On classical irregular $q$-difference equations

Julien Roques


doi:10.1112/S0010437X12000292
On classical irregular $q$-difference equations

Julien Roques

ABSTRACT

The primary aim of this paper is to (provide tools to) compute Galois groups of classical irregular $q$-difference equations. We are particularly interested in quantizations of certain differential equations that arise frequently in the mathematical and physical literature, namely confluent generalized $q$-hypergeometric equations and $q$-Kloosterman equations.

1. Introduction

Throughout this paper, $q$ is a nonzero complex number such that $|q| < 1$. For all $\alpha \in \mathbb{C}$, we set $q^\alpha = e^{\alpha \log(q)}$ where $\log(q)$ is a fixed logarithm of $q$. We denote by $\mathbb{C}(z)(\sigma_q, \sigma_q^{-1})$ the noncommutative algebra of noncommutative Laurent polynomials with coefficients in $\mathbb{C}(z)$ such that $\sigma_q z = qz \sigma_q$.

1.1 Motivation

Here are some examples of computations of $q$-difference Galois groups derived from the main results of this paper.

The generalized $q$-hypergeometric operator $L_q(a; b; \lambda)$ with parameters $a = (a_1, \ldots, a_r) \in (q^\mathbb{R})^r$ ($r \in \mathbb{N}$), $b = (b_1, \ldots, b_s) \in (q^\mathbb{R})^s$ ($s \in \mathbb{N}$) and $\lambda \in \mathbb{C}^*$ is given by

$$L_q(a; b; \lambda) = \prod_{j=1}^{s} \left( \frac{b_j}{q} \sigma_q - 1 \right) - z \lambda \prod_{i=1}^{r} (a_i \sigma_q - 1).$$

We assume that $r \neq s$ (see [Roq11] for the case where $r = s$). By replacing $z$ with $1/z$ if necessary, we can assume that $r > s$. For all $(i, j) \in \{1, \ldots, r\} \times \{1, \ldots, s\}$, we let $\alpha_i, \beta_j \in \mathbb{R}$ be such that $a_i = q^{\alpha_i}$ and $b_j = q^{\beta_j}$.

**Theorem.** Assume that $\beta_j - \alpha_i \notin \mathbb{Z}$ for all $(i, j) \in \{1, \ldots, r\} \times \{1, \ldots, s\}$ (this condition is empty if $s = 0$) and that the algebraic group generated by $\text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_r})$ is connected.

Then the Galois group of $L_q(a; b; \lambda)$ is $\text{GL}(\mathbb{C}^r)$.

**Example.** The Galois group of $(q^{1/2} \sigma_q - 1)^s - z(\sigma_q - 1)^r$ is $\text{GL}(\mathbb{C}^r)$.

The $q$-Kloosterman operator $K_{q(U, V)}$ associated to a pair $(U, V)$ of elements of $\mathbb{C}[X]$ such that $U(0) = 0$ and $V(0) \neq 0$ is given by

$$K_{q(U, V)} = U(\sigma_q) + V(z^{-1}).$$
ON CLASSICAL IRREGULAR $q$-DIFFERENCE EQUATIONS

We let $c_1, \ldots, c_{\deg U}$ be the complex roots of $X^{\deg U}(U(X^{-1}) + V(0)) \in \mathbb{C}[X]$ and, for all $i \in \{1, \ldots, \deg U\}$, we denote by $(u_i, \alpha_i)$ the unique element of $U \times \mathbb{R}$ such that $c_i = u_i q^{\alpha_i} (U \subset \mathbb{C}$ denotes the unit circle).

**Theorem.** Assume that $\deg U$ and $\deg V$ are relatively prime, that the algebraic group generated by $\text{diag}(u_1, \ldots, u_{\deg U})$ and $\text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_{\deg U}})$ is connected, and that there exists $z_0 \in \mathbb{C}^*$ such that $V(z_0) = 0$ and $V(q^k z_0) \neq 0$ for all $k \in \mathbb{Z}^*$. Then the Galois group of $\text{Kl}_q(U, V)$ is $\text{GL}(\mathbb{C}^{\deg U})$.

**Example.** For relatively prime integers $m$ and $n$, the Galois group of $(1 - \sigma_q)^n + (1 - z^{-1})^m - 1$ is $\text{GL}(\mathbb{C}^n)$.

**Proposition.** Let us consider $V \in q + X \mathbb{C}[X]$. Then, for any odd integer $n \geq 2$ coprime to $\deg V$, the Galois group of $\text{Kl}_q((q^{1/2} - X)^2(1 - X)^{n-2} - q, V)$ is $\text{GL}(\mathbb{C}^n)$.

In order to achieve these goals, we present our results in two parts.

Part I is devoted to the following problem: find simple and relevant characterizations of the classical linear algebraic groups.

Part II is a Galoisian study of $q$-difference operators $L \in \mathbb{C}(z)\langle \sigma_q, \sigma_q^{-1} \rangle$ of rank $n$ satisfying one of the following properties (see §4.2 for the notion of slope).

$(\mathcal{H}1)$ At 0, $L$ is isoclinic and its slope is of the form $m/n$ with $m \in \mathbb{Z}^*$ coprime to $n$.

$(\mathcal{H}2)$ At 0, $L$ has two slopes, 0 and $\mu$. Denoting by $r$ the multiplicity of $\mu$, we have $\mu = m/r$ for some $m \in \mathbb{Z}^*$ coprime to $r$. The exponents attached to the slope 0 belong to $q^\mathbb{R}$.

For instance, the generalized $q$-hypergeometric operators with $s > 0$ considered above satisfy $(\mathcal{H}2)$, whereas the generalized $q$-hypergeometric operators with $s = 0$ and the $q$-Kloosterman operators $\text{Kl}_q(U, V)$ with $\deg U$ coprime to $\deg V$ satisfy $(\mathcal{H}1)$.

Our starting point originates from the work of Katz [Kat87]; we exploit the structure of the local formal Galois groups. However, the $q$-difference and differential cases are rather different; in particular, the ‘theta torus’ is ‘poorer’ than its differential analogue, Ramis’s exponential torus. We make essential use of works by van der Put and Reversat [vdPR07], van der Put and Singer [vdPS97] and Sauloy [Sau04]. In the theory of (irregular) linear differential equations, another way of computing Galois groups was explored: the use of Ramis’s ‘wild fundamental group’ (see [DM89, Mit96]). It would be interesting to compute $q$-difference Galois groups using the $q$-analogue of the wild fundamental group introduced by Ramis and Sauloy in [RS07, RS09]. The crucial difference lies in the presence of a unipotent Stokes component (and hence in the analytic properties of the slopes filtration).

In some cases, the classical equations studied in this paper can be seen as $q$-deformations of certain classical differential equations (this is exploited by André in [And01]; see also [Sau00, §§3–5]), namely the confluent generalized hypergeometric equations and the Kloosterman equations. These differential equations were studied by Katz, with contributions from Gabber, in [Kat87, Kat90], by Katz and Pink in [KP87], by Beukers et al. in [BBH88], by Duval and Mitschi in [DM89] and by Mitschi in [Mit96].

The original interest of the author in the classical equations studied in the present paper comes from the discrete Morales–Ramis theory developed in [CR08, CR11] for deriving the nonintegrability of classical nonlinear $q$-difference equations, such as discrete Painlevé equations.
1. Organization of the paper

Part I essentially provides 'easily checkable' characterizations of the classical linear algebraic groups. In §2 we give a new characterization relying on pairs of semisimple elements with special spectra. In §3 we give consequences of results established by Katz and Kostant. Part II considers applications of these purely representation-theoretic results to the Galois theory of irregular difference equations. In §4 we present the elements of slopes theory and some useful Galoisian results. In §§5 and 6 we show that the connected algebraic groups occurring as Galois groups of irreducible equations that satisfy either (H1) or (H2) belong to a very short list of linear algebraic groups. In §7 we compute Galois groups of q-Kloosterman equations and of generalized q-hypergeometric equations. In §8 we give a ⊗-indecomposability criterion, which we apply to the calculation of q-difference Galois groups. In §9, combining several results of this paper, we give additional computations of Galois groups.

PART I. CHARACTERIZATIONS OF THE CLASSICAL LINEAR ALGEBRAIC GROUPS

2. A characterization of the classical linear algebraic groups

Let $E$ be a $\mathbb{C}$-vector space of finite dimension $n \geq 3$. Let us consider $\alpha$ and $\beta$ in $\mathbb{N}$ such that $\alpha \geq 1$, $\beta \geq 2$ and $n = \alpha + \beta$.

**Definition 1 (Property (P)).** A pair $f, g$ of semisimple elements of $\text{GL}(E)$ satisfies property $(P)$ if:

- the list of eigenvalues of $f$ is of the form $(a \text{ repeated } \alpha \text{ times}, b \text{ repeated } \beta \text{ times})$ where $a, b \in \mathbb{C^*}$ are such that $a \neq \pm b$;
- the list of eigenvalues of $g$ is of the form $(c \text{ repeated } \alpha + 1 \text{ times}, d_1, \ldots, d_{\beta-1})$ where $c, d_1, \ldots, d_{\beta-1}$ are pairwise distinct nonzero complex numbers.

This section is devoted to the proof of the following result.

**Theorem 2.** Let $G$ be a connected algebraic subgroup of $\text{GL}(E)$ which acts irreducibly on $E$. If $G$ contains a pair of semisimple elements $f, g$ satisfying $(P)$, then the derived subgroup $G'$ of $G$ is $\text{SL}(E)$, $\text{SO}(E)$ or (if $n = \dim(E)$ is even) $\text{Sp}(E)$. Furthermore, $G' \subset G \subset \mathbb{C^*}G'$.

**Proposition 3.** Let $G$ be a connected semisimple algebraic subgroup of $\text{GL}(E)$ which acts irreducibly on $E$. If $G$ contains a semisimple element $f$ whose list of eigenvalues is of the form $(a \text{ repeated } \alpha \text{ times}, b \text{ repeated } \beta \text{ times})$ for some $a, b \in \mathbb{C^*}$ such that $a \neq \pm b$, then its Lie algebra $g$ contains a semisimple element whose list of eigenvalues is $(\beta \text{ repeated } \alpha \text{ times}, -\alpha \text{ repeated } \beta \text{ times})$.

**Proof.** Gabber’s theorem [Kat90, Theorem 1.0] applied to the Lie subalgebra $g$ of $\text{End}(E)$ and the subgroup $H$ of $G$ generated by $f$ ensures that, for any $x, y$ in $\mathbb{C}$ such that $\alpha x + \beta y = 0$, $g$ contains a semisimple element whose list of eigenvalues is $(x \text{ repeated } \alpha \text{ times}, y \text{ repeated } \beta \text{ times})$.

**Proposition 4.** Let $G$ be a connected semisimple algebraic subgroup of $\text{SL}(E)$ which acts irreducibly on $E$. If $G$ contains a pair of semisimple elements $f, g$ satisfying $(P)$, then $G$ is simple (in the sense that its Lie algebra is simple).

**Proof.** Let $\rho : G \hookrightarrow \text{GL}(E)$ be the standard representation of $G$, which is irreducible by hypothesis. It comes from an irreducible representation $\bar{\rho} : \widetilde{G} \rightarrow G \hookrightarrow \text{GL}(E)$ of the universal
covering \( \tilde{G} \) of \( G \). We want to prove that \( G \) is simple, i.e. that its Lie algebra \( \text{Lie}(G) = \text{Lie}(\tilde{G}) = \mathfrak{g} \) is simple.

Assume to the contrary that \( \mathfrak{g} \) is not simple. Then it splits into a direct sum \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) of nontrivial semisimple Lie algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) in such a way that the irreducible representation \( \text{Lie}(\tilde{\rho}) : \mathfrak{g} \rightarrow \text{End}(E) \) (irreducible representation \( \mathfrak{g}_1 \rightarrow \text{End}(E_1) \)) \( \otimes \) (irreducible representation \( \mathfrak{g}_2 \rightarrow \text{End}(E_2) \)) with \( n_1 = \dim(E_1) \geq 2 \) and \( n_2 = \dim(E_2) \geq 2 \). Denoting by \( \tilde{G}_1 \) and \( \tilde{G}_2 \) the connected and simply connected semisimple Lie groups with respective Lie algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) and integrating the above representations of \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) into representations \( \tilde{\rho}_1 : \tilde{G}_1 \rightarrow \text{GL}(E_1) \) and \( \tilde{\rho}_2 : \tilde{G}_2 \rightarrow \text{GL}(E_2) \), we get that \( \tilde{G} \) is \( \tilde{G}_1 \times \tilde{G}_2 \) and \( \tilde{\rho} \) is \( \tilde{\rho}_1 \otimes \tilde{\rho}_2 \). So the list of eigenvalues of any element of \( G = \text{Im}(\tilde{\rho}) \) is of the form \( \{\lambda_i\mu_j : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \).

