Helgason’s number
and lacunarity constants

R.E. Edwards and Kenneth A. Ross

This paper studies the connection between the best possible value of a constant in the compact abelian case of a known inequality due to Helgason and the $A_2$-constants of sets of characters. Various estimates of and expressions for the best possible value are given.

1. Introduction; the numbers $W_G$ and $h$

1.1. Helgason ([7], p. 245; [8], (36.10)) shows that if $G$ is a CAG (= compact Hausdorff abelian group), then the inequality

\[(a) \quad \|h\|_2 \leq M \sup \{ \|h \ast f\|_1 : f \in L^1(G), \|f\|_1 \leq 1 \}\]

holds for all $h \in L^2(G)$ with $M = \sqrt{2}$. [Note that the supremum in (a) is unaltered if we write $f \in \mathbb{T}(G)$ in place of $f \in L^1(G)$, where $\mathbb{T}(G)$ denotes the set of complex-valued trigonometric polynomials on $G$.] Moreover (see 1.3 below), (a) is equivalent to the inequality

\[(b) \quad \|F\|_2 \leq M \sup \{ \|Ff\|_1 : f \in C(G), \|f\|_1 \leq 1 \}\]

holding for all $F \in \hat{C}$, where $\hat{C}$ denotes the character group of $G$ and

Received 30 March 1973. The second author was partly supported by a grant from the National Science Foundation of the USA. He also gratefully acknowledges help and facilities of the Australian National University during his visit from September to December 1972. The authors are grateful to Professor P.A.P. Moran for helpful discussions, and in particular for the indication of the relevance of the central limit theorem to Theorem 2.11.
the set of continuous complex-valued functions on $G$. Inequality (b) appears in Theorem (2.1) of [3].

For a given $G$, we will denote by $M_G$ the smallest number $M \geq 0$ for which (a) (or (b)) is true. Clearly, $M_G \geq 1$ for every CAG $G$.

In what follows we introduce a certain number $h$, defined in terms of $L_2$-constants of large finite sets (see 1.4 and 1.5 below), which we call the Helgason number. The reason for the name is that we shall prove the following facts:

(i) $M_G \leq h$ for every CAG $G$ (Corollary 1.8);

(ii) $M_G = h$ for certain specifiable CAGs $G$ (Corollary 1.12, Theorem 3.6, Corollary 3.8).

Helgason’s result is included in the inequalities

(iii) $2^{k-\frac{1}{2}} \leq h \leq 2^k$ (Theorem 2.11, Corollary 2.5),

which we shall prove on the way.

We introduce also a somewhat similarly-defined number $h_n$ for every positive integer $n$, showing that

(iv) $h_n \leq h_{n+1}$ and $h = \lim_{n \to \infty} h_n$ (Lemma 1.6).

We will also show that

(v) $h_2 = \sqrt{\pi}/2$ (Theorem 2.10), and that

(vi) $h_n = \sup \left\{ \frac{1}{\sqrt{\pi}} \right\} / E_n(a_1, \ldots, a_n) \leq (2^{-1/n})^k$,

where

$$E_n(a_1, \ldots, a_n) = (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |c_1 e^{i\theta_1} + \cdots + c_n e^{i\theta_n}| d\theta_1 \cdots d\theta_n$$

and $c_1, \ldots, c_n$ denote nonnegative real numbers, not all zero (Corollaries 2.7 and 2.4).

In Section 3, we show that each $h_n$ can be given in terms of sets of
characters of $T$ (the circle group) only.

We have been unable to evaluate $h$; it would be very interesting to know whether or not $h < \sqrt{2}$.

We start with a simple lemma.

**Lemma 1.2.** If $G$ is a CAG and $g \in \mathcal{T}(G)$, then

$$\|g\|_1 = \sup\left\{ \left\| \sum_{\chi \in \hat{G}} g^*(\chi) f^*(\chi) \right\| : f \in \mathcal{C}(G), \|f\|_u \leq 1 \right\}.$$  

