

TOPOLOGICAL DIMENSIONS OF GLOBAL ATTRACTORS
FOR SEMILINEAR PDE'S WITH DELAYS

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An estimate is obtained on the Hausdorff and fractal dimensions of global attractors of semilinear partial differential equations with delay: $\dot{x}(t) = Ax(t) + f(x_t)$. The method employed is to associate such an equation with a nonlinear semigroup on a product space and then appeal to the upper estimate due to Constantin, Foias and Teman on topological dimensions of global attractors for general nonlinear dynamical systems.

1. INTRODUCTION.

Recent studies of nonlinear dynamical systems show that many infinite-dimensional evolution equations have global attractors with finite Hausdorff and fractal dimensions. In particular, this fact has been demonstrated by Babin and Vishik [1, 2] and Constantin, Foias and Teman [3] for Navier-Stokes equations, Koppel and Ruelle [8], Marion [11, 12] for reaction diffusion equations and Mallet-Paret [10] for (ordinary) functional differential equations.

The purpose of this paper is to extend these results to a class of semilinear partial differential equations with delay of the form:

$$\begin{aligned} \text{(FDE)} \quad & \dot{x}(t) = Ax(t) + f(x_t), \quad t \geq 0 \\ & x(0) = h \in H, \quad x_0 = \phi \in L^2([-\tau, 0]; H) \end{aligned}$$

where H is a Hilbert space, $\tau > 0$ is a given constant, $A : D(A) \rightarrow H$ generates a strongly continuous semigroup on H and $f : L^2([-\tau, 0]; H) \rightarrow H$ is everywhere defined and Lipschitz continuous. As is well known, a major difficulty in obtaining such an extension is caused by the fact that the variational equation

$$\text{(VE)} \quad \dot{u}(t) = Au(t) + Df(x_t)u_t, \quad t > 0$$

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of (FDE) about a given solution $x(\cdot)$ is not formulated on the state space $X = L^2([-r, 0]; H) \times H$ so that the upper estimate due to Constantin, Foias and Teman [3] cannot be applied to our case directly.

Our approach is to extend the generator A (in H) to \bar{A} on the state space X in a natural way (Proposition 3.2) so that the variational equation (VE) can be reformulated as an evolution system on X of the form

$$(EVE) \quad \dot{U}(t) = \bar{A}U(t) + \{0, Df(x_t)P_1U(t)\}$$

where P_1 denotes the projection of X onto $L^2([-r, 0]; H)$. This is motivated by the papers by Travis and Webb [15, 16] and Webb [17, 18] where they constructed a generator for a nonlinear semigroup and associated such a semigroup with the solutions of (FDE) by using results from the general nonlinear semigroup theory.

The rest of the paper is organised as follows. In Section 2, we collect some general results from the theory of nonlinear semigroups, dissipative dynamical systems and topological dimensions of global attractors. In Section 3, we apply the general results of Webb [17, 18] to (FDE) and obtain a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on the product space X (Theorem 3.1). Then from the generation formula of semigroups due to Crandall and Liggett [4], we establish a variation-of-constants formula for variational equations of (FDE) (Proposition 3.4) which is then applied to yield the relation between (EVE) and (VE) (Proposition 3.6). Finally, we show that the semigroup $\{T(t)\}_{t \geq 0}$ is uniformly differentiable with respect to the global attractor and that, because $D(\bar{A})$ is dense in X , the upper estimate due to Constantin, Foias and Teman [3] can then be applied to obtain our main result (Theorem 3.10) on the topological dimensions of global attractors for (FDE).

The verification of the hypotheses in Theorem 3.10 for reaction-diffusion systems with delays is a non-trivial task and will be reported in a future paper.

2. SEMIGROUPS, EVOLUTION EQUATIONS AND GLOBAL ATTRACTORS.

In this section, we set forth some necessary preliminaries from the theory of nonlinear semigroups, dissipative dynamical systems and topological dimensions of global attractors.

Throughout this section, X is a Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $|\cdot|_X$.

A (nonlinear) *strongly continuous semigroup* $\{T(t)\}_{t \geq 0}$ on X is a family of (everywhere defined) continuous mappings $T(t) : X \rightarrow X$, $t \geq 0$, satisfying the following properties:

- (i) $T(0) = I$ (identity),

- (ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$, and
- (iii) for each $x \in X$, $T(\cdot)x : [0, \infty) \rightarrow X$

is continuous.

A strongly continuous semigroup has a densely defined generator. The following "generation" theorem is due to Crandall and Liggett [4]. (See also Kato [7].)

THEOREM 2.1. *Suppose that*

- (i) $A : D(A) \subset X \rightarrow X$, where $D(A)$ is a dense subset of X ,
- (ii) for some $\gamma \in \mathbb{R}$, $\gamma I - A : D(A) \rightarrow X$ is monotone (accretive), and
- (iii) $R(I - \lambda A) = X$ for all $\lambda > 0$ sufficiently small.

Then $T(t)x \equiv \lim_{n \rightarrow \infty} [I - t/n A]^{-n}x$ exists for every $x \in X$ and $t \geq 0$. Moreover, $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on X and

$$(2.1) \quad |T(t)x - T(t)y|_X \leq e^{\gamma t}|x - y|_X, \quad \text{for all } x, y \in X, \quad t \geq 0.$$

A is called a generator for the semigroup $\{T(t)\}_{t \geq 0}$.