Since \( f \) belongs to \( G \), its list of eigenvalues \((a \text{ repeated } \alpha \text{ times}, b \text{ repeated } \beta \text{ times})\) is of the form \( \{\lambda_i\mu_j : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \).

Note that either \( \text{card}\{\lambda_i \mid 1 \leq i \leq n_1\} = 1 \) or \( \text{card}\{\mu_j \mid 1 \leq j \leq n_2\} = 1 \). Otherwise, there would exist \( t, u \in \{\lambda_i \mid 1 \leq i \leq n_1\} \) and \( v, w \in \{\mu_j \mid 1 \leq j \leq n_2\} \) such that \( t \neq u \) and \( v \neq w \). The sublists \( \{tv, tw, uv, uw\} \) of \( \{\lambda_i\mu_j : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \) would be made up of at least three distinct numbers (otherwise, since \( \{tv, uw\} \cap \{tw, uv\} = \emptyset \), we would have \( tv = uw \) and \( tw = uv \) so that \( v/w = (tv)/(tw) = (uv)/(uw) = w/v \) hence \( v = w \) and \( t = -w \); therefore the inclusion \( \{tv, -tv\} = \{tv, tw, uv, uw\} \subset \{\lambda_i\mu_j : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} = \{a, b\} \) would be an equality, and so \( a = -b \), which is a contradiction). This contradicts the fact that \( f \) has two eigenvalues.

Up to relabeling, we can assume that \( \text{card}\{\lambda_i \mid 1 \leq i \leq n_1\} = 1 \) and \( \text{card}\{\mu_j \mid 1 \leq j \leq n_2\} = 2 \). Hence \( \alpha \) and \( \beta \) are nonzero integral multiples of \( n_1 \); in particular, \( n_1 \leq \alpha \) and \( n_1 \leq \beta \).

Since \( g \) belongs to \( G \), its list of eigenvalues \((c \text{ repeated } \alpha + 1 \text{ times}, d_1, \ldots, d_{\beta-1})\) is of the form \( \{\lambda_i'\mu_j' : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \). So there exist \( \alpha + 1 \) distinct indices \((i_1, j_1), \ldots, (i_{\alpha+1}, j_{\alpha+1})\) in \( \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \) such that \( c = \lambda_{i_1}'\mu_{j_1}' = \cdots = \lambda_{i_{\alpha+1}}'\mu_{j_{\alpha+1}}' \). Since \( n_1 \leq \alpha + 1 \), we get that there exist \( 1 \leq k \neq k' \leq \alpha + 1 \) such that \( i_k = i_{k'} \). Hence \( j_k \neq j_{k'} \) and \( \lambda_{i_k}'\mu_{j_k}' = \lambda_{i_{k'}}'\mu_{j_{k'}}' \), so \( \mu_{j_k}' = \mu_{j_{k'}}' \). Therefore, for all \( 1 \leq i \leq n_1 \), \( \lambda_i'\mu_{j_k}' = \lambda_i'\mu_{j_{k'}}' \), and so \( \lambda_i'\mu_{j_k}' = c \) (because \( c \) is the unique eigenvalue of \( g \) with multiplicity greater than \( 1 \)). Thus, \( \lambda_1' = \cdots = \lambda_{n_1}' \). So any element of \( \{\lambda_i'\mu_j' : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \) occurs at least \( n_1 \geq 1 \) times. But this is a contradiction (since \( g \) has at least one eigenvalue with multiplicity \( 1 \)), so \( \mathfrak{g} \) is simple.

We have proved that any connected semisimple algebraic subgroup of \( \text{GL}(E) \) that acts irreducibly on \( E \) and which contains a pair of semisimple elements \( f, g \) satisfying \((P)\) is simple and that its Lie algebra contains a morphism with exactly two eigenvalues. This restricts the possibilities for \( G \) by virtue of the following result of Serre. For the notion of minuscule representations, we refer to Bourbaki [Bou75].

**THEOREM 5** (Serre [Ser79, §3]). If a simple Lie subalgebra \( \mathfrak{g} \) of \( \text{End}(E) \) which acts irreducibly on \( E \) contains a morphism with exactly two eigenvalues, then \( \mathfrak{g} \) is a classical Lie algebra \((A_m, B_m, C_m \text{ or } D_m)\) and its weights in \( E \) are minuscule.

It is proved in [Bou75, ch. 8, §7.3] that the minuscule representations of classical Lie algebras are

\[ A_m, \quad m \geq 1; \quad \omega_1, \ldots, \omega_m \]
\[ B_m, \quad m \geq 3; \quad \omega_m \]
\[ C_m, \quad m \geq 2; \quad \omega_1 \]
\[ D_m, \quad m \geq 4; \quad \omega_1, \omega_{m-1}, \omega_m. \]
Remark 1. This list is slightly different from the one given in [Bou75] because (we are only interested in classical Lie algebras and) we have taken into consideration accidental isomorphisms.

The corresponding representations of connected Lie groups are conjugated to a factor of one of the following representations:

\[ \text{SL}_{m+1}(\mathbb{C}), \ m \geq 1; \text{std}, \wedge^2(\text{std}) \ldots, \wedge^m(\text{std}) \]
\[ \text{Spin}_{2m+1}(\mathbb{C}), \ m \geq 3; \text{spin representation} \]
\[ \text{Sp}_{2m}(\mathbb{C}), \ m \geq 2; \text{std} \]
\[ \text{Spin}_{2m}(\mathbb{C}), \ m \geq 4; \text{half-spin representations or \text{`std representation of SO}_{2m}(\mathbb{C}).} \]

For any subgroup \( G \) of \( \text{GL}(E) \), we denote by \( \text{std} \) the standard representation of \( G \), i.e. the inclusion \( G \hookrightarrow \text{GL}(E) \).

In what follows, we shall prove that among the above representations, the only ones whose image contains a pair of semisimple elements satisfying \((P)\) are \( \text{SL}_{m+1}(\mathbb{C}) \) in std or in \( \wedge^m(\text{std}) \), \( \text{Sp}_{2m}(\mathbb{C}) \) in std, and \( \text{Spin}_{2m}(\mathbb{C}) \) in the standard representation of \( \text{SO}_{2m}(\mathbb{C}). \)

**Proposition 6.** For \( 1 < k < m \) (so \( m \geq 3 \)), the image of \( \text{SL}_{m+1}(\mathbb{C}) \) in \( \wedge^k(\text{std}) \) does not contain a pair of semisimple elements satisfying \((P)\).

**Proof.** By duality, i.e. the fact that \( \wedge^k(\text{std}) \cong (\wedge^m(\text{std})^*)^* \), it is sufficient to consider the case where \( 1 < k \leq (m+1)/2 \).

Assume to the contrary that the image of \( \text{SL}_{m+1}(\mathbb{C}) \) in \( \wedge^k(\text{std}) \) contains a pair of semisimple elements \( f, g \) satisfying \((P)\).

Then, the list of eigenvalues \( (a \ \text{repeated} \ \alpha \ \text{times}, \ b \ \text{repeated} \ \beta \ \text{times}) \) of \( f \) is of the form

\[ (u_{i_1}, \ldots, u_{i_k} = u_{i_1} \cdots u_{i_k}; \ 1 \leq i_1 < i_2 < \cdots < i_k \leq m + 1). \]

We have \( \text{card}\{u_i \mid 1 \leq i \leq m + 1\} \geq 2 \) because \( a \neq b \). We claim that \( \text{card}\{u_i \mid 1 \leq i \leq m + 1\} > 2 \). Assume to the contrary that \( \text{card}\{u_i \mid 1 \leq i \leq m + 1\} = 2 \). Up to renumbering, we can assume that \( u_1, u_2 \) and \( u_3 \) are pairwise distinct. Then \( u_{3, \ldots, k+2}, u_{2,4, \ldots, k+2} \) and \( u_{1,4, \ldots, k+2} \) (note that \( k + 2 \leq (m + 1)/2 + 2 \leq m + 1 \) because \( m \geq 3 \)) would be pairwise distinct, and therefore \( \text{card}\{u_{i_1, \ldots, i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq m + 1\} > 3 \); this is a contradiction.

So, up to renumbering, we can assume that there exists \( i \in \{1, \ldots, m\} \) such that \( u := u_1 = \cdots = u_i \neq u_{i+1} = \cdots = u_{m+1} = v \).

We claim that \( i = 1 \) or \( i = m \). Indeed, assume to the contrary that \( 2 \leq i \leq m - 1 \) (recall that \( m \geq 3 \)) and denote by \( l \) the smallest nonnegative integer such that \( i \leq l + k \) (so \( l = 0 \) if \( i \leq k \) and \( l = i - k \) if \( i > k \)). Then \( u_{l+1, \ldots, l+k}, u_{l+2, \ldots, l+k+1} \) and \( u_{l+3, \ldots, l+k+2} \) would be pairwise distinct (indeed, there exists \( t \in \mathbb{C}^* \) such that \( u_{l+1, \ldots, l+k} = u^2 t, u_{l+2, \ldots, l+k+1} = u v t \) and \( u_{l+3, \ldots, l+k+2} = u^2 v t \), and these three numbers are pairwise distinct because \( u \neq \pm v \)), so \( \text{card}\{u_{i_1, \ldots, i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq m + 1\} > 3 \); this is a contradiction.

Consequently, we have that either \( u_1 \neq u_2 = \cdots = u_{m+1} \) or \( u_1 = \cdots = u_m \neq u_{m+1} \), so we have either \( (\alpha, \beta) = \left( \binom{m}{k-1}, \binom{m}{k} \right) \) or \( (\alpha, \beta) = \left( \binom{m}{k}, \binom{m}{k-1} \right) \). In any case, we have \( \alpha \geq \min \{\binom{m}{k-1}, \binom{m}{k}\} = \binom{m}{k} \) (the last equality holds because \( k \leq (m + 1)/2 \)).

On the other hand, the list of eigenvalues \( (c \ \text{repeated} \ \alpha + 1 \ \text{times}, \ d_1, \ldots, d_{\beta - 1}) \) of \( g \) is of the form

\[ (v_{i_1}, \ldots, v_{i_k} = v_{i_1} \cdots v_{i_k}; \ 1 \leq i_1 < i_2 < \cdots < i_k \leq m + 1). \]
ON CLASSICAL IRREGULAR q-DIFFERENCE EQUATIONS

This list is the concatenation of the \( \binom{m}{k-1} \) lists of the form
\[
(v_{i_1, \ldots, i_{k-1}, j} = v_{i_1} \cdots v_{i_{k-1}} v_j ; \ i_{k-1} < j \leq m + 1)
\]
indexed by \( 1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq m \).

Since \( \alpha + 1 > \binom{m}{k-1} \), we get that there exist \( 1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq m \) and \( i_{k-1} < j, j' \leq m + 1 \) with \( j \neq j' \) such that \( c = v_{i_1, \ldots, i_{k-1}, j} = v_{i_1, \ldots, i_{k-1}, j'} \). So \( v_j = v_{j'} \). Up to relabelling, we can assume that \( v_1 = v_2 \).

For all \( 3 \leq i_2 < \cdots < i_k \leq m + 1 \), we obviously have \( v_1 v_{i_2} \cdots v_{i_k} = v_{2} v_{i_2} \cdots v_{i_k} \). Since \( c \) is the only eigenvalue of \( g \) with multiplicity greater than 1, we necessarily have, for all \( 3 \leq i_2 < \cdots < i_k \leq m + 1 \), \( c = v_1 v_{i_2} \cdots v_{i_k} \). Therefore, \( v_3 = \cdots = v_{m+1} \).

If \( k > 2 \), then it is clear that any element of the list \( (v_{i_1, \ldots, i_k} ; \ 1 \leq i_1 < i_2 < \cdots < i_k \leq m + 1) \) occurs with multiplicity at least 2: this is a contradiction.

If \( k = 2 \), then any element of the list \( (v_{i_1, i_2} ; \ 1 \leq i_1 < i_2 \leq m + 1) \) occurs with multiplicity at least 2 except, possibly, the term corresponding to \( i_1 = 1 \) and \( i_2 = 2 \). In particular, \( c = v_1 v_3 = v_3 v_4 = v_5 \) and so \( v_1 = v_3 \), giving \( v_1 = \cdots = v_{m+1} \) and hence \( \text{card}\{v_{i_1, \ldots, i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq m + 1\} = 1 \): this is a contradiction.

**Proposition 7.** The image of \( \text{Spin}_{2m}(\mathbb{C}) \) with \( m \geq 4 \) in any of its 1/2-spin representations does not contain a pair of semisimple elements satisfying \( (P) \).