**Proof.** If $\lambda_G$ denotes normalised Haar measure on $G$, then

$$\|g\|_1 = \int |g| d\lambda_G = \sup\left\{ \left| \int g(x) f(x^{-1}) d\lambda_G(x) \right| : f \in \mathcal{C}(G), \|f\|_u \leq 1 \right\}$$

$$= \sup\left\{ \left| \sum_{\chi \in \hat{G}} g^*(\chi) \int x(x) f(x^{-1}) d\lambda_G(x) \right| : f \in \mathcal{C}(G), \|f\|_u \leq 1 \right\}$$

$$= \sup\left\{ \left| \sum_{\chi \in \hat{G}} g^*(\chi) f^*(\chi) \right| : f \in \mathcal{C}(G), \|f\|_u \leq 1 \right\}.$$

1.3. Now we verify the equivalence of (a) and (b) in 1.1. The supremum on the right of (a) is

$$\sup\left\{ \left\| \sum_{\chi \in \hat{G}} a(\chi) h^*(\chi) \right\|_1 : \text{suppa finite, } \|a\|_u \leq 1 \right\}$$

which, by Lemma 1.2, is equal to

$$\sup\left\{ \left| \sum_{\chi \in \hat{G}} a(\chi) h^*(\chi) f^*(\chi) \right| : \text{suppa finite, } \|a\|_u \leq 1, f \in \mathcal{C}(G), \|f\|_u \leq 1 \right\}$$

$$= \sup_{f} \sup_{a} \left\{ \left| \sum_{\chi \in \hat{G}} a(\chi) h^*(\chi) f^*(\chi) \right| \right\}$$

$$= \sup_{h} \left\{ \|h^* f^*\|_1 : f \in \mathcal{C}(G), \|f\|_u \leq 1 \right\}.$$  

Thus (a) is equivalent to (b) for $F (= h^*)$ in $L^2(\hat{G})$; but this is easily seen to be equivalent to (b) for arbitrary $F \in \mathcal{C}(\hat{G})$.

1.4. If $G$ is a CAG and $E$ is a subset of $\hat{G}$, we write $\mathcal{T}_E(G)$ for the set of $f \in \mathcal{T}(G)$ such that $f^*(\chi) = 0$ for every $\chi \in \hat{G} \setminus E$. We also write

$$\Lambda_G(E) = \sup\{\|f\|_2 : f \in \mathcal{T}_E(G), \|f\|_1 = 1 \} \leq \infty.$$
and call $\Lambda_G(E)$ the $\Lambda_2$-constant of $E$. It is easy to see that $\Lambda_G(E)$ is a finite assumed maximum whenever $E$ is finite. Moreover,
$$\Lambda_G(E) = \sup\{\Lambda_G(F) : F \text{ finite}, F \subseteq E\}.$$

1.5. Define sets $S$ and $S_n$ ($n$ a positive integer) of nonnegative real numbers as follows.

$S$ is the set of real numbers $\kappa \geq 0$ with the property that, for every positive integer $n$, there exists a CAG $K_n$ and an $n$-element subset $E_n$ of $\hat{K}_n$ such that
$$\Lambda_{K_n}(E_n) \leq \kappa.$$

$S_n$ is the set of real numbers $\kappa \geq 0$ with the property that there exists a CAG $K$ and an $n$-element subset $E$ of $\hat{K}$ such that
$$\Lambda_K(E) \leq \kappa.$$

The proof of Corollary 2.4 below shows incidentally that $2^{\frac{1}{n}} \in S$.

We now define
$$h_n = \inf S_n, \quad h = \inf S.$$

It is simple to verify that
$$S_{n+1} \subseteq S_n, \quad S = \bigcap_{n=1}^{\infty} S_n.$$

These observations render the next lemma obvious.

**LEMMA 1.6.** We have $h_n \leq h_{n+1}$ for every positive integer $n$, and
$$h = \lim_{n \to \infty} h_n.$$

**THEOREM 1.7.** Let $n$ be a positive integer. Then (b) of 1.1 holds with $M = h_n$ for every CAG $G$ and every $F \in \hat{G}$ whose support has cardinal $v(\text{supp} F)$ at most $n$.

