Canad. Math. Bull. Vol. **25** (2), 1982

A NOTE ON COMPLETELY AND ABSOLUTELY MONOTONE FUNCTIONS

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ABSTRACT. The solutions of a certain class of first order linear differential equations are shown to be either completely or absolutely monotone depending on the nature of its coefficients. This is a simple theorem which is used to deduce a number of new and interesting results dealing with the complete and absolute monotonicity of functions. In particular, a partial answer is supplied to a question posed by Askey and Pollard: "When is $x^{-c_0}(1^2+x^2)^{-c_1}\cdots(n^2+x^2)^{-c_n}$ completely monotone?"

1. **Introduction.** During the past decade it has become apparent that the notions of completely and absolutely monotone function play an important role in classical analysis. They occur, for example, in proving the positivity of integrals involving Bessel functions or the positivity of Cesàro means of certain Jacobi series; see Askey [1]. The notion of complete monotonicity has also been used in characterising the gamma function [6].

In the past various methods have been developed to prove that particular functions are either completely or absolutely monotone; see for example the work of Askey and Pollard [2] and of Fields and Ismail [4]. The purpose of this note is to show that some additional and some known results follow directly from a simple theorem dealing with the nature of the solutions to first order linear differential equations. In particular, it is shown that $x^{-c_0}(1^2+x^2)^{-c_1}\cdots(n^2+x^2)^{-c_n}$ is completely monotone for certain values of c_i.

To begin, recall that a function $f(x)$ is said to be k-times monotone on $(0,\infty)$ if all its derivatives $f^{(i)}(x)$ exist and satisfy

$$(1)\qquad (-1)^i f^{(i)}(x)\geq 0 \qquad (0<x<\infty;\ i=0,1,2,\ldots,k).$$

In case k is infinite, $f(x)$ is said to be completely monotone on $(0,\infty)$. On the other hand, a function $f(x)$ is said to be absolutely monotone on $(0,\infty)$ if (1)

Received by the editors September 27, 1979 and, in revised form, April 8, 1980, August 14, 1980 and November 26, 1980.

AMS subject classifications (1980). Primary 26D15, 34A10; Secondary 33A15, 33A40, 40G05.

Key words and phrases. Completely monotone, absolutely monotone, Inequalities, Bessel functions, gamma functions.

https://doi.org/10.4153/CMB-1982-020-x Published online by Cambridge University Press

can be replaced by

$$f^{(i)}(x) \geq 0 \qquad (0 < x < \infty;\ i = 0, 1, 2, \ldots).$$

An important and fundamental result which will be needed here is the so-called Bernstein-Widder theorem which states that a necessary and sufficient condition for a function $f(x)$ to be completely monotone on $(0, \infty)$ is that it should be representable as a Laplace-Stieltjes type integral of the form

$$f(x) = \int_0^\infty e^{-xt}\, du(t),$$

where $du(t) \geq 0$, and the integral converges for $0 < x < \infty$.

For further discussion on the properties and characteristics of completely and absolutely monotone functions, see Widder [8].

2. **Completely monotone functions.** Consider the following simple theorem:

THEOREM 1. *Let $p(x)$ be a completely monotone function defined on* $(0, \infty)$. *Then all solutions of the differential equation*

$$y'(x) + p(x)y(x) = 0 \tag{2}$$

which are nonnegative for a single $x_0 > 0$ are completely monotone on $(0, \infty)$.

The proof of this theorem is straightforward and depends on the fact that the solutions in question can be written in the form

$$y(x) = y(x_0) \exp\left[\int_{x_0}^{x} -p(t)\, dt\right] \geq 0, \qquad 0 < x < \infty.$$

The required result follows by differentiation of (2).

The above is an elementary theorem which can be used to obtain a number of known as well as new results on completely and absolutely monotone functions.

EXAMPLE 1. (The question posed by Askey and Pollard [2]). The function $y(x) = x^{-c_0}(1^2 + x^2)^{-c_1} \cdots (n^2 + x^2)^{-c_n}$ is completely monotone for $x > 0$ provided that

$$c_0 \geq 2c_1 \geq 0 \quad \text{and} \quad 0 \leq c_i \leq c_{i-1}\left[1 - \frac{1}{i+\alpha}\right], \qquad (i = 2, 3, \ldots, n) \tag{3}$$