For the rest of this section, we let $\{T(t)\}_{t \geq 0}$ be a (fixed) strongly continuous semigroup on X . The following notions leading to the concept of a global attractor and its existence can be found in Hale [6]. We include them here for the sake of convenience.

A subset $Y \subset X$ is said to be *positively invariant* (respectively *invariant*) if $T(t)Y \subset Y$ (respectively $T(t)Y = Y$) for all $t \geq 0$. A compact invariant set is said to be a *maximal compact invariant set* if it contains all compact invariant sets. A *global attractor* Ω is a maximal compact invariant set such that $\delta(T(t)B, \Omega) \rightarrow 0$ as $t \rightarrow \infty$, for every bounded set $B \subset X$. Here, $\delta(B_0, B_1)$ denotes *semi-distance* of two sets $B_0, B_1 \subset X$, that is $\delta(B_0, B_1) := \sup_{x \in B_0} \inf_{y \in B_1} |x - y|_X$.

Obviously, Ω is unique and the dynamics on a global attractor include all the possible asymptotic behaviour (that is, as $t \rightarrow \infty$) of the given semigroup $\{T(t)\}_{t \geq 0}$. To guarantee the existence of such an attractor, we introduce the following concepts. The semigroup $\{T(t)\}_{t \geq 0}$ is said to be *bounded dissipative* if, there is a bounded set $B_0 \subset X$ such that for every bounded set $B \subset X$ there exists $t(B_0) > 0$ such that $T(t)B \subset B_0$ for all $t \geq t(B_0)$. The semigroup $\{T(t)\}_{t \geq 0}$ is *asymptotically smooth* if for any non-empty positively invariant set $B \subset X$ there exists a compact set $J \subset B$ such that $\delta(T(t)B, J) \rightarrow 0$ as $t \rightarrow \infty$. An example of an asymptotically smooth semigroup is one which is *uniformly compact*, that is, for every bounded set B , there exists $t_0 \geq 0$ such that $\bigcup_{t \geq t_0} T(t)B$ has compact closure.

The following existence result for global attractor can be found in Hale [6].

THEOREM 2.2. *If a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X is asymptotically smooth and bounded dissipative, then $\{T(t)\}_{t \geq 0}$ possesses a global attractor Ω .*

Next, we will recall the concept of Hausdorff and fractal dimensions of a compact set $\Omega \subset X$. More details can be found in Teman [14], Federler [5] and Mandelbrot [9].

For any $d > 0$, let $\mu_H(\Omega, d) = \lim_{\epsilon \rightarrow 0} \mu_H(\Omega, d, \epsilon)$ denote the d -dimensional Hausdorff measure of the set $\Omega \subset X$, where $\mu_H(\Omega, d, \epsilon) = \inf \sum_i r_i^d$ and the infimum is taken over all coverings of Ω by balls of radius $r_i \leq \epsilon$. It can be shown that there exists $d_H(\Omega) \in [0, +\infty]$ such that $\mu_H(\Omega, d) = 0$ for $d > d_H(\Omega)$ and $= \infty$ for $d < d_H(\Omega)$. $d_H(\Omega)$ is called the Hausdorff dimension of Ω .

The fractal dimension (or capacity) of Ω is defined as

$$d_F(\Omega) = \inf\{d > 0 : \mu_F(\Omega, d) = 0\},$$

where $\mu_F(\Omega, d) = \limsup_{\epsilon \rightarrow 0} \epsilon^d n_F(\Omega, \epsilon)$ and $n_F(\Omega, \epsilon)$ is the minimum number of balls of radius $\leq \epsilon$ which is necessary to cover Ω .

In order to use linearisation to estimate the Hausdorff and fractal dimension of the global attractor Ω of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ later on, we need the following notions. $\{T(t)\}_{t \geq 0}$ is said to be uniformly differentiable in Ω if, for every $x \in \Omega$ and $t \geq 0$ there exists a bounded linear operator $L(t, x) : X \rightarrow X$ such that

$$(2.2) \quad \sup_{z, y \in \Omega, 0 < |z - y| \leq \epsilon} \frac{|T(t)y - T(t)x - L(t, x)(y - x)|_X}{|y - x|_X} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

and

$$(2.3) \quad \sup_{x \in \Omega} |L(t, x)|_{BL(X, X)} < +\infty,$$

where $BL(X, X)$ is the Banach space of bounded linear operators on X with the usual operator norm $|\cdot|_{BL(X, X)}$.

Let $\{T(t)\}_{t \geq 0}$ be uniformly differentiable in Ω , $x \in \Omega$ and let N be a given positive integer. We denote by $\omega_N(L(t, x))$ the norm of the exterior product $\wedge^N L(t, x) : \wedge^N X \rightarrow \wedge^N X$, that is,

$$(2.4) \quad \omega_N(L(t, x)) = \sup_{\xi^1, \dots, \xi^N \in X, |\xi^i| \leq 1} |L(t, x)\xi^1 \wedge \dots \wedge L(t, x)\xi^N|_{\wedge^N X}.$$

Set

$$(2.5) \quad \bar{\omega}_N(t) = \sup_{x \in \Omega} \omega_N(L(t, x)), \quad t \geq 0$$

and

$$(2.6) \quad \pi_N = \lim_{t \rightarrow \infty} \bar{\omega}_N(t)^{1/t}.$$

The *uniform Lyapunov numbers* are defined as follows:

$$(2.7) \quad \mu_1 = \ln \pi_1$$

and

$$(2.8) \quad \mu_N = \ln \pi_N - \ln \pi_{N-1}, \quad \text{for } N \geq 2.$$

The following upper estimate on the Hausdorff and fractal dimensions of a global attractor was proved in Constantin, Foias and Teman [3].