**Proof.** Assume to the contrary that the image \( G \) of \( \text{Spin}_{2m}(\mathbb{C}) \) in one of its 1/2-spin representations contains a pair of semisimple elements \( f, g \) satisfying \( (P) \).

Let us first treat the case of the 1/2-spin representation \( \rho_0 \) whose weights have an odd number of minus signs.

Proposition 3 ensures that \( \text{Lie}(G) = \mathfrak{g} \) contains an element \( u \) whose list of eigenvalues is \( E_u = (\beta \text{ repeated } \alpha \text{ times}, -\alpha \text{ repeated } \beta \text{ times}) \). There exist \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \) such that
\[
E_u = (\epsilon_1 \lambda_1 + \cdots + \epsilon_m \lambda_m ; (\epsilon_1, \ldots, \epsilon_m) \in \{-1, 1\}^m \text{ such that } \epsilon_1 \cdots \epsilon_m = -1).
\]

Since \( (\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_2, \ldots, \lambda_1 + \cdots + \lambda_m - 2\lambda_m) \) is a sublist of \( E_u \), we get that \( \text{card}\{\lambda_i \mid 1 \leq i \leq m\} \leq 2 \).

Assume that \( \text{card}\{\lambda_i \mid 1 \leq i \leq m\} = 1 \), i.e. that \( \lambda := \lambda_1 = \cdots = \lambda_m \). Note that \( \lambda \neq 0 \). If \( m \geq 5 \), then
\[
(\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_1 - 2\lambda_2 - 2\lambda_3, \\
\lambda_1 + \cdots + \lambda_m - 2\lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - 2\lambda_5) \\
= ((m-2)\lambda, (m-6)\lambda, (m-10)\lambda)
\]
is a sublist of \( E_u \) made up of three distinct numbers, which is a contradiction. If \( m = 4 \), then \( E_u \) is \((2\lambda \text{ repeated } 4 \text{ times}, -2\lambda \text{ repeated } 4 \text{ times}) \). In particular, \( \alpha = \beta = 2^{m-2} \).

Assume that \( \text{card}\{\lambda_i \mid 1 \leq i \leq m\} = 2 \), i.e. that \( \lambda := \lambda_1 = \cdots = \lambda_i \) and \( \lambda_{i+1} = \cdots = \lambda_m =: \mu \) for some \( 1 \leq i < m \) and some distinct complex numbers \( \lambda \) and \( \mu \). Since \( m \geq 4 \), up to relabelling we can assume that \( i \geq 2 \). Then
\[
(\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_1 - 2\lambda_2 - 2\lambda_3, \\
\lambda_1 + \cdots + \lambda_m - 2\lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 - 2\lambda_5 - 2(2\lambda + \mu))
\]
is a sublist of \( E_u \). Since \( \lambda \neq \mu \), we have \( \lambda_1 + \cdots + \lambda_m - 2\lambda \neq \lambda_1 + \cdots + \lambda_m - 2\mu \); so, since \( E_u \) is composed of two elements, \( \lambda_1 + \cdots + \lambda_m - 2(\lambda + \mu) \) is equal to either \( \lambda_1 + \cdots + \lambda_m - 2\lambda \)
or \(\lambda_1 + \cdots + \lambda_m - 2\mu\), that is, \(\lambda = 0\) or \(\mu = -\lambda\). If \(\lambda = 0\) and \(i < m - 1\), then
\[
(\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_2 - 2\lambda_3, \lambda_1 + \cdots + \lambda_m - 2\lambda_4 - 2\lambda_m)
= ((m - i)\mu, (m - i - 2)\mu, (m - i - 4)\mu)
\]
is a sublist of \(E_u\) made up of three pairwise distinct complex numbers (because \(\mu \neq \lambda = 0\)); but this is impossible. If \(\lambda = 0\) and \(i = m - 1\), then \(E_u\) has the form \((\mu\) repeated \(2^{m-2}\) times, \(-\mu\) repeated \(2^{m-2}\) times\) and hence \(\alpha = \beta = 2^{m-2}\). If \(\mu = -\lambda\) and \(i \geq 3\), then
\[
(\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_2 - 2\lambda_3, \lambda_1 + \cdots + \lambda_m - 2\lambda_4 - 2\lambda_m)
= (\lambda_1 + \cdots + \lambda_m - 2\lambda, \lambda_1 + \cdots + \lambda_m - 6\lambda, \lambda_1 + \cdots + \lambda_m + 2\lambda)
\]
is a sublist of \(E_u\) made up of three pairwise distinct complex numbers, which is impossible. Similarly, the case where \(\lambda = -\mu\) and \(m - i \geq 3\) is impossible. So, since \(m \geq 4\), the only possibility that is compatible with \(\lambda = -\mu = m = 4\) and \(i = 2\), in which case \(E_u\) is of the form \((2\lambda\) repeated \(4\) times, \(-2\lambda\) repeated \(4\) times\); thus, in particular, \(\alpha = \beta = 2^{m-2}\).

Therefore, in any possible case, we have \(\alpha = \beta = 2^{m-2}\).

On the other hand, since \(g\) belongs to \(G\), its list of eigenvalues \(E_g = (c\) repeated \(\alpha + 1\) times, \(d_1, \ldots, d_{\beta - 1}\)\) has the form
\[
E_g = (\mu_{1}^{\epsilon_1}, \ldots, \mu_{m}^{\epsilon_m}; (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^m \text{ such that } \epsilon_1 \cdots \epsilon_m = -1).
\]
This list is the concatenation of the \(2^{m-2}\) lists of the form
\[
\left(\prod_{i \in \{1, \ldots, m\}\setminus\{1, \ldots, i_{p-1}, i_p\}} \mu_i \cdot \prod_{i \in \{i_1, \ldots, i_{p-1}, i_p\}} \mu_{i_p}^{-1}; i_p < i_p \leq m\right)
\]
indexed by \(1 \leq i_1 < \cdots < i_{p-1} \leq m - 1\) with \(1 \leq p \leq m\) an odd number. Since \(\alpha + 1 > 2^{m-2}\), we see that there exist \(1 \leq i_1 < \cdots < i_{p-1} < m - 1\) and \(i_{p-1} < j, j' \leq m\) with \(j \neq j'\) such that
\[
c = \prod_{i \in \{1, \ldots, m\}\setminus\{i_1, \ldots, i_{p-1}, j, j'\}} \mu_i \cdot \prod_{i \in \{i_1, \ldots, i_{p-1}, j, j'\}} \mu_{i_p}^{-1}
\]
and so \(\mu_{j'}^{2} = \mu_{j}^{2}\), i.e. \(\mu_{j} = \pm \mu_{j'}\). Up to renumbering, we can assume that \(\mu_1 = \pm \mu_2\). So, for all \(3 \leq k, l \leq m\) with \(k \neq l\) (recall that \(m \geq 4\)), we have
\[
\mu_1 \mu_2 \mu_k \mu_l - 1 \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k, l\}} \mu_i = \mu_1^{-1} \mu_2 \mu_k \mu_l^{-1} \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k, l\}} \mu_i.
\]
Thus \(\mu_1 \mu_2 \mu_k \mu_l^{-1} \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k, l\}} \mu_i\) occurs with multiplicity greater than 1 in \(E_g\), and hence
\[
c = \mu_1 \mu_2^{-1} \mu_k \mu_l^{-1} \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k, l\}} \mu_i.
\]
Similarly, for all \(3 \leq k, l \leq m\) with \(k \neq l\),
\[
c = \mu_1 \mu_2^{-1} \mu_k \mu_l \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k, l\}} \mu_i.
\]
So, for all \(3 \leq k, l \leq m\) with \(k \neq l\), we have \(\mu_{k}^{2} \mu_{l}^{2} = 1\). If \(m \geq 5\), then for all \(3 \leq k, l \leq m\) there exists \(3 \leq k' \leq m\) such that \(k' \neq k, l\); so \(\mu_{k}^{2} / \mu_{l}^{2} = (\mu_{k}^{2} \mu_{l}^{2}) / (\mu_{k}^{2} \mu_{l}^{2}) = 1\) / 1 = 1, i.e. \(\mu_{k}^{2} = \mu_{l}^{2}\). Therefore, we get \(\mu_{3}^{2} = \cdots = \mu_{m}^{2} = \pm 1\). This implies that any element of
\[
E_g = (\mu_{1}^{\epsilon_1}, \ldots, \mu_{m}^{\epsilon_m}; (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^m \text{ such that } \epsilon_1 \cdots \epsilon_m = -1)
\]
1630
has multiplicity at least 2 because \( \mu_1^\epsilon \cdots \mu_m^\epsilon = \mu_1^{\epsilon_1} \cdots \mu_m^{\epsilon_m} \mu_{m-1}^{\epsilon_{m-1}} \mu_{m-2}^{\epsilon_{m-2}} \cdots \mu_2^{\epsilon_2} \mu_1^{\epsilon_1} \); this is a contradiction.

If \( m = 4 \), then it is easily seen that \( E_g \) is of the form \((\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \) (this is more generally true if \( m \) is even). If \( m = 4 \) and \( c^{-1} = b \), then \( \alpha + 1 \) would be an even number (because if \( c \in \{\nu_i, \nu_i^{-1}\} \), then \( \{\nu_i, \nu_i^{-1}\} \) and the number \( \alpha + 1 \) of occurrences of \( c \) in \( E_g = (\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \) must be even); so \( \alpha \) would be an odd number and hence would not be an integral power of 2, which is a contradiction. If \( m = 4 \) and \( c^{-1} \neq b \), then the fact that \( c \) occurs with multiplicity \( \alpha + 1 \) in \( E_g = (\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \) implies that \( c^{-1} \) occurs with multiplicity \( \alpha + 1 > 1 \) in \( E_g \), so \( c = c^{-1} \) (because \( c \) is the unique eigenvalue of \( g \) with multiplicity greater than 1); this is again a contradiction.

Let us now treat the case of the 1/2-spin representation \( \rho_+ \) whose weights have an even number of minus signs.

Since \( \rho_+ \) is dual to \( \rho_- \) when \( m \) is odd, it is sufficient to consider the case where \( m \) is even. As mentioned above, the fact that \( m \) is even implies that the list \( E_f = (a \text{ repeated } \alpha \text{ times}, \ b \text{ repeated } \beta \text{ times}) \) of eigenvalues of \( f \) is of the form \( E_f = (\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \). We claim that \( \alpha = \beta = 2^m - 2 \). Indeed, assume first that \( a = a^{-1} \), i.e. that \( a = \pm 1 \). This implies that \( b^{-1} \neq b \) and \( b^{-1} \neq a \), because \( b \neq \pm a = \pm 1 \). So \( b^{-1} \) does not belong to \( E_f = (a \text{ repeated } \alpha \text{ times}, \ b \text{ repeated } \beta \text{ times}) \), and hence \( b \) itself does not belong to \( E_f = (\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \), which is a contradiction. A similar argument shows that \( b \neq b^{-1} \). Therefore \( a = a^{-1} \) and \( b \neq b^{-1} \). Since \( b \) belongs to \( E_f = (\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \), \( b^{-1} \) belongs to \( E_f \). Since \( b^{-1} \neq b \), the only possibility is that \( a = b^{-1} \), and hence the number of occurrences of \( a \) and of \( b \) in \( E_f = (\nu_1, \nu_1^{-1}, \ldots, \nu_2^{m-2}, \nu_2^{-1}) \) are the same. Thus \( \alpha = \beta = 2^m - 2 \). Now, the same argument as for the \( m = 4 \) case treated above allows us to conclude the proof.

**Proposition 8.** The image of \( \text{Spin}_{2m+1}(\mathbb{C}) \) in its spin representation does not contain a pair of semisimple elements satisfying \((\mathcal{P})\).

**Proof.** Assume that the image \( G \) of \( \text{Spin}_{2m+1}(\mathbb{C}) \) in its spin representation contains a pair of semisimple elements \( f, g \) satisfying \((\mathcal{P})\).

Proposition 3 ensures that \( \text{Lie}(G) = \mathfrak{g} \) contains an element \( u \) whose list of eigenvalues is \( E_u = (\beta \text{ repeated } \alpha \text{ times}, \ -\alpha \text{ repeated } \beta \text{ times}) \). So there exist \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \) such that

\[
E_u = (\epsilon_1 \lambda_1 + \cdots + \epsilon_m \lambda_m; \ (\epsilon_1, \ldots, \epsilon_m) \in \{-1, 1\}^m).
\]

Since \( (\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_2, \ldots, \lambda_1 + \cdots + \lambda_m - 2\lambda_m) \) is a sublist of \( E_u \), we get that \( \text{card}\{\lambda_i \mid 1 \leq i \leq m\} \leq 2 \).