**Proof.** Let $\kappa \in S_n$ and let $K$ be a CAG such that there exists an
$n$-element subset $E = \{\xi_1, \ldots, \xi_n\}$ of $\hat{K}$ for which $\Lambda_K(E) \leq \kappa$. Suppose $\nu(\text{supp} F) = r \leq n$ and enumerate $\text{supp} F$ as $\{x_1, \ldots, x_r\}$. Then, for every $x \in G$, we have

$$\left\{ \sum_{j=1}^{r} |F(x_j)|^2 \right\}^{\frac{1}{2}} \leq \kappa \int_K \left\{ \sum_{j=1}^{r} F(x_j) \xi_j(x) \lambda_j(y) \right\} d\lambda_K(y).$$

Integrating over $G$ and using Fubini's Theorem, this gives

$$\left\{ \sum_{j=1}^{r} |F(x_j)|^2 \right\}^{\frac{1}{2}} \leq \kappa \left\| \sum_{j=1}^{r} F(x_j) \xi_j(y_0) x_j \right\|_{L^1(G)} \int d\lambda_K(y),$$

which shows that

$$\left\{ \sum_{j=1}^{r} |F(x_j)|^2 \right\}^{\frac{1}{2}} \leq \kappa \left\| \sum_{j=1}^{r} F(x_j) \xi_j(y_0) x_j \right\|_{L^1(G)}$$

for some $y_0 \in K$. Using Lemma 1.2, it follows that

$$\left\{ \sum_{j=1}^{r} |F(x_j)|^2 \right\}^{\frac{1}{2}} \leq \kappa \sup \left\{ \left\| \sum_{j=1}^{r} F(x_j) \xi_j(y_0) f(x_j) \right\|_{L^1(G)} : f \in C(G), \|f\|_u \leq 1 \right\}$$

$$\leq \kappa \sup \left\{ \left\| \sum_{j=1}^{r} F(x_j) f(x_j) \right\|_{L^1(G)} : f \in C(G), \|f\|_u \leq 1 \right\}.$$

Since this is true for every $k \in S_n$, it remains true with $h_n$ in place of $\kappa$. Thus, (b) of 1.1 is true with $M = h_n$ for the stated functions $F$.

**COROLLARY 1.8.** We have $M_G \leq h_n$ for every CAG $G$.

**Proof.** By Lemma 1.6 and Theorem 1.7, (b) of 1.1 holds with $M = h$ for every $f \in C(G)$ having a finite support. But then (b) holds with $M = h$ for every $F \in C(G)$, and so $M_G \leq h$.

**REMARK.** From Theorem 1.7 it follows that, if $G$ is of finite order $n$, then $M_G \leq h_n$ which, by Corollary 2.4, is at most $(2-1/n)^{\frac{1}{2}} < 2^{\frac{1}{2}}$.

Thus Helgason's inequality (that is, 1.1 (a) with $M = 2^{\frac{1}{2}}$) is not best possible when only groups of given finite order $n$ are considered. In
addition it can be shown that, if $G$ is the subgroup \{-1, 1\} of $T$, then $M_G = 1$ whereas (by Theorem 2.10) $h_G = \pi \sqrt{2}/4 > 1$.

**Theorem 1.9.** Suppose that $G$ is a CAG, that $E$ is a Sidon subset of $\hat{G}$, and that $S_G(E)$ is the Sidon constant of $E$, that is, the smallest nonnegative real number $\kappa$ for which

$$\|f\|_1 \leq \kappa \|f\|_u$$

for every $f \in T_E(G)$. Then

$$\Lambda_G(E) \leq M_G S_G(E).$$

**Proof.** Let $f \in T_E(G)$. Using (b) of 1.1 with $M = M_G$, we have

$$\|f\|_2 = \|f^\wedge\|_2 \leq M_G \sup \bigg\{ \sum_{\chi \in G} |f^\wedge(\chi) g^\wedge(\chi)| : g \in \mathcal{C}(G), \|g\|_u \leq 1 \bigg\}$$

$$= M_G \sup \bigg\{ \sum_{\chi \in E} f^\wedge(\chi) \omega(\chi) g^\wedge(\chi) : g \in \mathcal{C}(G), \|g\|_u \leq 1, \omega \in \Omega \bigg\},$$

where $\Omega = \hat{G}$. Writing $\kappa$ for $S_G(E)$, a known property of Sidon sets ([18], (37.2)) asserts that every $\omega \in \Omega$ agrees on $E$ with $\omega^\wedge$ for some $\mu_\omega \in \mathcal{M}(G)$ satisfying $\|\mu_\omega\| \leq \kappa$. It follows that

$$\|f\|_2 \leq M_G \kappa \sup \bigg\{ \sum_{\chi \in G} f^\wedge(\chi) \kappa^\wedge(\chi) : k \in \mathcal{C}(G), \|k\|_u \leq \kappa \bigg\}$$

$$= M_G \kappa \|f\|_1,$$

the last step by Lemma 1.2. Thus $\Lambda_G(E) \leq M_G \kappa$.