THEOREM 2.3. *Assume that the hypotheses in Theorem 2.2 hold and the strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ is uniformly differentiable in the global attractor $\Omega \subset X$. If there exists an integer $N \geq 1$ such that*

$$(2.9) \quad \mu_1 + \dots + \mu_N < 0,$$

then

$$d_H(\Omega) \leq N$$

and

$$d_F(\Omega) \leq N \max_{1 \leq i \leq N-1} \left\{ 1 + \frac{(\mu_1 + \dots + \mu_i)_+}{|\mu_1 + \dots + \mu_N|} \right\}.$$

In order to investigate the uniform differentiability of a semigroup, we need to consider the following *semilinear initial value problem*:

$$(2.10) \quad \begin{cases} du(t)/dt = Au(t) + f(t, u(t)), & t > 0 \\ u(0) = u_0 \in X, \end{cases}$$

where $A : D(A) \subset X \rightarrow X$ generates a strongly continuous semigroup of bounded linear operators $T(t) : X \rightarrow X$, $t \geq 0$ and the mapping $f : [0, \infty) \times X \rightarrow X$ satisfies the following properties: $f(\cdot, x) : [0, \infty) \rightarrow X$ is continuous for each fixed $x \in X$, and for any constant $t_0 > 0$, $f(t, x)$ is uniformly Lipschitz continuous in $x \in X$ for all $t \in [0, t_0]$.

A *strong solution* of (2.10) is a continuous function $u : [0, \infty) \rightarrow X$ which is (i) continuously differentiable on $(0, \infty)$, (ii) $u(t) \in D(A)$ for $t > 0$ and (iii) (2.10) is satisfied on $(0, \infty)$. A *mild solution* is a continuous function $u : [0, \infty) \rightarrow X$ satisfying the integral equation

$$(2.11) \quad u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds, \quad t \geq 0.$$

The following existence and regularity result can be found in Pazy [13].

THEOREM 2.4. *For every $u_0 \in X$, the initial value problem (2.10) has a unique mild solution $u : [0, \infty) \rightarrow X$ and the mapping $u_0 \mapsto u$ is Lipschitz continuous from X to $C([0, t_0]; X)$ for each $t_0 > 0$. Moreover, if $f : [0, \infty) \times X \rightarrow X$ is continuously differentiable, then the mild solution of (2.10) with $u_0 \in D(A)$ is a strong solution.*

3. DIMENSIONS OF GLOBAL ATTRACTORS FOR ABSTRACT FDEs.

We now consider the following initial value problem of abstract semilinear functional differential equations (FDEs)

$$(FDE) \quad \begin{cases} \dot{x}(t) = Ax(t) + f(x_t), & t > 0 \\ x(0) = h \in H, \quad x_0 = \phi \in L^2([-r, 0]; H) \end{cases}$$

where H is a Hilbert space with an inner product $(\cdot, \cdot)_H$ and norm $|\cdot|_H$, $r > 0$ is fixed, x_t is the standard notation meaning that if $x : [-r, \infty) \rightarrow H$ and $t \geq 0$, then $x_t : [-r, 0] \rightarrow H$ is defined as $x_t(\theta) = x(t + \theta)$ for almost all $\theta \in [-r, 0]$. $f : L^2([-r, 0]; H) \rightarrow H$ is everywhere defined and Lipschitz continuous with Lipschitz constant β . $A : D(A) \subset H \rightarrow H$ is a densely defined linear operator such that $\alpha I - A$ is monotone in H for some $\alpha \in \mathbb{R}$ and that $R(I - \lambda A) = A$ for sufficiently small $\lambda > 0$.

Under the above assumptions on A , by Theorem 2.1 due to Crandall and Liggett [4], A generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ of bounded linear operators defined by

$$(3.1) \quad S(t)h = \lim_{n \rightarrow \infty} [I - \frac{t}{n}A]^{-n}h, \quad t \geq 0, \quad h \in H.$$

Moreover, $|S(t)|_{BL(H;H)} \leq e^{\alpha t}, \quad t \geq 0.$

Following Webb [17, 18], we will treat the initial value problem (FDE) in the Hilbert space $X = L^2([-r, 0]; H) \times H$ with inner product

$$(\{\phi, h\}, \{\psi, k\})_X = \int_{-r}^0 (\phi(\theta), \psi(\theta))_H d\theta + (h, k)_H \quad \text{for } \{\phi, h\}, \{\psi, k\} \in X$$

and norm

$$|\{\phi, h\}|_X = (\{\phi, h\}, \{\phi, h\})_X^{1/2} \quad \text{for } \{\phi, h\} \in X.$$

Define $B : D(B) \subset X \rightarrow X$ by

$$D(B) = \{ \{\phi, h\} \in X : \phi : [-r, 0] \rightarrow H \text{ is absolutely continuous, } \dot{\phi} \in L^2([-r, 0]; H) \text{ and } h = \phi(0) \in D(A) \},$$

$$B\{\phi, h\} = \{ \dot{\phi}, Ah + f(\phi) \}, \quad \text{for } \{\phi, h\} \in D(B).$$

The following result was proved in Webb [17].