Assume that \( \text{card}\{\lambda_i \mid 1 \leq i \leq m\} = 1 \), i.e. that \( \lambda := \lambda_1 = \cdots = \lambda_m \). We have \( \lambda \neq 0 \). Then

\[
(\lambda_1 + \cdots + \lambda_m, \lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_2, \ldots, \\
\lambda_1 + \cdots + \lambda_m - 2\lambda_1 - \cdots - 2\lambda_m) = ((m - 2j)\lambda; \ 0 \leq j \leq m)
\]

is a sublist of \( E_u \) made of \( m + 1 > 2 \) mutually distinct numbers, and this is a contradiction.

Assume that \( \text{card}\{\lambda_i \mid 1 \leq i \leq m\} = 2 \), i.e. that \( \lambda := \lambda_1 = \cdots = \lambda_i \) and \( \lambda_{i+1} = \cdots = \lambda_m =: \mu \) for some \( 1 \leq i < m \) and some distinct complex numbers \( \lambda \) and \( \mu \). Up to renumbering, we can assume that \( i \geq 2 \). Using the fact that \((\pm \lambda + \lambda_3 + \cdots + \lambda_m)\) is a sublist of \( E_u \), we see that \( \lambda = 0 \). Moreover, \( i = m - 1 \), because otherwise \((\lambda_1 + \cdots + \lambda_{m-2} \pm \mu \pm \mu)\) would be a sublist of \( E_u \) made up of four distinct elements (as \( \mu \neq \lambda = 0 \)), which is impossible. So \( E_u \) has the form \( (\mu \text{ repeated } 2^{m-1} \text{ times}, \ -\mu \text{ repeated } 2^{m-1} \text{ times}) \), hence \( \alpha = \beta = 2^{m-1} \).
On the other hand, since \( g \) belongs to \( G \), its list of eigenvalues \( E_g = (c \text{ repeated } \alpha + 1 \text{ times, } d_1, \ldots, d_{2^{m-1}}) \) is of the form \( E_g = (\mu_1^{i_1} \cdots \mu_m^{i_m}; (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n) \). This list is the concatenation of the \( 2^m - 1 \) lists

\[
\left( \prod_{i \in \{1, \ldots, m\}\setminus\{i_1, \ldots, i_{p-1}, i_p\}} \mu_i \cdot \prod_{i \in \{i_1, \ldots, i_{p-1}, i_p\}} \mu_i^{-1}; \ i_p < i_p' \leq m \right)
\]

indexed by \( 1 \leq i_1 < \cdots < i_{p-1} \leq m - 1 \) with \( 0 \leq p \leq m \). Since \( \alpha + 1 > 2^m - 1 \), we see that there exist \( 1 \leq i_1 < \cdots < i_{p-1} \leq m - 1 \) and \( i_{p-1} < j, j' \leq m \) with \( j \neq j' \) such that

\[
\prod_{i \in \{1, \ldots, m\}\setminus\{i_1, \ldots, i_{p-1}, j\}} \mu_i \cdot \prod_{i \in \{i_1, \ldots, i_{p-1}, j\}} \mu_i^{-1} = \prod_{i \in \{1, \ldots, m\}\setminus\{i_1, \ldots, i_{p-1}, j'\}} \mu_i \cdot \prod_{i \in \{i_1, \ldots, i_{p-1}, j'\}} \mu_i^{-1}
\]

and so \( \mu_j^2 = \mu_j'^2 \). Up to renumbering, we can assume that \( \mu_2^2 = \mu_2^2 \). So, for all \( 3 \leq k \leq m \), we have

\[
\mu_1 \mu_2^{-1} \mu_k^{-1} \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k\}} \mu_i \text{ occurs with multiplicity greater than 1 in } E_g,
\]

and hence

\[
c = \mu_1 \mu_2^{-1} \mu_k^{-1} \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2, k\}} \mu_i.
\]

Similarly, we have, for all \( 3 \leq k \leq m \),

\[
c = \mu_1 \mu_2^{-1} \prod_{i \in \{1, \ldots, m\}\setminus\{1, 2\}} \mu_i.
\]

Therefore, for all \( 3 \leq k \leq m \), \( \mu_k^2 = 1 \), i.e. \( \mu_k = \pm 1 \). This clearly implies that any element of \( E_g = (\mu_1^{i_1} \cdots \mu_m^{i_m}; (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n) \) occurs with multiplicity at least 2, which is a contradiction.

**Proof of Theorem 2.** Since \( G \) acts irreducibly on \( E \), we have \( G = Z(G)^{0}G' \) where \( Z(G)^{0} \) denotes the connected center of \( G \) and \( G' \) the derived subgroup of \( G \). Moreover, \( Z(G)^{0} \) is included in the scalars, so \( G' \subset G \subset C^*G' \) and \( G' \) is a connected semisimple algebraic subgroup of \( SL(E) \) which acts irreducibly on \( E \). Let \( f, g \) be a pair of semisimple elements of \( G \) satisfying (\( P \)). Then there exist \( t_f, t_g \in \mathbb{C}^* \) such that \( f' = t_f f \) and \( g' = t_g g \) belong to \( G' \). It is clear that \( f', g' \) is a pair of semisimple elements of \( G' \) satisfying (\( P \)). Proposition 4 ensures that \( G' \) is simple. Proposition 3 and Theorem 5 ensure that \( G' \) is classical and that, as a representation of \( G' \), \( E \) is minuscule. In view of the classification of minuscule representations given after Theorem 5, the result follows from Propositions 6, 7 and 8.

**3. Additional results**

We let \( E \) be a \( \mathbb{C} \)-vector space of finite dimension \( n \geq 2 \).

**Theorem 9.** Let \( G \) be a connected algebraic subgroup of \( GL(E) \). Assume that \( G \) contains a semisimple element \( u \) having \( n \) distinct eigenvalues and an element \( v \) which permutes cyclically the \( n \) eigenspaces of \( u \). Then the derived subgroup \( G' \) of \( G \) is either the image of \( \prod_{i=1}^l \text{SL}(\mathbb{C}^{n_i}) \) in \( \bigotimes_{i=1}^l \) std for some \( l \in \mathbb{N}^* \) and some pairwise coprime numbers \( n_1, n_2, \ldots, n_l > 1 \) or the image of \( \text{Sp}(\mathbb{C}^{n_1}) \times \prod_{i=2}^l \text{SL}(\mathbb{C}^{n_i}) \) in \( \bigotimes_{i=1}^l \) std for some \( l \in \mathbb{N}^* \) and some pairwise coprime numbers \( n_1 \geq 4 \) even and \( n_2, \ldots, n_l > 1 \). Moreover, \( G' \subset G \subset C^*G' \).
ON CLASSICAL IRREGULAR $q$-DIFFERENCE EQUATIONS

Proof. The fact that $G$ contains a semisimple element $u$ having $n$ distinct eigenvalues and an element $v$ which permutes cyclically the corresponding eigenspaces implies that $G$ acts irreducibly on $E$. So $G' \subset G \subset \mathbb{C}^* G'$ and $G'$ is a connected semisimple algebraic subgroup of $\text{SL}(E)$ which acts irreducibly on $E$ (see the beginning of the proof of Theorem 2 for details) and contains an element $u' (=\xi u$ for some $\xi \in \mathbb{C}^*)$ with $n$ distinct eigenvalues and an element $v' (=\xi v$ for some $\xi \in \mathbb{C}^*)$ that permutes cyclically the corresponding eigenspaces.

By virtue of [Kat87, Corollary 3.2.8], to conclude the proof it suffices to find a maximal torus $T$ in $G'$ and an element $w$ in the normalizer $N(T)$ of $T$ such that, as a representation of $T, E$ is the direct sum of $n$ distinct characters which are cyclically permuted by the conjugation action of $w$. But since $u'$ is a semisimple element of $G'$, it is contained in a maximal torus $T$ of $G'$. By commutativity, this maximal torus leaves invariant the $n$ eigenspaces of $u'$. It is now clear that $T$ and $w = v' \in N(T)$ have the required properties. 

\[ \blacksquare \]

**Theorem 10.** Let $G$ be a connected algebraic subgroup of $\text{GL}(E)$ which acts irreducibly on $E$. If $G$ contains a semisimple element $f$ whose list of eigenvalues is of the form $(a, b$ repeated $n - 1$ times) for some $a, b \in \mathbb{C}^*$ such that $a \neq \pm b$, then the derived subgroup $G'$ of $G$ is $\text{SL}(E)$. Furthermore, $G' \subset G \subset \mathbb{C}^*G'$.

**Proof.** Since $G$ acts irreducibly on $E$, $G' \subset G \subset \mathbb{C}^* G'$ and $G'$ is a connected semisimple algebraic subgroup of $\text{SL}(E)$ which acts irreducibly on $E$ (see the beginning of the proof of Theorem 2 for details) and contains $f' = tf$ for some $t \in \mathbb{C}^*$. Proposition 3 ensures that the semisimple Lie algebra $\mathfrak{g}'$ of $G'$ contains a semisimple morphism whose list of eigenvalues is $(n - 1, -1$ repeated $n - 1$ times). Since $G'$ acts irreducibly on $E$, so does $\mathfrak{g}'$. Kostant’s characterization of $\mathfrak{sl}(E)$ given in [Kos58] then ensures that $\mathfrak{g}' = \mathfrak{sl}(E)$ and hence that $G' = \text{SL}(E)$.

\[ \blacksquare \]

**Part II. Applications to $q$-difference Galois theory**

**4. Review of useful facts and results**

**4.1 $q$-difference modules and systems**

Let $(K, \sigma)$ be a difference field and let $\mathcal{D}_{(K, \sigma)}$ be the noncommutative algebra $K\langle \sigma, \sigma^{-1} \rangle$ of noncommutative Laurent polynomials with coefficients in $K$ satisfying the relation $\sigma a = \sigma(a)\sigma$ for any $a \in K$. The full subcategory of the category of $\mathcal{D}_{(K, \sigma)}$-modules whose objects are the $\mathcal{D}_{(K, \sigma)}$-modules of finite length is denoted by $\mathcal{E}_{(K, \sigma)}$. It is a $K^\sigma$-linear abelian tensor category, where $K^\sigma = \{a \in K \mid \sigma(a) = a\}$ is the subfield of constants of $(K, \sigma)$.

It will sometimes be convenient to choose specific bases. We introduce the category $\mathcal{E}'_{(K, \sigma)}$, which is tensor-equivalent to $\mathcal{E}_{(K, \sigma)}$, described as follows: its objects are difference systems $(\sigma Y = AY)$ where $A \in \text{GL}_n(K)$, and its morphisms from $(\sigma Y = AY)$, $A \in \text{GL}_n(K)$, to $(\sigma Y = BY)$, $B \in \text{GL}_m(K)$, are the matrices $F \in M_{m,n}(K)$ such that $BF = \sigma(F)A$.

We refer to [vdPS97, Chapter 1, especially §1.4] or to [Sau04, §1.1] for details. In particular, the tensor product, denoted by $\otimes$, and the dual, denoted by $\vee$, are defined there.

We denote by $\mathbb{C}(z)$ the local ring of germs of analytic functions at 0 and by $\mathbb{C}\{z\}$ its field of fractions; we denote by $\mathbb{C}[[z]]$ the local ring of formal series in $z$ and by $\mathbb{C}(\!(z)\!)$ its field of fractions.

For $K = \mathbb{C}(z)$, $\mathbb{C}\{\{z\}\}$ or $\mathbb{C}(\!(z)\!)$, we denote by $\sigma_q$ the automorphism of $K$ defined by $\sigma_q(a(z)) = a(qz)$. Then $(K, \sigma_q)$ is a difference field with field of constants $\mathbb{C}$.
4.2 Slopes

Our main reference for slopes theory is [Sau04], where it is assumed that \(|q| > 1\) (in opposition to our hypothesis of \(|q| < 1\)). The slopes defined in this paper are thus the opposite of those defined in [Sau04]; but since we use only the formal part of [Sau04], this has no impact on what follows.

The Newton polygon \(\mathcal{N}(L)\) of \(L = \sum_i a_i \sigma_q^{j} \in \mathcal{D}(C((z)),\sigma_q)\) is the convex hull in \(\mathbb{R}^2\) of \(\{(i, j) \mid i \in \mathbb{Z} \text{ and } j \geq v_z(a_i)\}\) where \(v_z\) denotes the \(z\)-adic valuation on \(C((z))\). This polygon is made up of two vertical half-lines and \(k\) vectors \((r_1, d_1), \ldots, (r_k, d_k) \in \mathbb{N}^* \times \mathbb{Z}\) having pairwise distinct slopes, called the slopes of \(L\). For any \(i \in \{1, \ldots, k\}\), \(r_i\) is called the multiplicity of the slope \(d_i/r_i\).