**Corollary 1.10.** Let $G$ be a CAG. Then

$$(1) \quad M_G \leq h \leq M_G \lim_{n \to \infty} \inf \inf \{S_G(E) : E \subseteq \hat{G}, \chi(E) = n\}.$$ 

(The infimum of the empty set is understood to be $\infty$.)

**Proof.** The first inequality in (1) is just Corollary 1.8. For the rest, let $t_n$ denote the infimum appearing in (1), which we may assume to
be finite. If \( E \subseteq \hat{G} \) and \( \nu(E) = n \), then \( \Lambda_E(\hat{G}) \in S_n \) and so
\[ \Lambda_E(\hat{G}) \geq h_{\pi(n)} \]. By Theorem 1.9 we therefore have
\[ h_{\pi(n)} \leq \Lambda_E(\hat{G}) \leq M_{\hat{G}}(E) \).

From this it follows that
\[ (2) \quad h_{\pi(n)} \leq M_{\hat{G}}(n) \].

The second inequality in (1) follows from (2) and Lemma 1.6.

1.11. If \( G \) is a CAG, a subset \( E \) of \( \hat{G} \) will be termed strongly independent if, whenever \( \chi_1, \ldots, \chi_n \) denote distinct elements of \( E \) and \( m_1, \ldots, m_n \) denote integers, the relation
\[ m_1 \chi_1 \cdots m_n \chi_n = 1 \]
implies that \( m_1 = \ldots = m_n = 0 \). For example, if \( I \) is any set and \( G = T^I \), then the set of projections
\[ \pi_{i_0} : \langle x_i \rangle_{i \in I} \mapsto x_{i_0} \]
with \( i_0 \in I \) is a strongly independent subset of \( \hat{G} \).

We list several properties of strongly independent sets which will be useful in the sequel.

(i) If \( G \) is a CAG and \( E \) a subset of \( \hat{G} \), then \( E \) is strongly independent if and only if the mapping \( \phi : x \mapsto (\chi(x))_{\chi \in E} \) maps \( G \) onto \( T^E \), where \( T \) denotes the circle group.

Proof. The image \( H = \phi(G) \) is a closed subgroup of \( T^E \). If the character group of \( T \) be identified with \( Z \) (the additive group of integers) in the usual fashion, the annihilator \( A \) in \( (T^E)^\ast \) of \( H \) is precisely the set of \( Z \)-valued functions \( \chi \mapsto m(\chi) \) on \( E \) having finite supports and such that
\[ \prod_{\chi \in E} m(\chi) = 1 \).
The strong independence of $E$ is equivalent to the assertion that $A$ is the trivial subgroup of $\prod_{\chi \in E} Z$. Since $H$ is the annihilator in $\prod_{\chi \in E}$ of $A$, this occurs if and only if $H = \prod_{\chi \in E}$.

(ii) If $G$ is a CAG and $E$ a strongly independent subset of $\hat{G}$, then $S_G(E) = 1$.

Proof. This follows at once from (i) and the definition of $S_G(E)$ in 1.9.

(iii) Suppose that $G$ is a CAG and that $E$ is a strongly independent subset of $\hat{G}$. If $\chi_1, \ldots, \chi_n$ are distinct elements of $E$ and $c_1, \ldots, c_n$ are complex numbers, then

$$\int_G \left| \sum_{k=1}^n c_k \chi_k \right| d\lambda_G = \int_G \left| \sum_{k=1}^n |c_k| \chi_k \right| d\lambda_G.$$

Proof. For $k \in \{1, 2, \ldots, n\}$, choose $\omega_k \in T$ such that $c_k = |c_k| \omega_k$. By (i), there exists $a \in G$ such that $\chi_k(a) = \omega_k$ for $k \in \{1, 2, \ldots, n\}$. Then $\sum_{k=1}^n c_k \chi_k$ is the $a$-translate of $\sum_{k=1}^n |c_k| \chi_k$, and the stated equality follows from translation-invariance of $\lambda_G$.