THEOREM 3.1. *Under the above hypotheses, we have*

- (i) *B is densely defined, $\gamma I - B$ is monotone in X with $\gamma = \max\{0, 1/2 + \alpha\} + \beta$, and $R(I - \lambda B) = X$ for sufficiently small $\lambda > 0$.*
- (ii) *B generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X defined by*

$$(3.2) \quad T(t)\{\phi, h\} = \lim_{n \rightarrow \infty} [I - \frac{t}{n}B]^{-n}\{\phi, h\}, \quad \{\phi, h\} \in X, \quad t \geq 0.$$

- (iii) *$|T(t)\{\phi, h\} - T(t)\{\psi, k\}|_X \leq e^{\gamma t}|\{\phi, h\} - \{\psi, k\}|_X$, for $\{\phi, h\}, \{\psi, k\} \in X$ and $t \geq 0$.*
- (iv) *Let P_1 and P_2 denote the projections of X onto $L^2([-r, 0]; H)$ and H respectively and for every $\{\phi, h\} \in X$, define $x^{\phi, h} : [-r, \infty) \rightarrow H$ by*

$$x^{\phi, h}(t) = \begin{cases} \phi(t) & -r \leq t < 0, \\ P_2 T(t)\{\phi, h\} & t \geq 0. \end{cases}$$

Then $x_t^{\phi, h} = P_1 T(t)\{\phi, h\}$ for $t \geq 0$.

- (v) *If $\{\phi, h\} \in D(B)$, then*

$$\dot{x}^{\phi, h}(t) = Ax^{\phi, h}(t) + f(x_t^{\phi, h}), \quad \text{for almost everywhere } t \geq 0.$$

REMARK: As a consequence of (iv), we have

$$P_1 T(t)\{\phi, h\}(\theta) = \phi(t + \theta) \quad \text{for } t \geq 0, t + \theta < 0.$$

To reformulate (FDE) as an abstract ordinary differential equation we introduce the following operator \bar{A} defined in X:

$$D(\bar{A}) = \{ \{\phi, h\} \in X : \phi \text{ is absolutely continuous on } [-r, 0], \\ \dot{\phi} \in L^2([-r, 0]; H) \text{ and } h = \phi(0) \in D(A) \}$$

$$\bar{A}\{\phi, h\} = \{\dot{\phi}, Ah\}, \quad \{\phi, h\} \in D(\bar{A}).$$

By Theorem 3.1, it follows immediately that \bar{A} generates a strongly continuous semigroup $\{\bar{S}(t)\}_{t \geq 0}$ of bounded linear operators on X by the formula:

$$(3.3) \quad \bar{S}(t)\{\phi, h\} = \lim_{n \rightarrow \infty} [I - \frac{t}{n}\bar{A}]^{-n}\{\phi, h\}, \quad \{\phi, h\} \in X, \quad t \geq 0.$$

Moreover, we have

$$(3.4) \quad |\bar{S}(t)|_{BL(X, X)} \leq e^{\max\{0, 1/2 + \alpha\}t}, \quad t \geq 0.$$

The following result shows that $\{S(t)\}_{t \geq 0}$ is exactly the projection onto H of the semigroup $\{\bar{S}(t)\}_{t \geq 0}$.

PROPOSITION 3.2. $S(t)h = P_2 \bar{S}(t)\{\phi, h\}$ for all $\{\phi, h\} \in X$.

PROOF: It is easily seen that $P_2[I - t/n \bar{A}]^n\{\phi, h\} = [I - t/n A]^{-n}h$ for all $\{\phi, h\} \in X$, $t \geq 0$ and positive integer n . Therefore by (3.1) and (3.3), we get $P_2 \bar{S}(t)\{\phi, h\} = P_2 \lim_{n \rightarrow \infty} [I - t/n \bar{A}]^{-n}\{\phi, h\} = \lim_{n \rightarrow \infty} P_2 [I - t/n \bar{A}]^{-n}\{\phi, h\} = \lim_{n \rightarrow \infty} [I - t/n A]^{-n}h = S(t)h$. □

Throughout the remainder of this section, we assume that $\{T(t)\}_{t \geq 0}$ satisfies the following conditions.

- (H1) $\{T(t)\}_{t \geq 0}$ is bounded dissipative,
- (H2) $\{T(t)\}_{t \geq 0}$ is uniformly compact, and
- (H3) $T(t)X \subset D(B)$ for all $t \geq 2r$.

PROPOSITION 3.3. The semigroup $\{T(t)\}_{t \geq 0}$ has a global attractor Ω and $\Omega \subset D(B)$.

PROOF: The existence of Ω is an immediate consequence of Theorem 2.2. The conclusion that $\Omega \subset D(B)$ follows from (H3) and the invariance of Ω . □

To study the differentiability of $\{T(t)\}_{t \geq 0}$ with respect to initial data, we consider the variational equation:

$$(3.5) \quad \begin{cases} dU(t)/dt &= \bar{A}U(t) + \{0, Df(x_t^{\phi, h})P_1U(t)\}, \quad t > 0 \\ U(0) &= \{\psi, k\} \in X \end{cases}$$

where $\{\phi, h\} \in \Omega$ is given.

PROPOSITION 3.4. If $f : L^2([-r, 0]; H)$ is twice continuously differentiable (C^2), then for each $\{\phi, h\} \in \Omega$ and $\{\psi, k\} \in X$, there exists a continuous function $U = U(\phi, h, \psi, k) : [0, \infty) \rightarrow X$ such that

$$(3.6) \quad U(t) = \bar{S}(t)\{\psi, k\} + \int_0^t \bar{S}(t-s)\{0, Df(x_s^{\phi, h})P_1U(s)\}ds, \quad t \geq 0$$

Moreover, if $\{\psi, k\} \in D(\bar{A})$, then $U(t)$ is a strong solution of (3.5).

