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On Locating Isometric $\ell_1^{(n)}$

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Abstract. Motivated by a question of Per Enflo, we develop a hypercube criterion for locating linear isometric copies of $\ell_1^{(n)}$ in an arbitrary real normed space X.

The said criterion involves finding 2^n points in X that satisfy one metric equality. This contrasts nicely to the standard classical criterion wherein one seeks n points that satisfy 2^{n-1} metric equalities.

1 Introduction and Terminology

The study of hypercubes (or, more simply, n-cubes) spans several areas of mathematics, ranging from Banach space theory to problems in complexity. See, for example, the papers of Enflo [E], Bourgain, Milman, and Wolfson [BMW], Lövblom [L], Lennard, Tonge, and Weston [LTW2], and the monographs of Deza and Laurent [DL], and Pisier [P].

In order discuss our hypercubes, which are *a priori* non-linear, and to motivate the hypercube criterion that is the topic of this paper, it will be helpful to comment on the properties of some well-known linear cubes. To this end, the 2^n points $\mathcal{C} = \{0,1\}^n$ form a natural n-dimensional cube in $\ell_p^{(n)}$, $p \geq 1$. In relation to $\mathcal{C}$, the terms "edge" and "diagonal" have their natural and well established meaning. If E denotes the set of edges in $\mathcal{C}$ and D denotes the set of diagonals in $\mathcal{C}$, then

$$\sum_{e \in E} \|e\|_1 = \sum_{d \in D} \|d\|_1 \tag{1.1}$$

while

$$\sum_{e \in E} \|e\|_p > \sum_{d \in D} \|d\|_p \tag{1.2}$$

if $p > 1$.

In this paper, we shall be interested in the following situation: consider an arbitrary collection $\mathcal{C}$ of 2^n distinct points in a normed space $(X, \|\cdot\|)$ where each point in $\mathcal{C}$ is indexed by a distinct $\{0,1\}^n$ vector. $\mathcal{C}$ now presents itself as a non-linear n-dimensional cube, with edge and diagonal terminology determined via the indexing vectors in the natural way. Let E denote the set of edges in $\mathcal{C}$, and let D denote the set of diagonals in $\mathcal{C}$. In a private communication, Per Enflo asked the following question. If such a non-linear cube $\mathcal{C}$ satisfies the equality

$$\sum_{e \in E} \|e\| = \sum_{d \in D} \|d\| \tag{1.3}$$

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does it follow that X contains a linear isometric copy of $\ell_1^{(n)}$? Our main result (Theorem 2.4) answers Enflo's question affirmatively.

Since hypercube terminology and notation is not entirely standardized we shall presage the ensuing discussion with an important list of definitions and conventions. This list includes some cube-combinatorial conventions that are likely peculiar to this paper.

Throughout this paper, our scalar field is the set of real numbers.

Definition 1 An n-dimensional *hypercube* (or, more simply, an *n-cube*) in a normed space X is a set of 2^n distinct points in X where each point in the set is indexed by a distinct n-vector with coordinates chosen from the set $\{0, 1\}$. The points in such an n-cube will be denoted by their indexing vectors only. In other words, $\{0, 1\}^n \subseteq X$, unless context dictates otherwise.

Two points, or vertices, in an n-cube will be called a *edge* if their two (indexing) vectors vary in precisely one coordinate.

Suppose e_1 and e_2 are the endpoints of a given edge in an n-cube, with $e_1(k) = 0$ and $e_2(k) = 1$ for some k, $1 \leq k \leq n$. Let v denote the $(n-1)$-vector obtained by deleting the k-th coordinate (of the indexing vector) of e_1 (or e_2). Then x_v^k will denote the *edge vector* that initiates at e_1 and terminates at e_2. In other words, $x_v^k = e_2 - e_1$. This notation will be very expedient later.

Two points, or vertices, in an n-cube will be called a *diagonal* if their two (indexing) vectors vary at each coordinate.

The *opposite* of a vertex $e \in \{0, 1\}^n$ is the vertex $\bar{e} \in \{0, 1\}^n$ for which the pair $(e, \bar{e})$ forms a diagonal.