where $-1 < \alpha \leq A$ and $A = 4.5678018 \cdots$ is the positive root of

$$9x^7 + 55x^6 - 14x^5 - 948x^4 - 3247x^3 - 5013x^2 - 3780x - 1134 = 0.$$

Consider the logarithmic derivative of $y(x)$ as given by

$$\begin{aligned} \frac{-y'(x)}{y(x)} &= \frac{c_0}{x} + \frac{2c_1 x}{(1+x^2)} + \cdots + \frac{2c_n x}{(n^2+x^2)} \\ &= 2\int_0^\infty e^{-xt}\left[\frac{c_0}{2} + c_1 \cos t + \cdots + c_n \cos nt\right] dt \end{aligned} \tag{4}$$

https://doi.org/10.4153/CMB-1982-020-x Published online by Cambridge University Press

then, because

$$a_0+a_1\cos t+\frac{a_2(1+\alpha)}{(2+\alpha)}\cos 2t+\cdots+\frac{a_n(1+\alpha)}{(n+\alpha)}\cos nt\geq 0$$

for $a_0\geq a_1\geq\cdots\geq a_n\geq 0$ and under the above conditions stated on α, see Gasper [5, Theorem 2] (where A has been replaced by α in order to correct a misprint), it follows that the right side of (4) is completely monotone. Hence, by Theorem 1 the function $y(x)$ is completely monotone.

Example 2. The conditions referred to in (3) require that c_1 be positive. However, it is possible to include instances where $c_1<0$. To begin, consider the function $\phi(x)=x^{-c_0}(1^2+x^2)^{-c_1}$ with $c_0\geq 2|c_1|$ then

$$\frac{-\phi'(x)}{\phi(x)}=\int_0^\infty e^{-xt}[c_0+2c_1\cos t]\,dt$$

which, together with Theorem 1, implies that $\phi(x)$ is completely monotone on $(0,\infty)$ since the condition $c_0\geq 2|c_1|$ ensures that the integrand is positive. On using

$$x^{-1}(1^2+x^2)^{-1}=\int_0^\infty e^{-xt}[1-\cos t]\,dt$$

and the fact that a product of completely monotone functions is itself completely monotone, it appears that $x^{-c_0-m}(1+x^2)^{-c_1-m}$ is also completely monotone for $x>0$, $c_0\geq 2\,|c_1|$ and $m=0,1,\ldots$.

Corollaries to Example 2. (a) *The function $x^{-2|\tau|}(1+x^2)^{-\tau}$ is completely monotone for any τ real, $x>0$. This result was proved by Askey and Pollard* [2] *who used two different methods, the first making use of a theorem of I. J. Schoenberg and the second making use of an inductive argument.* (*Incidentally, these authors also point out that the function $x^{-\kappa\tau}(1+x^2)^{-\tau}$ cannot be completely monotone for all $\tau>0$ if $\kappa<2$.*)

A similar result, namely the complete monotonicity of $x^{-2\tau}(1+x^2)^{-\tau}$, when $\tau>0$ and $x>0$, was proved independently by Fields and Ismail [3]. In another paper by the same authors, [4], it is shown that the function $x^{2\tau-2\rho\tau-b}(1+x^2)^{-\tau}$ is completely monotone for $b\geq 0$, $\rho\geq 2$, $\tau>0$, $x>0$ and other more complicated results are deduced by using an argument of Darboux type and fractional integration. However, the result mentioned here is a trivial consequence of Corollary (a).

(b) *The function $x^{-d-m}(1+x^2)^{-c-m}$ is completely monotone for $d=2c$, $0<c\leq \frac{1}{2}$, $m=0,1,\ldots$, and the function $x^{-(2c-1)}(1+x^2)^{-c}$ is completely monotone for $c\geq 1$. These results follow from Example 2 and also appear in Askey* [1].

The significance of the notion of complete monotonicity is that it can be combined with the earlier mentioned Bernstein-Widder theorem to verify that certain integrals involving special functions are positive. By way of example we follow Askey [1] and use the previous results. Thus, because

$$(1+x^2)^{-\alpha-1/2}=\frac{\Gamma(\frac{1}{2})}{2^\alpha\Gamma(\alpha+\frac{1}{2})}\int_0^\infty e^{-xt}t^\alpha J_\alpha(t)\,dt,\quad \text{for}\quad \alpha>-\tfrac{1}{2},\quad x>0,$$

see Watson [7, p. 386] it follows, on account of the convolution theorem for Laplace transforms, that

$$\int_0^\infty e^{-xt}\left(\int_0^t (t-u)^\beta u^\alpha J_\alpha(u)\,du\right)dt=\frac{2^\alpha\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+1)}{\Gamma(\frac{1}{2})x^{\beta+1}(1+x^2)^{\alpha+\frac{1}{2}}}$$

when $\beta>-1$ and since $x^{-(\beta+1)}(1+x^2)^{-(\alpha+\frac{1}{2})}$ is completely monotone for $\beta\geq 2\alpha$ that

$$\int_0^x (x-t)^{2\alpha+\varepsilon}t^\alpha J_\alpha(t)\,dt\geq 0,$$

for $\alpha>-\frac{1}{2}$, $\varepsilon\geq 0$ and $x>0$.