PROOF: Let $L_1 = \sup_{\Phi \in P_1\Omega} |Df(\Phi)|$, where

$$|Df(\Phi)| = \sup_{\Psi \in L^2([-r, 0]; H), |\Psi|_{L^2([-r, 0]; H)} \leq 1} |Df(\Phi)\Psi|_H.$$

Since f is C^1 and $P_1\Omega$ is compact, $L_1 < \infty$. Given any $\{\phi, h\} \in \Omega$, define $F_{\phi, h} : [0, \infty) \times X \rightarrow X$ by

$$F_{\phi, h}(t, \{\psi, k\}) = \{0, Df(x_t^{\phi, h})\psi\}, \quad t \geq 0, \quad \{\psi, k\} \in X.$$

Since $\{\phi, h\} \in \Omega \subset D(B)$, $x^{\phi, h} : [0, \infty) \rightarrow L^2([-r, 0]; H)$ is C^1 . Therefore, by the assumption that $f : L^2([-r, 0]; H) \rightarrow H$ is C^2 , it follows that $F_{\phi, h}$ is C^1 . Moreover, the invariance of Ω for the semigroup $\{T(t)\}_{t \geq 0}$ guarantees that $x_t^{\phi, h} \in P_1\Omega$ and thus $|Df(x_t^{\phi, h})| \leq L_1 < \infty$, for all $t \geq 0$. This implies that $F_{\phi, h}(t, \cdot)$ is Lipschitz continuous on X , uniformly for all $t \in [0, \infty)$. Therefore the conclusion follows from Theorem 2.4. \square

From now on, we assume that f is C^2 . The following result shows that $P_1U^{\phi, h, \psi, k}(t)$ is the segment of $P_2U^{\phi, h, \psi, k}(t)$ on $[t - r, t]$.

PROPOSITION 3.5. *Let $u^{\phi, h, \psi, k}(t) = P_2U^{\phi, h, \psi, k}(t)$, where $\{\phi, h\} \in \Omega$ and $\{\psi, k\} \in X$. Then $u_t^{\phi, h, \psi, k} = P_1U^{\phi, h, \psi, k}(t)$.*

PROOF: To simplify notation, we set $U(t) = U^{\phi, h, \psi, k}(t)$ and $u(t) = u^{\phi, h, \psi, k}(t)$. Applying (iv) of Theorem 3.1 to the semigroup $\{\bar{S}(t)\}_{t \geq 0}$, we get $P_1\bar{S}(t)\{\psi, k\}(\theta) = P_2\bar{S}(t + \theta)\{\psi, k\}$, for $t \geq 0$, $\theta \in [-r, 0]$ and $t + \theta \geq 0$. Applying P_2 on both sides of (3.6) at $t + \theta$, we get

$$\begin{aligned} P_2U(t + \theta) &= P_2\bar{S}(t + \theta)\{\psi, k\} + P_2 \int_0^{t+\theta} \bar{S}(t + \theta - s)\{0, Df(x_s^{\phi, h})P_1U(s)\}ds \\ &= P_1\bar{S}(t)\{\psi, k\}(\theta) + \int_0^{t+\theta} P_1\bar{S}(t - s)\{0, Df(x_s^{\phi, h})P_1U(s)\}(\theta)ds \\ &= P_1\bar{S}(t)\{\psi, k\}(\theta) + P_1 \int_0^t \bar{S}(t - s)\{0, Df(x_s^{\phi, h})P_1U(s)\}(\theta)ds \\ &= P_1U(t)(\theta), \end{aligned}$$

where the equality

$$\int_0^{t+\theta} P_1\bar{S}(t - s)\{0, Df(x_s^{\phi, h})P_1U(s)\}(\theta)ds = P_1 \int_0^t \bar{S}(t - s)\{0, Df(x_s^{\phi, h})P_1U(s)\}(\theta)ds$$

holds because by the Remark following Theorem 3.1, $P_1\bar{S}(t - s)\{0, Df(x_s^{\phi, h})P_1U(s)\}(\theta) = 0$, whenever $t + \theta - s \leq 0$. The conclusion then follows from the definition of P_2 and u_t . \square

PROPOSITION 3.6. *If $\{\phi, h\} \in \Omega$ and $\{\psi, k\} \in D(\bar{A})$ then $u^{\phi, h, \psi, k} : [0, \infty) \rightarrow H$ is differentiable and*

$$(3.7) \quad \dot{u}^{\phi, h, \psi, k}(t) = Au^{\phi, h, \psi, k} + Df(x_t^{\phi, h})u_t^{\phi, h, \psi, k}, \quad t \geq 0.$$

PROOF: Since $\{\phi, h\} \in \Omega$ and $\{\psi, k\} \in D(\bar{A})$, by Proposition 3.4, $U^{\phi, h, \psi, k}$ is a strong solution of (3.5). Therefore $u^{\phi, h, \psi, k} = P_2U^{\phi, h, \psi, k} : [0, \infty) \rightarrow H$ is differentiable. Applying P_2 on both sides of (3.5) and using Proposition 3.5, we obtain (3.7). \square