For a given k, $1 \leq k \leq n$, there are 2^{n-1} edges in an n-cube whose endpoint indexing vectors (pairwise) vary in the k-th coordinate. These edges will be called *parallel*, or *k-parallel* if we need to be more precise.

Let $\mathcal{F}_1$ denote the set of vertices in an n-cube whose first coordinate is $+1$. We shall refer to $\mathcal{F}_1$ as the *distinguished face* of the n-cube. The distinguished face allows us to write the set of *diagonal vectors* of an n-cube as $\{d_e \mid e \in \mathcal{F}_1\}$ where $d_e = e - \bar{e}$ is the adopted convention.

Between any two vertices in an n-cube there are connected paths of non-parallel edges. Such paths will be referred to as *short*. If the vertices in question are the endpoints of a diagonal, then there are $n!$ distinct short paths between them.

Given $n \geq 2$, we say that a normed space X contains a *non-linear* $\ell_1^{(n)}$*-cube* if there exists an n-cube $\mathcal{C}$ in X such that the sum of the lengths of the $n \cdot 2^{n-1}$ edges equals the sum of the lengths of the 2^{n-1} diagonals. In other words, (1.3) holds for $\mathcal{C}$.

As noted in Weston [W], it follows from the proof of Enflo [E, Theorem 2.1], together with roundness computations from Lennard, Tonge, and Weston [LTW1] for $p > 2$, that a normed space X which contains non-linear $\ell_1^{(n)}$-cubes for all $n \geq 2$ cannot be uniformly homeomorphic to any L_p-space (be it commutative or otherwise) with $1 < p < \infty$. For example, X could be ℓ_1, ℓ_∞, or any infinite dimensional C^*-algebra.

To prove the result stated in the abstract the following theorem, from Weston [W, Theorem 2.1], will be helpful.

Theorem 1.1 **([W, Theorem 2.1])** *Consider a non-linear $\ell_1^{(n)}$-cube $\mathcal{C} = \{0,1\}^n$ contained in a normed space $(X, \|\cdot\|)$. Then*

(a) for a fixed k, $1 \le k \le n$, any two k-parallel edges have the same length, and
(b) any two diagonals have the same length. Moreover,
(c) if $\mathcal{S}$ is a short path between the endpoints e and $\bar{e}$ of a diagonal, then

$$\sum \|x_\nu^k\| = \|e - \bar{e}\|$$

where the sum is taken over all edge vectors x_ν^k that lie in the path $\mathcal{S}$.

We will refer to Theorem 1.1(c) as the *Short Path Property*.

Theorem 1.2 shows that the equality (1.3) imposes some 'linear-like' metric structure on a non-linear n-cube $\mathcal{C}$, but falls short of showing the existence of a linear isometry from a subspace of X onto $\ell_1^{(n)}$. In order to prove our main result (Theorem 2.4) in the next section, we introduce a combinatorial strategy, based on short paths, which allows the construction of a linear isometry from a subspace of the linear span of $\mathcal{C}$ onto $\ell_1^{(n)}$.

2 Parity and the Main Result

In this section, given an n-cube, we will be considering *oriented short paths* from $\bar{e}$ to e, $e \in \mathcal{F}_1$. More precisely, we adopt the convention that our short path initiates at $\bar{e}$ and terminates at e. This means that an edge vector x_ν^k in the oriented short path will either be traversed positively or negatively.

Lemma 2.1 *Consider a given n-cube $\{0,1\}^n$. Let $e \in \mathcal{F}_1$ be given, and consider the diagonal vector $d_e = e - \bar{e}$. Consider an arbitrary edge vector x_ν^k with endpoint vertices e_1 and e_2, where $e_1(k) = 0$ and $e_2(k) = 1$. (Note that e_1 and e_2, as $\{0,1\}$-vectors, are encoded in the edge vector notation x_ν^k.)*

(i) If $\bar{e}(k) = e_2(k)$, then there are oriented short paths from $\bar{e}$ to e that traverse $-x_\nu^k$, and none that traverse $+x_\nu^k$.
(ii) If $\bar{e}(k) = e_1(k)$, then there are oriented short paths from $\bar{e}$ to e that traverse $+x_\nu^k$, and none that traverse $-x_\nu^k$.

Proof We give the proof of (i), noting that the proof of (ii) is similar.