Let \(M\) be an object of \(\mathcal{E}(C((z)),\sigma_q)\). The cyclic vector lemma [DiV02, Lemma 1.3.1] ensures that there exists \(L \in \mathcal{D}(C((z)),\sigma_q)\) such that \(M \cong \mathcal{D}(C((z)),\sigma_q)/\mathcal{D}(C((z)),\sigma_q) \cdot L\). One can define the slopes of \(M\) to be the slopes of \(L\) and the multiplicity of a slope \(\lambda\) of \(M\) to be the multiplicity of \(\lambda\) as a slope of \(L\). This definition is independent of the chosen \(L\) (see [Sau04, Théorème et définition 2.2.5]). An object \(M\) of \(\mathcal{E}(C((z)),\sigma_q)\) is pure isoclinic if it has a unique slope.

For instance, for \(a \in C((z))^*\), the Newton polygon of \(M = \mathcal{D}(C((z)),\sigma_q)/\mathcal{D}(C((z)),\sigma_q)(\sigma_q - a)\) is the convex subset of \(\mathbb{R}^2\) delimited by the vertical half-lines \(\{0\} \times \mathbb{R}^+\) and \(\{1\} \times \{v_z(a), +\infty\}\) together with the segment from \((0, 0)\) to \((1, v_z(a))\). So \(M\) is pure isoclinic with slope \(v_z(a)\). To give another example, \(M = \mathcal{D}(C((z)),\sigma_q)/\mathcal{D}(C((z)),\sigma_q)(q z \sigma_q^2 - (1 + z) \sigma_q + 1)\) has two slopes, namely \(0\) and \(1\), both with multiplicity \(1\).

4.3 Galois groups

Let \(\mathcal{E}\) be a tannakian category over \(C\), and let \(\omega\) be a \(C\)-fiber functor on \(\mathcal{E}\). For any object \(M\) of \(\mathcal{E}\), we let \(\langle M \rangle\) denote the tannakian category generated by \(M\) in \(\mathcal{E}\) and let \(\text{Gal}(M, \omega)\) denote the complex linear algebraic group \(\text{Aut}^\otimes(\omega|_{\langle M \rangle})\). The choice of a specific fiber functor is of no consequence: since \(C\) is algebraically closed, any two \(C\)-fiber functors on \(\mathcal{E}\) are isomorphic. For the theory of tannakian categories, we refer to Deligne and Milne’s paper [DM81].

4.3.1 Connectedness. Let \(M\) be an object of \(\mathcal{E}(C((z)),\sigma_q)\).

The categories \(\mathcal{E}(C((z)),\sigma_q)\) and \(\mathcal{E}(C(z),\sigma_q)\) are neutral tannakian over \(C\) (see [vdPS97, §1.4]). Let \(\hat{\omega}\) be a \(C\)-fiber functor on \(\mathcal{E}(C((z)),\sigma_q)\). The formalization functor \(\hat{\cdot} : \mathcal{E}(C((z)),\sigma_q) \to \mathcal{E}(C(z),\sigma_q)\) being an exact and faithful \(\otimes\)-functor, \(\omega = \hat{\omega} \circ \hat{\cdot}\) is a \(C\)-fiber functor on \(\mathcal{E}(C(z),\sigma_q)\).

The following result is [vdPS97, Proposition 12.2] (compare with Gabber’s result [Kat87, Proposition 1.2.5]).

**Proposition 11.** The natural closed immersion \(\text{Gal}(\hat{M}, \hat{\omega}) \hookrightarrow \text{Gal}(M, \omega)\) of the local formal Galois group \(\text{Gal}(\hat{M}, \hat{\omega})\) of \(M\) at 0 into the Galois group \(\text{Gal}(M, \omega)\) of \(M\) induces a surjective morphism \(\text{Gal}(\hat{M}, \hat{\omega})/\text{Gal}(\hat{M}, \hat{\omega})^0 \to \text{Gal}(M, \omega)/\text{Gal}(M, \omega)^0\).

**Corollary 12.** If \(\text{Gal}(\hat{M}, \hat{\omega})\) is connected, then \(\text{Gal}(M, \omega)\) is connected.

We give an additional corollary for later use.
COROLLARY 13. Assume that $M$ satisfies (H1) and is regular singular at $\infty$ with exponents in \( \{ e \in \mathbb{C}^* \mid e^{n'} \in q\mathbb{Z} \} \) for some $n' \in \mathbb{Z}^+$ coprime to the rank $n$ of $M$. Then $\text{Gal}(M, \omega)$ is connected.

**Proof.** We set $G = \text{Gal}(M, \omega)$ and denote by $G_0$ and $G_\infty$ the local formal Galois groups of $M$ at $0$ and $\infty$, respectively. Proposition 16 below and [vdPR07, Example 5.6 in §5.2] ensure that $G_0/G_0^0 \cong (\mathbb{Z}/n^2\mathbb{Z})$. Proposition 11 implies that the order of any element of $G/G_0$ divides $n^2$. Moreover, using [vdPS97, ch. 12] or [Sau03, §2.2], we see that the order of any element of $G_\infty/G_\infty^0$ divides $n'$. Proposition 11 ensures that the same property holds for the elements of $G/G_0$. Therefore, $G/G_0$ is trivial. \( \square \)

4.3.2 *Lie-irreducibility.*

**Definition 14.** We say that a list $c_1, \ldots, c_n$ of nonzero complex numbers is $q$-Kummer induced if there exist a divisor $d \geq 2$ of $n$ and a permutation $\nu$ of \( \{ 1, \ldots, n \} \) such that, for all $i \in \{ 1, \ldots, n \}$, $c_i = q^{1/d} c_{\nu(i)} \bmod q\mathbb{Z}$.

**Proposition 15.** If $M$ is an irreducible object of $\mathcal{E}_{(\mathbb{C}(z), \sigma_q)}$ which is of rank $n$ and regular singular at $\infty$ with non-$q$-Kummer-induced exponents $c_1, \ldots, c_n \in q\mathbb{R}$, then $M$ is Lie-irreducible, i.e. the action of $\text{Gal}(M, \omega)^0$ on $\omega(M)$ is irreducible.

**Proof.** For all $i \in \{ 1, \ldots, n \}$, let $\gamma_i \in \mathbb{R}$ be such that $c_i = q^{\gamma_i}$. It follows from [vdPS97, ch. 12] or [Sau03, §2.2] that the local formal Galois group of $M$ at $\infty$ is generated, as an algebraic group, by its neutral component and by a semisimple morphism $f$ with list of eigenvalues $e^{2\pi i \gamma_1}, \ldots, e^{2\pi i \gamma_n}$. Proposition 11 implies that $G = \text{Gal}(M, \omega)$ is generated, as an algebraic group, by $G^0$ and $f$. So, since the action of $G$ on $\omega(M)$ is irreducible, its restriction to the abstract group $H$ generated by $G^0$ and $f$ is still irreducible. Assume that $M$ is not Lie-irreducible and let $V = \{ 0 \}$, $\omega(M)$ be a minimal invariant subspace of $\omega(M)$ for the action of $G^0$. For all $k \in \mathbb{Z}$, $f^k V$ is an invariant subspace of $\omega(M)$ for the action of $G^0$, because $G^0$ is a normal subgroup of $G$. Therefore $\sum_{k \in \mathbb{Z}} f^k V$ is an invariant subspace of $\omega(M)$ for the action of $H$ and hence $\omega(M) = \sum_{k \in \mathbb{Z}} f^k V$. Let $d$ be the smallest integer greater than $1$ such that $\omega(M) = \sum_{k=0}^{d-1} f^k V$. It is easily seen that $\omega(M) = \bigoplus_{k=0}^{d-1} f^k V$. This implies that $f$ and $e^{2\pi i/d} f$ are conjugate. Considering the eigenvalues of $f$, we see that there exists a permutation $\nu$ of \( \{ 1, \ldots, n \} \) such that, for all $i \in \{ 1, \ldots, n \}$, $e^{2\pi i \gamma_i} = e^{2\pi i/d} e^{2\pi i \gamma_{\nu(i)}}$, i.e. $c_i = q^{1/d} c_{\nu(i)} \bmod q\mathbb{Z}$.

5. Main theorem in the one-slope case

**Proposition 16 (Reformulation of (H1)).** Let $\widehat{M}$ be an object of $\mathcal{E}_{(\mathbb{C}(z), \sigma_q)}$ of rank $n \geq 2$. The following properties are equivalent:

(a) $\widehat{M}$ is irreducible (i.e. simple);

(b) $\widehat{M} \cong M_q(n, m, a) := D_{(\mathbb{C}(z), \sigma_q)}/D_{(\mathbb{C}(z), \sigma_q)}(\sigma_q^n - q_m^{mn(n-1)/2} a z^m)$ for some $m \in \mathbb{Z}^*$ coprime to $n$ and some $a \in \mathbb{C}^*$;

(c) $\widehat{M}$ satisfies (H1).

**Proof.** The equivalence (a) $\iff$ (b) is [vdPR07, Proposition 1.3], and (b) $\implies$ (c) is obvious. It remains to prove (c) $\implies$ (a). Assume that $\widehat{M}$ satisfies (H1). Let $\widehat{M}'$ be a nonzero subobject of $\widehat{M}$. Then $\widehat{M}'$ is pure isoclinic with slope $\mu$ (see [Sau04, Théorème 2.3.1]). In order to prove that $\widehat{M} = \widehat{M}'$, it is sufficient to prove that the rank $n'$ of $\widehat{M}'$ is greater than or equal to $n$. 

1635
This is indeed the case as $n' \mu$ has to be a relative integer (immediate from the definition of the slopes of $\tilde{M}$).

**Lemma 17.** If $M_1, \ldots, M_t$ are objects of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ of rank greater than 1 such that $M = M_1 \otimes \cdots \otimes M_t$ satisfies $(\mathcal{H}1)$, then $M_1, \ldots, M_t$ satisfy $(\mathcal{H}1)$.

**Proof.** Let $n, n_1, \ldots, n_t$ be the respective ranks of $M, M_1, \ldots, M_t$. Note that $n = n_1 \cdots n_t$. Since $M = M_1 \otimes \cdots \otimes M_t$ is pure isoclinic at 0 with slope $\mu = m/n$, $M_1, \ldots, M_t$ are pure isoclinic at 0 with respective slopes $\mu_1, \ldots, \mu_t$ such that $\mu = \mu_1 + \cdots + \mu_t$ (see [Sau04, Théorème 2.3.1]).

For any $i \in \{1, \ldots, l\}$, $\mu_i$ has the form $m_i/n_i$ for some $m_i \in \mathbb{Z}$. The equalities $m/n = \mu_1 + \cdots + \mu_t = m_1/n_1 + \cdots + m_t/n_t$ and $n = n_1 \cdots n_t$, together with the fact that $m$ is coprime to $n$, imply that for any $i \in \{1, \ldots, l\}$, $m_i$ is coprime to $n_i$.

**Lemma 18.** Let $M$ be an object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ which is of rank $n$ and satisfies $(\mathcal{H}1)$. Assume that $M \cong M_1 \otimes M_2$ for some objects $M_1$ and $M_2$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ with respective ranks $n_1 > 1$ and $n_2$. If $M'' \cong U_1 \otimes M_1$ for some rank-one object $U_1$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$, then $n_1 = 2$.

**Proof.** We have $M'' \cong M_1' \otimes M_2' \cong U_1 \otimes M_1 \otimes M_2'$. Lemma 17 ensures that both $M_1$ and $M_2$ satisfy $(\mathcal{H}1)$. Denoting by $\mu_1, \mu_2$ and $\nu$ the respective slopes of $M_1, M_2$ and $U_1$ at 0, we get that the unique slope $-\mu_1 - \mu_2 \nu$ of $M''$ at 0 is equal to the unique slope $\nu + \mu_1 - \mu_2$ of $U_1 \otimes M_1 \otimes M_2'$ at 0. So $2\mu_1 = -\nu \in \mathbb{Z}$ (because $U_1$ has rank one). Since $M_1$ satisfies $(\mathcal{H}1)$, we get $n_1 = 2$.

This following result was (essentially) proved by van der Put and Singer in [vdPS97, §1.2]. Following the referees’ suggestion, we shall give a sketch of the proof here.

**Proposition 19.** If $(\sigma_q Y =AY)$ is an object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ which is of rank $n$ and has a connected Galois group $G$, then there exists an object $(\sigma_q Y =BY)$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ isomorphic to $(\sigma_q Y =AY)$ such that $B$ belongs to $G(\mathbb{C}(z))$.

