{M}_{k_2} = \int_{P_{21}} \delta \operatorname{Irr} \mathcal{M}_{k_1}.$$

Since P_{21} and P_1 are two rational points of X_{k_1}/k_1 lying over P, there exists an automorphism σ of k_1 over k such that $\sigma(P_1) = P_{21}$. Thus,

$$\int_{P_{21}} \delta \operatorname{Irr} \mathcal{M}_{k_1} = \int_{P_1} \delta \operatorname{Irr} \sigma^+\mathcal{M}_{k_1} = \int_{P_1} \delta \operatorname{Irr} \mathcal{M}_{k_1}.$$

From Lemma A.5, we deduce

$$\int_{Q_1} \delta \operatorname{Irr} \mathcal{M}_{K_1} = \int_{P_1} \delta \operatorname{Irr} \mathcal{M}_{k_1} = \int_{P_2} \delta \operatorname{Irr} \mathcal{M}_{k_2} = \int_{Q_2} \delta \operatorname{Irr} \mathcal{M}_{K_2}$$

and Lemma A.6 is proved. □

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Remark A.7. Applying Lemma A.6 to the case where $K_1 = K_2 = K$, we conclude that the integral $\int_Q \delta \operatorname{Irr} \mathcal{M}_K$ does not depend on the choice of a rational point Q of X_K/K lying over P.

LEMMA A.8. *Let (X, D) be a geometrically connected normal crossing surface over k. Let $k \subset K_1$ and $k \subset K_2$ be field extensions where K_1 and K_2 are algebraically closed. Let Z be a locally closed subset of X. Then*

$$\int_{Z_{K_1}} \delta \operatorname{Irr} \mathcal{M}_{K_1} = \int_{Z_{K_2}} \delta \operatorname{Irr} \mathcal{M}_{K_2}.$$

Proof. We are going to use Lemma A.6. We first argue that the points of Z_{K_1} and Z_{K_2} not sent to a closed point of Z do not contribute to the above integrals. From Remark 1.29, we have $\mathrm{TL}(\mathcal{M}_{K_1}) = \mathrm{TL}(\mathcal{M})_{K_1}$. Hence, Lemma 5.12 yields

$$\int_{Z_{K_1}} \delta \operatorname{Irr} \mathcal{M}_{K_1} = \int_{(Z \cap \mathrm{TL}(\mathcal{M}))_{K_1}} \delta \operatorname{Irr} \mathcal{M}_{K_1}.$$

Since $\mathrm{TL}(\mathcal{M})$ is a finite set of closed points of X, so is $Z \cap \mathrm{TL}(\mathcal{M})$. In particular, $(Z \cap \mathrm{TL}(\mathcal{M}))_{K_1}$ is a finite set of rational points of X_{K_1} lying above closed points of X. For every point P of $Z \cap \mathrm{TL}(\mathcal{M})$, choose a rational point $Q_1(P)$ of X_{K_1} above P and let us denote by $\deg P$ the degree of P over k. Since K_1 is algebraically closed, X_{K_1} admits exactly $\deg P$ rational points above P. From Remark A.7, we deduce

$$\int_{Z_{K_1}} \delta \operatorname{Irr} \mathcal{M}_{K_1} = \sum_{Q_1 \in (Z \cap \mathrm{TL}(\mathcal{M}))_{K_1}} \int_{Q_1} \delta \operatorname{Irr} \mathcal{M}_{K_1} = \sum_{P \in Z \cap \mathrm{TL}(\mathcal{M})} \deg P \cdot \int_{Q_1(P)} \delta \operatorname{Irr} \mathcal{M}_{K_1}.$$

Since the same formula holds with K_2, Lemma A.8 follows from Lemma A.6. □

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