In much the same way, by making use of the identity

$$x(1+x^2)^{-\alpha-3/2}=\frac{2^{-\alpha-1}\Gamma(\frac{1}{2})}{\Gamma(\alpha+3/2)}\int_0^\infty e^{-xt}t^{\alpha+1}J_\alpha(t)\,dt,\ \alpha>-1$$

Watson [7, p. 386], we may obtain the inequality

$$\int_0^x (x-t)^{2\alpha+2+\varepsilon}t^{\alpha+1}J_\alpha(t)\,dt\geq 0,$$

for $\alpha\geq-\frac{1}{2}$, $\varepsilon\geq 0$ and $x>0$.

For $\varepsilon=0$ these reduce to known results, see Fields and Ismail [4] and Askey [1].

EXAMPLE 3. During an investigation of the monotonicity of the function $\psi(x)=[x^\alpha\Gamma(x)(e/x)^x]^\beta$ on $(0,\infty)$ for various values of α and β, Muldoon [6, Theorem 2.1] uses Stirling's formula together with some properties of completely and absolutely monotone functions. The same result can be obtained more simply by using Theorem 1 and noting that

$$\begin{aligned}\frac{-\psi'(x)}{\psi(x)}&=-\beta\left[\alpha/x+\frac{\Gamma'(x)}{\Gamma(x)}-\log x\right]\\&=\beta\int_0^\infty e^{-xt}\left[\frac{1}{1-e^{-t}}-1/t-\alpha\right]dt\end{aligned}$$

(Binet's formula) and that the integrand is positive for $\alpha\leq\frac{1}{2}$ and where $\beta\geq 0$. In case $\alpha>\frac{1}{2}$ it is known from the asymptotic form of $\psi(x)$, see Muldoon [6], that this function is ultimately increasing and so cannot be completely monotone for all $x>0$.

3. **Absolutely monotone functions.** A corresponding theorem for absolutely monotone functions is the following:

THEOREM 2. *Let $q(x)$ be an absolutely monotone function defined on* $(0, \infty)$. *Then all the solutions of the differential equation*

$$y'(x) - q(x)y(x) = 0$$

which are nonnegative for a single $x_0 > 0$, are absolutely monotone on $(0, \infty)$.

EXAMPLE 4. Here we shall use the above theorem to reproduce a result of Askey and Pollard [2, Theorem 2], namely, that the function $\zeta(x) = (1-x)^{2\tau}(1-2xh+x^2)^{-\tau}$ is absolutely monotone for $\tau < 0$ and with $-1 \le h = \cos\theta \le 1$.

To see this, consider the logarithmic derivative of $\zeta(x)$, namely,

$$\frac{\zeta'(x)}{\zeta(x)} = \frac{-2\tau(1-h)(1+x)}{(1-x)(1-2xh+x^2)}$$

and since Askey and Pollard [2] have pointed out that the absolute monotonicity of $\dfrac{(1+x)}{(1-x)(1-2xh+x^2)}$ follows easily from Fejér's theorem on positivity of the $(C, 1)$ means for Fourier series, it follows that $\dfrac{\zeta'(x)}{\zeta(x)}$ is absolutely monotone and hence by Theorem 1 the required result follows.

We conclude this note by giving just one more theorem of this kind. Obviously, many such theorems can be obtained for first order differential equations.

THEOREM 3. *Consider the first order inhomogeneous linear differential equation*

$$y'(x) - f(x)y(x) = g(x) \tag{5}$$

where $f(x)$ and $g(x)$ are continuous on $x > 0$. Suppose further that $f(x)$ and $g(x)$ are absolutely monotone for $x > 0$. Then (5) *has a nontrivial solution which is also absolutely monotone on this interval.*

ACKNOWLEDGEMENTS. The authors wish to thank the referee for his comments. One of the authors (A.M.) is grateful to the Australian Government for a Postgraduate Research Award.

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