Consider the sets $M_1 = \{j \mid \bar{e}(j) \neq e_2(j)\}$ and $M_2 = \{j \mid e(j) \neq e_1(j)\}$. Then $\{M_1, M_2, \{k\}\}$ is a partition of $\{1, 2, \dots, n\}$. We begin by noting how to construct oriented short paths from $\bar{e}$ to e that traverse $-x_\nu^k$.

Start by changing the M_2 coordinates of $\bar{e}$, one at a time, until e_2 is reached. Then change the k-th coordinate of e_2 to get e_1. Finally, change the M_1 coordinates of e_1, one at a time, to reach e. Such paths are clearly short. (Note that if, for example, $M_2 = \varnothing$, then $\bar{e} = e_2$ and the process is simpler.)

To see that there is no oriented short path from $\bar{e}$ to e that traverses $+x_\nu^k$, note the following: in going from $\bar{e}$ to e_1, the k-th coordinate must change. The k-th

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coordinate will then have to change again in order to go from e_1 to e_2. This is not a short path. ■

Lemma 2.1 gives us the basis for a new concept that will play an important role in the proof of our main result, Theorem 2.4.

Definition 2 Let x_v^k be an arbitrary edge vector, and let d_e ($e \in \mathcal{F}_1$) be an arbitrary diagonal, in a given n-cube $\{0,1\}^n$. We define the *parity of* x_v^k w.r.t. d_e, denoted (x_v^k, d_e), to be $+1$ if there is an oriented short path from $\bar{e}$ to e that traverses $+x_v^k$, and to be -1 if there is an oriented short path from $\bar{e}$ to e that traverses $-x_v^k$.

As an immediate consequence of Lemma 2.1, together with an examination of its proof, we get the final lemma of this section.

Lemma 2.2 *Let x_v^k be an arbitrary edge vector, and let d_e ($e \in \mathcal{F}_1$) be an arbitrary diagonal, in a given n-cube $\{0,1\}^n$. Then the parity (x_v^k, d_e) is well defined with*

$$(x_v^k, d_e) = \begin{cases} +1: & \text{if } e(k) = 1 \\ -1: & \text{if } e(k) = 0. \end{cases}$$

Moreover, if $\mathcal{S}$ is a short path connecting the vertices $\bar{e}$ and e, then

$$d_e = \sum_{x_v^k \in \mathcal{S}} (x_v^k, d_e) x_v^k.$$

Theorem 2.3 *If a normed space X contains a non-linear $\ell_1^{(n)}$-cube $\{0,1\}^n$ for some $n \geq 2$, then it contains a linear isometric copy of $\ell_1^{(n)}$.*

In fact, given any oriented short path $\mathcal{S}$ between the endpoints of any given diagonal in the n-cube, the set $\{x_v^k \mid \pm x_v^k \in \mathcal{S}\}$ forms a basis for a linear isometric copy of $\ell_1^{(n)}$ in X.

Proof A standard result in linear functional analysis says that if a normed space X contains non-zero vectors $x^1, x^2, \ldots, x^n$ such that

$$\|x^1 \pm x^2 \pm \cdots \pm x^n\| = \sum_{k=1}^{n} \|x^k\| \tag{2.1}$$

for all possible combinations of sign, then the set $\{x^1, x^2, \ldots, x^n\}$ forms a basis for a linear isometric copy of $\ell_1^{(n)}$ in X. We will appeal to this criterion shortly.

Since the linear span of $\{0,1\}^n$ is finite dimensional, we may assume that our ambient space is $X = (\ell_\infty, \|\cdot\|_\infty)$.

Without loss of generality, by altering our definition of parity to reflect edge length if necessary, we may assume that each edge in our n-cube $\{0,1\}^n$ has length one. Consequently, each diagonal of the n-cube will have length n by the Short Path Property.