PROPOSITION 3.7. *For any constant $t_0 > 0$, there exists a constant $L > 0$ such that*

$$|x_t^{\phi,h} - x_t^{\psi,k}|_X \leq L|\{\phi, h\} - \{\psi, k\}|_X, \quad \text{for all } \{\phi, h\}, \{\psi, k\} \in \Omega, \quad t \in [0, t_0].$$

PROOF: To simplify notation, we set $x(t) = x^{\phi,h}(t)$ and $y(t) = x^{\psi,k}(t)$. Since $\{\phi, h\}, \{\psi, k\} \in D(B)$, by Proposition 3.1, we have

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x_t), \quad t \in [0, t_0] \quad \text{and} \\ \dot{y}(t) &= Ay(t) + f(y_t), \quad t \in [0, t_0], \end{aligned}$$

from which it follows that

$$(3.8) \quad x(t) = S(t)h + \int_0^t S(t-s)f(x_s)ds, \quad t \in [0, t_0]$$

$$(3.9) \quad y(t) = S(t)k + \int_0^t S(t-s)f(y_s)ds, \quad t \in [0, t_0].$$

Therefore

$$x(t) - y(t) = S(t)(h - k) + \int_0^t S(t-s)[f(x_s) - f(y_s)]ds, \quad t \in [0, t_0].$$

Since $|S(t)|_{BL(H;H)} \leq e^{\alpha t}$ for $t \in [0, t_0]$, we have

$$(3.10) \quad \begin{aligned} |x(t) - y(t)|_H &\leq e^{\alpha t}|h - k|_H + \int_0^t e^{\alpha(t-s)}\beta|x_s - y_s|_{L^2}ds \\ &\leq \max\{e^{\alpha t_0}, 1\} \left[|h - k|_H + \beta \int_0^t |x_s - y_s|_{L^2}ds \right], \end{aligned}$$

where for any $\rho \in L^2([-r, 0]; H)$, $|\rho|_{L^2} = \left(\int_{-r}^0 |\rho(\theta)|_H^2 d\theta \right)^{1/2}$. Thus by using the

inequality $(a + b)^2 \leq 2(a^2 + b^2)$ ($a, b \in \mathbb{R}$), we obtain

$$\begin{aligned} & \int_{-r}^0 |x(t + \theta) - y(t + \theta)|_H^2 d\theta \\ &= \int_{[t-r,t] \cap [-r,0]} |x(\xi) - y(\xi)|_H^2 d\xi + \int_{[t-r,t] \setminus [-r,0]} |x(\xi) - y(\xi)|_H^2 d\xi \\ &\leq \int_{[t-r,t] \cap [-r,0]} |\phi(\xi) - \psi(\xi)|_H^2 d\xi \\ &+ 2(\max\{e^{\alpha t_0}, 1\})^2 \int_{[t-r,t] \setminus [-r,0]} \left[|h - k|_H^2 + \beta^2 \left(\int_0^\xi |x_s - y_s|_{L^2} ds \right)^2 \right] d\xi \\ &\leq \int_{-r}^0 |\phi(\xi) - \psi(\xi)|_H^2 d\xi + 2(\max\{e^{\alpha t_0}, 1\})^2 \int_{-r}^0 \left[|h - k|_H^2 + \beta^2 \left(\int_0^t |x_s - y_s|_{L^2} ds \right)^2 \right] d\xi \\ &\leq \max\{1, 2r(\max\{e^{\alpha t_0}, 1\})^2\} |\{\phi, h\} - \{\psi, k\}|_X^2 \\ &+ 2(\max\{e^{\alpha t_0}, 1\})^2 \beta^2 r \left(\int_0^t |x_s - y_s|_{L^2} ds \right)^2. \end{aligned}$$

From the inequality $\sqrt{a^2 + b^2} \leq |a| + |b|$ ($a, b \in \mathbb{R}$), it follows that

$$|x_t - y_t|_{L^2} \leq \sqrt{\max\{1, M_0\}} |\{\phi, h\} - \{\psi, k\}|_X + \sqrt{M_0 \beta} \int_0^t |x_s - y_s|_{L^2} ds,$$

where $M_0 = 2(\max\{e^{\alpha t_0}, 1\})^2 r$. By the well-known Gronwall inequality, we obtain

$$(3.11) \quad |x_t - y_t|_{L^2} \leq \sqrt{\max\{1, M_0\}} |\{\phi, h\} - \{\psi, k\}|_X e^{\sqrt{M_0 \beta} t}.$$

Substituting (3.11) into (3.10), we get

$$(3.12) \quad |x(t) - y(t)|_H \leq \max\{e^{\alpha t_0}, 1\} \left[|h - k|_H + \sqrt{\frac{\max\{1, M_0\}}{M_0}} |\{\phi, h\} - \{\psi, k\}|_X e^{\sqrt{M_0 \beta} t} \right].$$

Therefore the conclusion follows from (3.11) and (3.12). □

PROPOSITION 3.8. *For any $t_0 > 0$, there exists $M > 0$ such that if $\{\phi, h\}, \{\psi, k\} \in \Omega$, then*

$$|x_t^{\psi, k} - x_t^{\phi, h} - u_t^{\phi, h, \psi - \phi, k - h}|_X \leq M |\{\psi, k\} - \{\phi, h\}|_X^2$$

for all $t \in [0, t_0]$.