**Proof.** We keep, and specialize to our situation, the notation of [vdPS97, §1.2]: let $k = \mathbb{C}(z)$, $\phi = \sigma_q$ and $C = \mathbb{C}$. The Galois group $G$ can be seen as the group of $k$-automorphisms which commute with $\phi$ of some Picard–Vessiot ring $R$ over $k$ of $\sigma_q Y = AY$. We consider the algebraic group $G = G \otimes \mathbb{C} k$ in $\text{GL}(n,k)$. Also, we consider the reduced algebraic subset $Z$ of $\text{GL}(n,k)$ corresponding to $R$. From [vdPS97, Theorem 1.13] it follows that $Z/k$ has a natural structure of $G$-torsor: the morphism $Z \times_k G \to G \times_k G \times_k G$ given by $(z, g) \mapsto (z g, g)$ is an isomorphism. But $k = \mathbb{C}(z)$ is a $\mathbb{C}^1$-field and $G$ is connected, so [vdPS97, Corollary 1.18] and the discussion following it ensure that $Z/k$ is a trivial $G$-torsor. Therefore $Z(k)$ is nonempty, and for $U \in Z(k)$ we have $Z(\bar{\mathbb{F}}) = UG(\bar{\mathbb{F}})$. We now use the $\tau$-invariance of $Z$ (the map $\tau$ is defined at the beginning of [vdPS97, §1.2]) and the $\tau$-invariance property is [vdPS97, Lemma 1.10]): since $\tau Z(\bar{\mathbb{F}}) = Z(\bar{\mathbb{F}})$, we have $\tau (UG(\bar{\mathbb{F}})) = UG(\bar{\mathbb{F}})$, i.e. $A^{-1} \phi(U) G(\bar{\mathbb{F}}) = UG(\bar{\mathbb{F}})$ (where we have used the fact that $\tau(U G(\bar{\mathbb{F}})) = A^{-1} \phi(U) \phi G(\bar{\mathbb{F}}) = A^{-1} \phi(U) G(\bar{\mathbb{F}})$). Hence $\phi(U) A^{-1} A U \in G(k)$. □

**Theorem 20** (Main theorem in the one-slope case). Let $M$ be an object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ which is of rank $n$, has a connected Galois group and satisfies $(\mathcal{H}1)$. Then $\text{Gal}(M, \omega)$ is the image of $\Pi_{l=1}^{n} \text{GL}(\mathbb{C}^{n_l})$ in $\prod_{l=1}^{n} \text{std}$ for some $l \in \mathbb{N}^*$ and some pairwise coprime numbers $n_1, \ldots, n_l > 1$ such that $n = n_1 \cdots n_l$.

**Proof.** We set $G = \text{Gal}(M, \omega)$. Proposition 16 and [vdPR07, Example 5.6 in §5.2] show that the hypotheses of Theorem 9 are satisfied by $G$ and hence that the derived subgroup $G'$
of $G$ is either the image of $\prod_{i=1}^l \text{SL}(\mathbb{C}^{n_i})$ in $\bigotimes_{i=1}^l \text{std}$ for some $l \in \mathbb{N}^*$ and some pairwise coprime numbers $n_1, n_2, \ldots, n_l > 1$ or the image of $\text{Sp}(\mathbb{C}^{n_1}) \times \prod_{i=2}^l \text{SL}(\mathbb{C}^{n_i})$ in $\bigotimes_{i=1}^l \text{std}$ for some $l \in \mathbb{N}^*$ and some pairwise coprime numbers $n_1 \geq 4$ even and $n_2, \ldots, n_l > 1$ and that $G' \subset G \subset \mathbb{C}^*G'$. Since $\det(M)$ is a rank-one irregular object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$, its Galois group is $\mathbb{C}^*$, so $G = \mathbb{C}^*G'$. Therefore, $G$ is either the image of $\prod_{i=1}^l \text{GL}(\mathbb{C}^{n_i})$ in $\bigotimes_{i=1}^l \text{std}$ or the image of $\mathbb{C}^* \text{Sp}(\mathbb{C}^{n_1}) \times \prod_{i=2}^l \text{GL}(\mathbb{C}^{n_i})$ in $\bigotimes_{i=1}^l \text{std}$. It remains to exclude the second case. Assume to the contrary that $G$ is $\mathbb{C}^* \text{Sp}(\mathbb{C}^{n_1}) \times \prod_{i=2}^l \text{GL}(\mathbb{C}^{n_i})$ in $\bigotimes_{i=1}^l \text{std}$. Using Proposition 19, we would get $M \cong M_1 \otimes \cdots \otimes M_l$ for some objects $M_1, \ldots, M_l$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$, where $M_1$ is such that $M_1^\vee \cong U_1 \otimes M_1$ for some rank-one object $U_1$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$. Lemma 18 would then imply that $n_1 = 2$. This is a contradiction. 

**Definition 21.** An object $M$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ is $\otimes$-decomposable if there exist two objects $M_1$ and $M_2$ of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ of rank at least 2 such that $M \cong M_1 \otimes M_2$. 

**Corollary 22.** Let $M$ be an object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ which is of rank $n$, has a connected Galois group and satisfies $(\mathcal{H}1)$. If $M$ is $\otimes$-indecomposable, then $\text{Gal}(M, \omega) = \text{GL}(\omega(M))$.

**Proof.** This is a direct consequence of Theorem 20 and Proposition 19. 

---

**6. Main theorem in the two-slopes case**

**Lemma 23.** Let $M$ be an object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ of rank $\geq 3$ satisfying $(\mathcal{H}2)$. Then $\text{Gal}(M, \omega)$ is neither a subgroup of $\mathbb{C}^*\text{SO}(\omega(M))$ nor a subgroup of $\mathbb{C}^*\text{Sp}(\omega(M))$ (for some bilinear forms).

**Proof.** Let $H$ be either $\text{SO}(\omega(M))$ or $\text{Sp}(\omega(M))$ and set $G = \mathbb{C}^*H$. Assume that $\text{Gal}(M, \omega)$ is a subgroup of $G$. Let $\rho$ be the representation of $\text{Gal}(M, \omega)$ corresponding to $M$ by tannakian duality. Let $\chi$ be the character of $G$ defined, for any $t \in \mathbb{C}^*$ and any $A \in H$, by $\chi(tA) = t^2$. The dual $\rho^\vee$ of $\rho$ is conjugated to $\rho \otimes (\chi^{-1} \circ p)$. Therefore, there exists a rank-one object $U$ of $(M)$ such that $M^\vee \cong U \otimes M$. But at 0 (see [Sau04, Théorème 2.3.1]), $M^\vee$ has two slopes, namely 0 with multiplicity $n - r$ and $-\mu$ with multiplicity $r$, while $U \otimes M$ has two slopes, namely $\nu$ with multiplicity $n - r$ and $\mu + \nu$ with multiplicity $r$ where $\nu \in \mathbb{Z}$ denotes the unique slope of $U$. The only possibility is $\mu = 0$, which gives a contradiction.

**Theorem 24.** (Main theorem in the two-slopes case). Let $M$ be an irreducible object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ which is of rank $n$, has a connected Galois group and satisfies $(\mathcal{H}2)$. Then $\text{Gal}(M, \omega) = \text{GL}(\omega(M))$.

**Proof.** The formal slopes decomposition [Sau04, Théorème 3.1.7] ensures that $\hat{M} \cong \hat{M}_0 \oplus \hat{M}_\mu$, where $\hat{M}_0$ is a regular singular object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ with exponents in $q^a$ and $\hat{M}_\mu$ is a pure isoclinic object of $\mathcal{E}(\mathbb{C}(z), \sigma_q)$ of slope $\mu$ and rank $r$. Proposition 16 ensures that $\hat{M}_\mu \cong \hat{M}_q^\vee(r, m, a)$ for some $a \in \mathbb{C}^*$, so $\hat{M} \cong \hat{M}_0 \oplus \hat{M}_q^\vee(r, m, a)$. Thus $\text{Gal}(M, \omega)$ contains, with respect to a suitable basis, $I_{n-r} \oplus \mathbb{C}^*I_r$ and $I_{n-r} \oplus \text{diag}(1, \zeta, \ldots, \zeta^{r-1})$ where $\zeta$ is a primitive $r$th root of 1 (a consequence of applying [vdPR07, §5] or [RS07, §3.2] to $[r] \star \hat{M} \cong [r] \star \hat{M}_0 \oplus \hat{M}_q(1, 0, c) \oplus \hat{M}_q(1, m, 1)$). If $r \geq 2$, Theorem 2 implies that $G \subset \text{Gal}(M, \omega) \subset \mathbb{C}^*G$ with $G = \text{SL}(\omega(M))$, $\text{SO}(\omega(M))$ or $\text{Sp}(\omega(M))$. Note that the Galois group of $\det(M)$ is $\mathbb{C}^*$ because $\det(M)$ is irregular of rank one,
so \( \text{Gal}(M, \omega) = \mathbb{C}^*G \). Lemma 23 leads to the conclusion. If \( r = 1 \), the result follows from Theorem 10. \( \square \)

7. Some computations of Galois groups

7.1 Generalized \( q \)-hypergeometric equations with two slopes

We keep the notation of § 1 (and the hypothesis that \( r > s \)) for the generalized \( q \)-hypergeometric operator with parameters \( \underline{a} = (a_1, \ldots, a_r) \in (\mathbb{Q}^r)^r \), \( \underline{b} = (b_1, \ldots, b_s) \in (\mathbb{Q}^s)^s \) and \( \lambda \in \mathbb{C}^* \), and we set

\[
\mathcal{H}_q(\underline{a}; \underline{b}; \lambda) = \mathcal{D}(\mathbb{C}^2, \sigma_q)/\mathcal{D}(\mathbb{C}^2, \sigma_q) \mathcal{L}_q(\underline{a}; \underline{b}; \lambda).
\]

If \( s > 0 \), then \( \mathcal{H}_q(\underline{a}; \underline{b}; \lambda) \) satisfies (\( \mathcal{K} 2 \)) (its slopes at 0 are 0 with multiplicity \( s \) and 1/(\( r - s \)) with multiplicity \( r - s \)). Theorem 24 leads to the following.

**Theorem 25.** The general linear group \( \text{GL}(\mathbb{C}^r) \) is the unique connected algebraic group occurring as the Galois group of some irreducible generalized \( q \)-hypergeometric module \( \mathcal{H}_q(\underline{a}; \underline{b}; \lambda) \) with parameters \( \underline{a} = (a_1, \ldots, a_r) \in (\mathbb{Q}^r)^r \) and \( \underline{b} = (b_1, \ldots, b_s) \in (\mathbb{Q}^s)^s \) with \( r > s > 0 \).

We now turn to explicit computations of \( q \)-hypergeometric Galois groups. For all \( i \in \{1, \ldots, r\} \), we denote by \( \alpha_i \) the unique element of \( \mathbb{R} \) such that \( a_i = q^{\alpha_i} \).

**Theorem 26.** Assume that \( s > 0 \), that \( \beta_j - \alpha_i \notin \mathbb{Z} \) for all \( (i,j) \in \{1, \ldots, r\} \times \{1, \ldots, s\} \), and that the algebraic group generated by \( \text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_r}) \) is connected. Then \( \text{Gal}(\mathcal{H}_q(\underline{a}; \underline{b}; \lambda), \omega) = \text{GL}(\mathbb{C}^r) \).

**Proof.** Since, for all \( (i,j) \in \{1, \ldots, r\} \times \{1, \ldots, s\} \), \( \beta_j - \alpha_i \notin \mathbb{Z} \), we have that \( \mathcal{H}_q(\underline{a}; \underline{b}; \lambda) \) is irreducible (using the same arguments as in [Roq11, § 5.1]). Moreover, \( \mathcal{H}_q(\underline{a}; \underline{b}; \lambda) \) is regular singular at \( \infty \) with exponents \( \alpha_1, \ldots, \alpha_r \). It follows easily from [vdPS97, ch. 12] or [San03, § 2.2] that if the algebraic group generated by \( \text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_r}) \) is connected, then the local formal Galois group of \( \mathcal{H}_q(\underline{a}; \underline{b}; \lambda) \) at \( \infty \) is connected; hence, by virtue of (the variant at \( \infty \) of) Corollary 12, \( \text{Gal}(\mathcal{H}_q(\underline{a}; \underline{b}; \lambda), \omega) \) is connected. Theorem 25 leads to the desired result. \( \square \)

For instance, the algebraic group generated by \( \text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_r}) \) is connected if \( \underline{a} \in (\mathbb{Q}^r)^r \) or if \( \alpha_1, \ldots, \alpha_r \) are \( \mathbb{Z} \)-linearly independent.