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Globally, the rest of our proof will be concerned with the specific diagonal $d_f = f - \bar{f}$, where $f = (1, 1, \dots, 1)$. However, all other diagonals in the n-cube will play important roles. Moreover, in the global sense, we shall be specifically concerned with the oriented short path $x^1_{v_1}, x^2_{v_2}, \dots, x^n_{v_n}$ from $\bar{f}$ to f defined by the following $(n-1)$-dimensional vectors:

$$
\begin{aligned}
v_1 &= (0, 0, 0, \dots, 0)\\
v_2 &= (1, 0, 0, \dots, 0)\\
v_3 &= (1, 1, 0, \dots, 0)\\
&\ \ \vdots\\
v_n &= (1, 1, 1, \dots, 1)
\end{aligned}
$$

For symbolic economy, we shall set $x^k = x^k_{v_k}$ for each k, $1 \leq k \leq n$. (Note that the following arguments can be cosmetically altered to suit any oriented short path connecting the endpoints of any diagonal in the n-cube.)

Our aim is to show that $\{x^1, x^2, \dots, x^n\}$ forms a basis for a linear isometric copy of $\ell_1^{(n)}$ via the above-mentioned criterion (2.1) of linear analysis.

Let an arbitrary $e \in \mathcal{F}_1$ be given, and set

$$
\tilde{e}(k) = (x^k_v, d_e) = \begin{cases} +1: & \text{if } e(k) = 1 \\ -1: & \text{if } e(k) = 0 \end{cases}
$$

for each k, $1 \leq k \leq n$. Each $\tilde{e}(1) = +1$ on account of having $e \in \mathcal{F}_1$. Moreover, varying $e \in \mathcal{F}_1$, ensures that any and all of the 2^{n-1} combinations of sign $\tilde{e}(2) = \pm 1$, $\tilde{e}(3) = \pm 1, \dots, \tilde{e}(n) = \pm 1$ will be attained. So our proof will be complete if we can establish that $\| \sum_{k=1}^n \tilde{e}(k) x^k \| = \sum_{k=1}^n \|x^k\|$.

If $\mathcal{S}$ is a short path from $\bar{e}$ to e, we know from Lemma 2.3 that

$$
d_e = \sum_{x^k_v \in \mathcal{S}} (x^k_v, d_e) x^k_v. \tag{2.2}
$$

In addition, the Short Path Property says that

$$
\|d_e\| = \sum_{x^k_v \in \mathcal{S}} \|x^k_v\|. \tag{2.3}
$$

Consequently, viewing d_e as a point in ℓ_∞, there must be a natural number $j = j(\epsilon)$ such that $d_e(j)$ is within (any prescribed) $\epsilon > 0$ of n (or $-n$, as the case may be), meaning that: $n - \epsilon < d_e(j) \leq n$ (or $-n \leq d_e(j) < -n + \epsilon$). We will express this analytic condition by writing $d_e \sim_\epsilon n$ (or $d_e \sim_\epsilon -n$, in which case the arguments are similar). The nature of $\| \cdot \|_\infty$, together with (2.2) and (2.3), then imply that $x^k_v(j) \sim_\epsilon (x^k_v, d_e) = \tilde{e}(k)$ for each $x^k_v \in \mathcal{S}$.

Now observe that, given any $k \in \{1, 2, \ldots, n\}$, precisely one of $+x^k$ or $-x^k$ will be traversed in an oriented short path from $\bar{e}$ to e by Lemma 2.1 with, moreover, the alternative being encoded by the parity (x^k, d_e). This gives $x^k(j) \sim_\epsilon (x^k, d_e)$ for all $k \in \{1, 2, \ldots, n\}$, with the upshot being that $\sum_{k=1}^n \tilde{e}(k) x^k(j) \sim_\epsilon n$. In other words, $\| \sum_{k=1}^n \tilde{e}(k) x^k \| = n = \sum_{k=1}^n \|x^k\|$, completing the proof. ■

Remark 3 The set of n-dimensional binary vectors $\{0, 1\}^n$, equipped with the metric d_{ℓ_1} induced by the standard norm on $\ell_1^{(n)}$, is often referred to as the *Hamming n-cube*. In Bourgain *et al.* [BMW, Corollary 5.9(i)] it is shown that a Banach space contains Hamming n-cubes uniformly if and only if it contains $\ell_1^{(n)}$'s uniformly.

Since Hamming n-cubes are particular examples of non-linear $\ell_1^{(n)}$-cubes, it follows that Theorem 2.4 implies an isometric version of [BMW, Corollary 5.9(i)]. Namely, a normed space contains Hamming n-cubes isometrically if and only if it contains $\ell_1^{(n)}$'s isometrically.

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