PROOF: Let

$$(3.13) \quad L_2 := \sup_{\Phi \in \overline{C\partial}(P_1 \Omega)} |D^2 f(\Phi)|.$$

Since f is C^2 and $\overline{\text{co}}(P_1\Omega)$ is compact, $L_2 < \infty$. To simplify the presentation, we set $x(t) = x^{\phi,h}(t)$, $y(t) = x^{\psi,k}(t)$, $u(t) = u^{\phi,h,\psi-\phi,k-h}(t)$, and $w(t) = y(t) - x(t) - u(t)$ for $t \geq -r$. Applying P_2 on both sides of (3.6) and using Proposition 3.2 and 3.5, we obtain

$$u(t) = S(t)u(0) + \int_0^t S(t-s)Df(x_s)u_s ds, \quad t \geq 0.$$

Therefore from (3.8) and (3.9), it follows that

$$\begin{aligned} w(t) &= \int_0^t S(t-s)[f(y_s) - f(x_s) - Df(x_s)u_s] ds \\ &= \int_0^t S(t-s) \left[\int_0^1 Df(\tau y_s + (1-\tau)x_s)(y_s - x_s) d\tau - Df(x_s)u_s \right] ds \\ &= \int_0^t S(t-s) \left[\int_0^1 Df(\tau y_s + (1-\tau)x_s) d\tau - Df(x_s) \right] (y_s - x_s) ds \\ &\quad + \int_0^t S(t-s)Df(x_s)w_s ds \\ &= \int_0^t S(t-s) \left[\int_0^1 \int_0^1 D^2 f(\theta(\tau y_s + (1-\tau)x_s) \right. \\ &\quad \left. + (1-\theta)x_s) d\theta \tau(y_s - x_s) d\tau \right] (y_s - x_s) ds + \int_0^t T(t-s)Df(x_s)w_s ds. \end{aligned}$$

Since $x_s, y_s \in P_1\Omega$ for all $s \geq 0$, $\theta(\tau y_s + (1-\tau)x_s) + (1-\theta)x_s \in \overline{\text{co}}(P_1\Omega)$, the closed convex hull of $P_1(\Omega)$, for all $\theta, \tau \in [0, 1]$. Therefore by (3.13) we get

$$\begin{aligned} |w(t)|_H &\leq \int_0^t e^{\alpha(t-s)} L_2 |y_s - x_s|_{L^2}^2 ds + \int_0^t e^{\alpha(t-s)} L_1 |w_s|_{L^2} ds \\ &\leq t_0 \max\{e^{\alpha t_0}, 1\} L_2 \max_{s \in [0, t_0]} |y_s - x_s|_{L^2}^2 + \int_0^t L_1 \max\{e^{\alpha t_0}, 1\} |w_s|_{L^2} ds. \end{aligned}$$

By Proposition 3.7, we have

$$(3.14) \quad |w(t)|_H \leq M_1 |\{\psi, k\} - \{\phi, h\}|_X^2 + M_2 \int_0^t |w_s|_{L^2} ds,$$

where $M_1 = t_0 \max\{e^{\alpha t_0}, 1\} L_2 L_1^2$ and $M_2 = L_1 \max\{e^{\alpha t_0}, 1\}$. Therefore,

$$\begin{aligned} |w_t|_{L^2}^2 &= \int_{-r}^0 |w(t+\theta)|_H^2 d\theta \\ &\leq 2M_1^2 \tau |\{\psi, k\} - \{\phi, h\}|_X^4 + 2M_2^2 \tau \left(\int_0^t |w_s|_{L^2} ds \right)^2, \end{aligned}$$

from which it follows that

$$|w_t|_{L^2} \leq \sqrt{2r}M_1|\{\psi, k\} - \{\phi, h\}|_X^2 + M_2\sqrt{2r} \int_0^t |w_s|_{L^2} ds.$$

This implies, by the Gronwall inequality, that

$$(3.15) \quad |w_t|_{L^2} \leq \sqrt{2r}M_1|\{\psi, k\} - \{\phi, h\}|_X^2 e^{M_2\sqrt{2r}t}.$$

Substituting (3.15) into (3.14), we get

$$(3.16) \quad |w(t)|_H \leq M_1|\{\psi, k\} - \{\phi, h\}|_X^2 + M_1|\{\psi, k\} - \{\phi, h\}|_X^2 e^{M_2\sqrt{2r}t}$$

for all $t \in [0, t_0]$. The conclusion then follows from (3.15) and (3.16). □

THEOREM 3.9. *Assume that f is C^2 . The semigroup $\{T(t)\}_{t \geq 0}$ is uniformly differentiable in Ω .*

PROOF: For any given $\{\phi, h\} \in \Omega$ and $t \geq 0$, define the linear operator $L(t, \{\phi, h\}) : X \rightarrow X$ by:

$$L(t, \{\phi, h\})\{\psi, k\} = U^{\phi, h, \psi, k}(t), \quad \text{for } \{\psi, k\} \in X.$$

By Proposition 3.8, we have

$$\sup_{\{\phi, h\}, \{\psi, k\} \in \Omega, 0 < |\{\phi, h\} - \{\psi, k\}|_X < \varepsilon} \frac{|T(t)\{\psi, k\} - T(t)\{\phi, h\} - L(t, \{\phi, h\})(\{\psi, k\} - \{\phi, h\})|_X}{|\{\psi, k\} - \{\phi, h\}|_X} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Therefore, it suffices to show the linear operator $L(t, \{\phi, h\})$ is bounded.