7.2 \( q \)-Kloosterman equations

We retain the notation of § 1 for the \( q \)-Kloosterman operators and set

\[
\mathcal{K}_q(U, V) = \mathcal{D}(\mathbb{C}^2, \sigma_q)/\mathcal{D}(\mathbb{C}^2, \sigma_q) \mathcal{K}_q(U, V).
\]

Note that \( \mathcal{K}_q(U, V) \) is pure isoclinic at 0 with slope \( \text{deg } V/\text{deg } U \). In particular, if \( \text{deg } U \) is coprime to \( \text{deg } V \), then \( \mathcal{K}_q(U, V) \) satisfies (\( \mathcal{K} 1 \)). Theorem 20 and Corollary 22 lead to the following result.

**Theorem 27.** Let \( G \) be a connected algebraic group occurring as the Galois group of some \( q \)-Kloosterman module \( \mathcal{K}_q(U, V) \) such that \( \text{deg } U \) is coprime to \( \text{deg } V \). Then \( G \) is the image of \( \prod_{i=1}^l \text{GL}(\mathbb{C}^{n_i}) \) in \( \text{std } \) for some \( l \in \mathbb{N}^* \) and some pairwise coprime numbers \( n_1, \ldots, n_l > 1 \) such that \( \text{deg } U = n_1 \cdots n_l \). If, moreover, \( \mathcal{K}_q(U, V) \) is \( \otimes \)-indecomposable, then \( G \) is \( \text{GL}(\mathbb{C}^{\text{deg } U}) \).

We denote by \( c_1, \ldots, c_{\text{deg } U} \) the roots of \( X^u(U(X^{-1}) + V(0)) \in \mathbb{C}[X] \). For all \( i \in \{1, \ldots, \text{deg } U\} \), we denote by \( (u_i, \alpha_i) \) the unique element of \( U \times \mathbb{R} \) such that \( c_i = u_i q^{\alpha_i} \).
ON CLASSICAL IRREGULAR $q$-DIFFERENCE EQUATIONS

Theorem 28. If $\deg U$ is coprime to $\deg V$ and if the algebraic group generated by $\text{diag}(u_1, \ldots, u_{\deg U})$ and $\text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_{\deg U}})$ is connected, then $\text{Gal}(Kl_q(U, V), \omega)$ is the image of $\prod_{i=1}^{l} \text{GL}(C_{n_i})$ in $\bigotimes_{i=1}^{l} \text{std}$ for some $l \in \mathbb{N}^*$ and some pairwise coprime numbers $n_1, \ldots, n_l > 1$ such that $\deg U = n_1 \cdots n_l$. If, moreover, $Kl_q(U, V)$ is $\otimes$-indecomposable, then $\text{Gal}(Kl_q(U, V), \omega)$ is $\text{GL}(C_{\deg U})$.

Proof. Note that $Kl_q(U, V)$ is regular singular at $\infty$ with exponents $c_1, \ldots, c_{\deg U}$. It follows easily from [vdPS97, ch. 12] or [Sau03, §2.2] that if the algebraic group generated by $\text{diag}(u_1, \ldots, u_{\deg U})$ and $\text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_{\deg U}})$ is connected, then the local formal Galois group of $Kl_q(U, V)$ at $\infty$ is connected and hence, by virtue of (the variant at $\infty$ of) Corollary 12, $\text{Gal}(Kl_q(U, V), \omega)$ is connected. Theorem 27 leads to the desired result. \hfill \square

Note that a $q$-Kloosterman module $Kl_q(U, V)$ with $\deg U$ coprime to $\deg V$ is not necessarily $\otimes$-indecomposable. For instance,

$$Kl_q(X^6, -(1 + q^{-1}X)(1 + q^{-3}X)(1 + q^{-2}X)(1 + X)^2) \cong Kl_q(X^2, -(1 + X)) \otimes Kl_q(X^3, -(1 + X)).$$

8. A $\otimes$-indecomposability criterion and application to $q$-Kloosterman operators

8.1 A $\otimes$-indecomposability criterion

Slopes theory leads to a simple proof of the $\otimes$-indecomposability of the Kloosterman differential modules with bidegree $(u, v)$ such that $u$ is coprime to $v$; see [Kat87]. In contrast, we gave at the end of §7.2 an example of $\otimes$-decomposable $q$-Kloosterman module $Kl_q(U, V)$ with $\deg U$ coprime to $\deg V$. In this section, we propose an obstruction to $\otimes$-decomposability (Theorem 31 below) coming from residues at points in $\mathbb{C}^*$ of intrinsic Birkhoff matrices. In [Roq11], we used related ideas to obtain an analogue of the usual notion of monodromy for the generalized $q$-hypergeometric equations.

We first work with $q$-difference systems.

Definition 29 (Property $(H_q)$). We say that an object $(\sigma_q Y = AY)$ of $E'_\mathbb{C}(z, \sigma_q)$ of rank $n$ satisfies the condition $(H_q)$ if:

1. there exists $z_0 \in \mathbb{C}^*$ such that $A$ is analytic at any point of $qZ \land z_0$, $A(z_0)$ has rank $n - 1$ and, for all $k \in \mathbb{Z}^*$, $A(q^k z_0) \in \text{GL}_n(C)$;
2. $(\sigma_q Y = AY)$ is pure isoclinic at both 0 and $\infty$.

Lemma 30. Let $(\sigma_q Y = AY)$ be an object of $E'_\mathbb{C}(z, \sigma_q)$ of rank $n$. If $(\sigma_q Y = AY)$ is pure isoclinic at 0 and $\infty$ with integral slopes denoted, respectively, by $\mu_0$ and $\mu_\infty$, then:

1. there exist $A^{(0)} \in \text{GL}_n(C)$ and $F^{(0)} \in \text{GL}_n(C \{ z \})$ such that $F^{(0)}$ is an isomorphism in $E'_\mathbb{C}(z, \sigma_q)$ from $(\sigma_q Y = z^{\mu_0} A^{(0)} Y)$ to $(\sigma_q Y = AY)$. Similarly, there exist $A^{(\infty)} \in \text{GL}_n(C)$ and $F^{(\infty)} \in \text{GL}_n(C \{ z^{-1} \})$ such that $F^{(\infty)}$ is an isomorphism in $E'_\mathbb{C}(z^{-1}, \sigma_q)$ from $(\sigma_q Y = z^{\mu_\infty} A^{(\infty)} Y)$ to $(\sigma_q Y = AY)$.

If, moreover, $(\sigma_q Y = AY)$ satisfies $(H_q)$, then:

1. for any $A^{(0)}$, $F^{(0)}$, $A^{(\infty)}$ and $F^{(\infty)}$ satisfying the conditions of (i), we have, for $z$ near $z_0$, $(F^{(0)})^{-1} F^{(\infty)}(z) = H \mod (z - z_0) M_n(C \{ z \} - z_0)$ for some $H \in M_n(C)$ with rank $n - 1$. 1639
Proof. For (i), we refer to [RS07, §2.2] and the references therein. We now prove that (ii) holds. Since $F(0)$ is an isomorphism from $(\sigma_q Y = z^\mu A(0)^0 Y)$ to $(\sigma_q Y = A Y)$, we have, for $z$ near $0$, $F(0)(qz) z^\mu A(0) = A(z) F(0)(z)$. Similarly, for $z$ near $\infty$, $F(\infty)(qz) z^{\mu_\infty} A(\infty) = A(z) F(\infty)(z)$. These equations, together with the fact that $F(0) \in GL_n(\mathbb{C}(\{z\}))$ and $F(\infty) \in GL_n(\mathbb{C}(\{z^{-1}\}))$, show that $F(0)$ and $F(\infty)$ can be extended meromorphically to $\mathbb{C}$ and $\mathbb{C}^*$, respectively, and that for all $m \in \mathbb{N}^+$ we have, over $\mathbb{C}^*$,

$$(F(0))^{-1} F(\infty)(z) = z^{-m_\mu q} (m(m-1)/2) \mu_\infty (A(0))^{-1} (q^m z) A(q^{m-1} z) \cdots A(z)$$

Now the result follows easily from the facts that $(F(0))^{-1} \in GL_n(\mathbb{C}(\{z\}))$, $F(\infty) \in GL_n(\mathbb{C}(\{z^{-1}\}))$, $A(z) = A(z_0)$ mod $(z-z_0) M_n(\mathbb{C} \{z-z_0\})$, and, for any $k \in \mathbb{Z}^*$, $A(q^k z) \in GL_n(\mathbb{C}) + (z-z_0) M_n(\mathbb{C} \{z-z_0\})$. □

**Theorem 31** ($\otimes$-indecomposability criterion for systems). Let $(\sigma_q Y = A Y)$ be an object of $\mathcal{E}'(\mathfrak{C}(z), \sigma_q)$ which satisfies $(H_q)$. Then $(\sigma_q Y = A Y)$ is $\otimes$-indecomposable.

Proof. Assume to the contrary that $(\sigma_q Y = A Y)$ is $\otimes$-decomposable. Then there exist $A_1 \in GL_{n_1}(\mathbb{C}(z))$ and $A_2 \in GL_{n_2}(\mathbb{C}(z))$ $(n_1, n_2 > 1)$ such that $(\sigma_q Y = A Y) \cong (\sigma_q Y = A_1 Y) \otimes (\sigma_q Y = A_2 Y)$. For further use, we denote by $R \in GL_n(\mathbb{C}(z))$ an isomorphism from $(\sigma_q Y = A_1 Y) \otimes (\sigma_q Y = A_2 Y)$ to $(\sigma_q Y = A Y)$. Since $(\sigma_q Y = A Y) \cong (\sigma_q Y = A_1 Y) \otimes (\sigma_q Y = A_2 Y)$ are pure isoclinic, both $(\sigma_q Y = A_1 Y)$ and $(\sigma_q Y = A_2 Y)$ are pure isoclinic (see [Sau04, Théorème 2.3.1]). Let $N \in \mathbb{N}^*$ be such that $[N]^*(\sigma_q Y = A_1 Y) \cong (\sigma_q Y = [N]^* A_1 Y)$, $[N]^*(\sigma_q Y = A_2 Y) \cong (\sigma_q Y = [N]^* A_1 Y)$, and $[N]^*(\sigma_q Y = A Y) \cong (\sigma_q Y = [N]^* A_1 Y) \otimes (\sigma_q Y = [N]^* A_2 Y)$ are all pure isoclinic with integral slopes. Lemma 30 ensures that there are $\mu_{1,0}, \mu_{1,\infty}, \mu_{2,0}, \mu_{1,\infty} \in \mathbb{Z}$ such that there exist:

- $A^0_1(0) \in GL_{n_1}(\mathbb{C})$ and $A^0_1(0) \in GL_{n_1}(\mathbb{C}(\{z_N\}))$ such that $F^0_1(0)$ is an isomorphism from $\sigma_{qN} Y = z_N^{\mu_{1,0}} A^0_1 Y$ to $\sigma_{qN} Y = [N]^* A_1 Y$;
- $A^\infty_1(0) \in GL_{n_1}(\mathbb{C})$ and $A^\infty_1(0) \in GL_{n_1}(\mathbb{C}(\{z_N^{-1}\}))$ such that $F^\infty_1(0)$ is an isomorphism from $\sigma_{qN} Y = z_N^{\mu_{1,\infty}} A^\infty_1 Y$ to $\sigma_{qN} Y = [N]^* A_1 Y$;
- $A^0_2(0) \in GL_{n_2}(\mathbb{C})$ and $A^0_2(0) \in GL_{n_2}(\mathbb{C}(\{z_N\}))$ such that $F^0_2(0)$ is an isomorphism from $\sigma_{qN} Y = z_N^{\mu_{2,0}} A^0_2 Y$ to $\sigma_{qN} Y = [N]^* A_2 Y$;
- $A^\infty_2(0) \in GL_{n_2}(\mathbb{C})$ and $F^\infty_2(0) \in GL_{n_2}(\mathbb{C}(\{z_N^{-1}\}))$ such that $F^\infty_2(0)$ is an isomorphism from $\sigma_{qN} Y = z_N^{\mu_{2,\infty}} A^\infty_2 Y$ to $\sigma_{qN} Y = [N]^* A_2 Y$.