Let

$$u(t) = \begin{cases} P_2 U^{\phi, h, \psi, k}(t) & \text{for } t \geq 0 \\ \psi(t) & \text{for } -r \leq t < 0. \end{cases}$$

If $\{\psi, k\} \in D(\overline{A})$, by Proposition 3.6, we have

$$u'(t) = Au(t) + Df(x_t)u_t \quad \text{for } t \geq 0,$$

where $x = x^{\phi, h}$. Thus

$$u(t) = S(t)u(0) + \int_0^t S(t-s)Df(x_s)u_s ds.$$

From this, it follows that

$$|u(t)|_H \leq e^{\alpha t}|k|_H + \int_0^t e^{\alpha(t-s)}L_1|u_s|_{L^2} ds.$$

Using a similar argument as in the proof of Propositions 3.7 and 3.8, we see that for any $t_0 > 0$, there exists a constant $V > 0$ such that

$$(3.17) \quad |u_t|_X \leq V|u_0|_X = V|\{\psi, k\}|_X \quad \text{for all } t \in [0, t_0].$$

Since the mild solution of the initial value problem (3.5) depends continuously on initial data and $D(\bar{A})$ is dense in X , (3.17) holds for all $\{\psi, k\} \in X$. Therefore $L(t, \{\phi, h\})$ is a bounded and

$$\sup_{\{\phi, h\} \in \Omega} |L(t, \{\phi, h\})|_{BL(X; X)} \leq V, \quad \text{for all } t \in [0, t_0].$$

□

We are now in a position to estimate the Lyapunov numbers and topological dimensions of the global attractor Ω .

Let $\{\phi, h\} \in \Omega$, $\{\psi_i, k_i\} \in D(\bar{A})$ and $U_i = U_i^{\phi, h, \psi_i, k_i}$, $i = 1, \dots, m$. By Proposition 3.4, we have

$$\frac{dU_i(t)}{dt} = \bar{A}U_i(t) + \{0, Df(x_i^{\phi, h})\}P_1U_i(t), \quad t \geq 0.$$

Therefore, by employing an argument similar to that for (2.40) in Teman [14, Chapter V], we obtain

$$(3.18) \quad \frac{1}{2} \frac{d}{dt} |U_1(t) \wedge \dots \wedge U_m(t)|_{\wedge^m X}^2 = |U_1(t) \wedge \dots \wedge U_m(t)|_{\wedge^m X}^2 \text{Tr}(G(t) \circ Q_m(t)),$$

where

$$(3.19) \quad Q_m(t) = Q_m(t, \phi, h, \psi_1, k_1, \dots, \psi_m, k_m)$$

is the orthogonal projection of X onto the space spanned by $U_1(t), \dots, U_m(t)$ and $G(t) = G(t, \phi, h) : X \rightarrow X$ is defined by

$$(3.20) \quad G(t)\{\psi, k\} = \bar{A}\{\psi, k\} + \{0, Df(x_i^{\phi, h})\}\psi.$$

Therefore

$$\begin{aligned} & |U_1(t) \wedge \dots \wedge U_m(t)|_{\wedge^m X} \\ &= |U_1(0) \wedge \dots \wedge U_m(0)|_{\wedge^m X} \exp\left(\int_0^t \text{Tr}(G(\tau) \circ Q_m(\tau))d\tau\right) \\ &= |\{\psi_1, k_1\} \wedge \dots \wedge \{\psi_m, k_m\}|_{\wedge^m X} \exp\left(\int_0^t \text{Tr}(G(\tau) \circ Q_m(\tau))d\tau\right). \end{aligned}$$

Let

$$(3.21) \quad q_m(t) = \sup_{\{\phi, h\} \in \Omega, \{\psi_i, k_i\} \in D(\bar{A}), |\{\psi_i, k_i\}|_X \leq 1, i=1, \dots, m} \frac{1}{t} \int_0^t \text{Tr}(G(\tau) \circ Q_m(\tau)) d\tau$$

and

$$(3.22) \quad q_m = \limsup_{t \rightarrow \infty} q_m(t).$$

Then we have

$$(3.23) \quad |U_1(t) \wedge \dots \wedge U_m(t)|_{\wedge^m X} \leq |\{\psi_1, k_1\} \wedge \dots \wedge \{\psi_m, k_m\}|_{\wedge^m X} \exp\{tq_m(t)\}.$$

Since the mild solution of the initial value problem (3.5) depends continuously on initial data and $D(\bar{A})$ is dense in X , (3.23) holds for all $\{\psi_i, k_i\} \in X, i = 1, \dots, m$. Therefore using the notations in (2.4)–(2.8), we have the following estimation:

$$(3.24) \quad \bar{\omega}_m(t) \leq \exp\{tq_m(t)\},$$

and

$$(3.25) \quad \pi_m \leq e^{q_m}.$$

Therefore

$$(3.26) \quad \mu_1 + \dots + \mu_m = \ln \pi_m \leq q_m.$$

By Theorem 2.3, we obtain our main result:

THEOREM 3.10. *Suppose that*

- (i) *(H1)–(H3) are satisfied, and*
- (ii) *$f : L^2([-r, 0]; H) \rightarrow H$ is twice continuously differentiable.*

If $q_m < 0$ for some m , then the Hausdorff dimension of Ω is less than or equal to m and the fractal dimension of Ω is less than or equal to $m \max_{1 \leq j \leq m-1} (1 + (q_j)_+ / |q_m|)$.

REMARK: Some smoothness condition on f (for example, (ii) in Theorem 3.10) seems to be necessary in order for Ω to be of finite dimension (see Yorke [19]).

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