So $F(0) = ([N]^* R)(F^0_1(0) \otimes F^0_2(0)) \in GL_n(\mathbb{C}(\{z_N\}))$ is an isomorphism from $(\sigma_{qN} Y = z_N^{\mu_{1,0}} A^0_1 Y) \otimes (\sigma_{qN} Y = z_N^{\mu_{2,0}} A^0_2 Y)$ to $(\sigma_{qN} Y = [N]^* A Y)$ and $F(\infty) = ([N]^* R)(F^\infty_1(0) \otimes F^\infty_2(0)) \in GL_n(\mathbb{C}(\{z_N^{-1}\}))$ is an isomorphism from $(\sigma_{qN} Y = z_N^{\mu_{1,\infty}} A^\infty_1 Y) \otimes (\sigma_{qN} Y = z_N^{\mu_{2,\infty}} A^\infty_2 Y)$ to $(\sigma_{qN} Y = [N]^* A Y)$. It is easy to see that $(\sigma_{qN} Y = [N]^* A Y)$ satisfies $(H_{qN})$. So Lemma 30 ensures that, near some $z_0 \in \mathbb{C}^*$, $(F(0))^{-1} F(\infty)(z_N) = H$ mod $(z_N - z_0) M_n(\mathbb{C} \{z-N - z_0\})$ for some $H \in M_n(\mathbb{C})$ with rank $n - 1$. Since $(F(0))^{-1} F(\infty) = (F(0))^{-1} F(\infty) \otimes (F(0))^{-1} F(\infty)$, $H$ has the form $H_1 \otimes H_2$ for some $H_1 \in M_{n_1}(\mathbb{C})$ and $H_2 \in M_{n_2}(\mathbb{C})$. Therefore the rank of $H$ is the product of the ranks of $H_1$ and $H_2$. This implies that either $n_1 = 1$ or $n_2 = 1$, which is a contradiction. □
Let us now switch to operators. Recall that the $q$-difference system $(\sigma_q Y = AY)$ associated to $L = \sum_{k=0}^{n} a_{n-k} \sigma_q^k \in \mathcal{D}(\mathbb{C}(z), \sigma_q)$ with $a_0 a_n \neq 0$ is given by:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n/a_0 & -a_{n-1}/a_0 & -a_{n-2}/a_0 & \cdots & a_2/a_0 & a_1/a_0 \end{pmatrix} \in \mathbb{GL}_n(\mathbb{C}(z)).$$

**Theorem 32** (⊗-indecomposability criterion for operators). Assume that $L = \sum_{k=0}^{n} a_{n-k} \sigma_q^k \in \mathcal{D}(\mathbb{C}(z), \sigma_q)$ with $a_0 a_n \neq 0$ is such that:

1. there exists $z_0 \in \mathbb{C}^*$ such that $a_n/a_0, \ldots, a_1/a_0$ are analytic at any point of $q^k z_0$, $a_n/a_0(z_0) = 0$ and, for all $k \in \mathbb{Z}^*$, $a_n/a_0(q^k z_0) \neq 0$;
2. $L$ is pure isoclinic at both 0 and $\infty$.

Then $L$ is ⊗-indecomposable.

**Proof.** Since $L$ is ⊗-indecomposable if and only if the associated $q$-difference system $(\sigma_q Y = AY)$ is ⊗-indecomposable, the result is an immediate consequence of Theorem 31. \hfill \square

### 8.2 Application to $q$-Kloosterman operators (including $\mathcal{H}_q(\alpha; \theta; \lambda)$)

We keep the notation of §7.2.

**Theorem 33.** The general linear group $\mathbb{GL}(\mathbb{C}^{\deg U})$ is the unique connected algebraic group occurring as the Galois group of some $q$-Kloosterman module $\mathcal{K}_q(U, V)$ such that $\deg U$ is coprime to $\deg V$ and such that there exists $z_0 \in \mathbb{C}^*$ satisfying $V(z_0) = 0$ and, for all $k \in \mathbb{Z}^*$, $V(q^k z_0) \neq 0$.

**Proof.** This is an immediate consequence of Theorems 32 and 27. \hfill \square

**Corollary 34.** The general linear group $\mathbb{GL}(\mathbb{C}^r)$ is the unique connected algebraic group occurring as the Galois group of some confluent generalized $q$-hypergeometric module $\mathcal{H}_q(\alpha; \theta; \lambda)$.

**Proof.** This is a special case of Theorem 33, since $\mathcal{L}_q(\alpha; \theta; \lambda) = z \mathcal{K}_q(-\lambda \prod_{i=1}^{r}(a_i X - 1) + (-1)^r \lambda, -(-1)^r \lambda + X)$. \hfill \square

In the following result, $c_1, \ldots, c_{\deg U}$ denote the complex roots of $X^{\deg U}(U(X^{-1}) + V(0)) \in \mathbb{C}[X]$ and, for all $i \in \{1, \ldots, \deg U\}$, $(u_i, \alpha_i)$ denotes the unique element of $\mathbb{U} \times \mathbb{R}$ such that $c_i = u_i q^{\alpha_i}$.

**Theorem 35.** Assume that $\deg U$ is coprime to $\deg V$, that the algebraic group generated by $\text{diag}(u_1, \ldots, u_{\deg U})$ and $\text{diag}(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_{\deg U}})$ is connected, and that there exists $z_0 \in \mathbb{C}^*$ such that $V(z_0) = 0$ and, for all $k \in \mathbb{Z}^*$, $V(q^k z_0) \neq 0$. Then, $\text{Gal}(\mathcal{K}_q(U, V), \omega)$ is $\mathbb{GL}(\mathbb{C}^{\deg U})$.

**Proof.** This is an immediate consequence of Theorems 32 and 28. \hfill \square

In the following result, for all $i \in \{1, \ldots, r\}$, $(u_i, \alpha_i)$ denotes the unique element of $\mathbb{U} \times \mathbb{R}$ such that $a_i = u_i q^{\alpha_i}$.
THEOREM 36. If the algebraic group generated by \( \text{diag}(u_1, \ldots, u_n) \) and \( \text{diag}(e^{2\pi i a_1}, \ldots, e^{2\pi i a_r}) \) is connected, then \( \text{Gal}(\mathcal{H}_q(a; \emptyset; \lambda), \omega) \) is \( \text{GL}(\mathbb{C}) \).

Proof. This is a special case of Theorem 35, since \( L_q(a; \emptyset; \lambda) = z K l_q(-\lambda \prod_{i=1}^{r} (a_i X - 1) + (-1)^j \lambda, -(-1)^j \lambda + X) \).

\( \square \)

8.3 Equations satisfying \( (\mathcal{H}1) \) with Galois group \( \bigotimes_{i=1}^l \text{GL}(\mathbb{C}^{n_i}) \)

THEOREM 37. For any \( l \in \mathbb{N}^* \), given any pairwise coprime numbers \( n_1, \ldots, n_l > 1 \), the image of \( \prod_{i=1}^l \text{GL}(\mathbb{C}^{n_i}) \) in \( \bigotimes_{i=1}^l \text{std} \) occurs as the Galois group of some object of \( \mathcal{E}(\mathbb{C}(z), \sigma_q) \) which is of rank \( n = n_1 \cdots n_l \) and satisfies \( (\mathcal{H}1) \).

Proof. Theorem 36 ensures that, for any \( i \in \{1, \ldots, l\} \), there exists an object \( M_i \) of \( \mathcal{E}(\mathbb{C}(z), \sigma_q) \) of rank \( n_i \) which satisfies \( (\mathcal{H}1) \) and whose Galois group is \( \text{GL}(\mathbb{C}^{n_i}) \). It is easily seen that \( \bigotimes_{i=1}^l M_i \) satisfies \( (\mathcal{H}1) \). For any \( i \in \{1, \ldots, l\} \), let \( \rho_i \) be the representation of \( \text{Gal}(\bigotimes_{i=1}^l M_i, \omega) \) corresponding to \( M_i \) by tannakian duality. Then, for any \( i \in \{1, \ldots, l\} \), the image of \( \rho_i \) is \( \text{GL}(\mathbb{C}^{n_i}) \) and \( \bigotimes_{i=1}^l \rho_i \) is a faithful representation (because it is the representation of \( \text{Gal}(\bigotimes_{i=1}^l M_i, \omega) \) corresponding to \( \bigotimes_{i=1}^l M_i \) itself). So the image of \( \bigotimes_{i=1}^l \rho_i \) coincides with the image of \( \prod_{i=1}^l \text{GL}(\mathbb{C}^{n_i}) \) in \( \bigotimes_{i=1}^l \text{std} \), by virtue of the Goursat–Kolchin–Ribet theorem [Kat90, Proposition 1.8.2].

\( \square \)

9. More computations

9.1 Non-\( q \)-Kummer-induced equations in the two-slopes case

THEOREM 38. Let \( M \) be an irreducible object of \( \mathcal{E}(\mathbb{C}(z), \sigma_q) \) which is of rank \( n \) and satisfies \( (\mathcal{H}2) \) with \( r \) coprime to \( n \). Assume that \( M \) is regular singular at \( \infty \) with exponents \( c_1, \ldots, c_n \in \mathbb{Q}^+ \). If the list \( c_1, \ldots, c_n \) is not \( q \)-Kummer induced, then \( \text{Gal}(M, \omega) = \text{GL}(\omega(M)) \).

Proof. We let \( G = \text{Gal}(M, \omega) \). Proposition 15 ensures that \( G^0 \), and hence its Lie algebra \( g \), acts irreducibly on \( \omega(M) \). Moreover, the proof of Theorem 24 shows that \( G^0 \) contains, with respect to some basis, \( I_{n-r} \oplus \mathbb{C} I_r \). So \( g \) contains, with respect to some basis, \( 0_{n-r} \oplus \mathbb{C} I_r \) and hence contains an element having two eigenvalues with relatively prime multiplicities. According to Serre [Ser67, § 4], this implies that \( g \) is either \( \mathfrak{sl}(\omega(M)) \) or \( \mathfrak{gl}(\omega(M)) \). Since \( \det(M) \) is irregular of rank one, its Galois group is \( \mathbb{C}^* \). So \( G = \text{GL}(\omega(M)) \).

An immediate application is the following (see § 7.1 for \( \mathcal{H}_q(a; \emptyset; \lambda) \)).

THEOREM 39. If \( a_1, \ldots, a_r \in \mathbb{Q}^+ \) is not \( q \)-Kummer induced and if \( r \) is coprime to \( s > 0 \), then \( \text{Gal}(\mathcal{H}_q(a; \emptyset; \lambda), \omega) = \text{GL}(\mathbb{C}^s) \).

9.2 Another example of a \( q \)-Kloosterman equation

The proof of the following \( \otimes \)-indecomposability criterion is left to the reader.

PROPOSITION 40. Let \( M \) be an object of \( \mathcal{E}(\mathbb{C}(z), \sigma_q) \) of rank \( n \). Assume that \( M \) is regular singular at \( \infty \) with exponents \( c_1, \ldots, c_n \in \mathbb{Q}^+ \). If \( M \) is \( \otimes \)-decomposable, then there exists a divisor \( 1 < d < n \) of such that \( c_1, \ldots, c_n \mod q^Z \) is of the form \( (c'_i c''_j ; 1 \leq i \leq d, 1 \leq j \leq n/d) \mod q^Z \) for some \( c'_1, \ldots, c'_d \in \mathbb{C}^* \) and some \( c''_1, \ldots, c''_{n/d} \in \mathbb{C}^* \).

We now give an illustration of the previous result. Note that we cannot apply Theorem 35 to \( K l_q((q^{1/2} - X)^2(1 - X)^{n-2} - q, V) \) where \( V \in \mathbb{C}[X] \) is such that \( V(0) = q \). However, we can obtain the following result.
ON CLASSICAL IRREGULAR \textit{q}-DIFFERENCE EQUATIONS

Proposition 41. Let us consider \( V \in \q + \mathbb{C}[X] \). Then, for any odd integer \( n \geq 2 \) coprime to \( \deg V \), the Galois group of \( K\ell_{q}(q^{1/2} - X)^{2}(1 - X)^{n-2} - q, V \) is \( \text{GL}(\mathbb{C}^{n}) \).

Proof. Recall (see § 7.2) that \( M = K\ell_{q}(q^{1/2} - X)^{2}(1 - X)^{n-2} - q, V \) is pure isoclinic at 0 with slope \( \deg V/n \) and is regular singular at \( \infty \), having exponents \( q^{1/2} \) with multiplicity 2 and 1 with multiplicity \( n - 2 \). Since \( n \) is odd, Corollary 13 ensures that the Galois group of \( M \) is connected. It is easily seen that \( M \) is \( \otimes \)-indecomposable by using Proposition 40. Theorem 27 leads to the conclusion.

References


RS09 J.-P. Ramis and J. Sauloy, \textit{The \textit{q}-analogue of the wild fundamental group. II}, Astérisque \textbf{323} (2009), 301–324.


J. Roques


Julien Roques Julien.Roques@ujf-grenoble.fr
Institut Fourier, Université Grenoble 1, UMR CNRS 5582,
100 rue des Maths, BP 74, 38402 St Martin d’Hères, France