COROLLARY 1.12. Let $G$ be a CAG with the property that, for every positive integer $n$, $\hat{G}$ contains an $n$-element strongly independent set. Then $M_G \in S$ and $h = M_G$.

Proof. For each positive integer $n$, let $I_n$ be an $n$-element strongly independent subset of $\hat{G}$. By 1.11 (ii), we have $S_G(I_n) = 1$ and so, by Theorem 1.9, $\lambda_G(I_n) \leq M_G$. Since this is the case for every positive integer $n$, it follows that $M_G \in S$. This entails that $h \leq M_G$ and the rest ensues from Corollary 1.8.

REMARK. From Corollaries 1.8 and 1.12 it follows that $h$ is the maximum of the numbers $M_G$ when $G$ ranges over the class of CAGs.
1.13. We insert here some remarks about the effect of continuous group homomorphisms.

Let \( G \) and \( K \) be CAGs and suppose that \( \phi \) is a continuous homomorphism of \( G \) onto \( K \). Write \( \phi^* \) for the dual isomorphism of \( \hat{K} \) into \( \hat{G} \) defined by \( \phi^*(\zeta) = \zeta \circ \phi \) for \( \zeta \in \hat{K} \), and let \( \hat{\phi} \) denote the mapping \( f \mapsto f \circ \phi \) of \( C(K) \) into \( C(G) \). In what follows, \( E \) denotes a subset of \( \hat{K} \) and \( F = \phi^*(E) \subseteq \hat{G} \). It is plain that

(1) \( \hat{\phi} \) preserves uniform norms

and that

(2) \( \hat{\phi} \) maps \( C_E(K) \) onto \( C_F(G) \).

(\( C_E(K) \) denotes the set of \( g \in C(K) \) such that \( g^*(\zeta) = 0 \) for \( \zeta \in \hat{K}\backslash E \), and \( C_F(G) \) is defined analogously.)

By considering the functional \( f \mapsto \int_G (\phi f) d\lambda_G \) and invoking the uniqueness of normalised Haar measure on \( K \), we infer that

(3) \( \int_G (f \circ \phi) d\lambda_G = \int_K f d\lambda_K \)

for every \( f \in C(K) \).

From (3) we may infer first that

(4) \( \hat{\phi} \) preserves \( L^p \)-norms \( (0 < p < \infty) \)

and second that, if \( \chi \in \hat{G} \) and \( f \in C(K) \), then

(5) \( (f \circ \phi)^*(\chi) = f^*(\phi^* \chi) \) if \( \chi \in \phi^*(\hat{K}) \) and 0 otherwise.

In particular,

(6) \( \| (f \circ \phi)^* \|_1 = \| f^* \|_1 \).

In view of (4) and (2), it follows that the \( \Lambda_2 \)-constant of \( F \) is equal to the \( \Lambda_2 \)-constant of \( E \). Similarly, from (1), (2) and (6) it appears that the Sidon constant of \( F \) is equal to the Sidon constant of \( E \).
From (3) it follows also that

\[(f \circ \phi) \ast (g \circ \phi) = (f \ast g) \circ \phi\]

for \(f\) and \(g\) in \(C(K)\). If \(\phi\) is an isomorphism (which occurs if and only if \(\phi^*\) maps \(\hat{K}\) onto \(\hat{G}\), that is, if and only if \(\phi\) maps \(C(K)\) onto \(C(G)\)), we infer from (7) and reference to 1.1 (a) that \(M_\phi = M_\phi^*\).

We end this section by recording another property of the number \(M_\phi\) for a given \(G\).

**Lemma 1.14.** Suppose that \(G\) is a CAG, that \(1 \leq p \leq 2\), and that \(q = 2p/(2-p)\). For \(F \in C(G)\) we have

\[\|F\|_q = \sup\{\|F\|_p : \|\phi\|_p = 1\}\].

**Proof.** We have

\[\sup\{\|F\|_2^2 : \|\phi\|_p = 1\} = \sup\{\|F\phi\|_2^2 : \|\phi\|_p = 1\}\].

Now \(\|\phi\|_p = 1\) if and only if \(\|\phi\|_{2p'} = 1\); and every nonnegative \(\psi\) satisfying \(\|\psi\|_{2p'} = 1\) has the form \(\phi^2\) for some \(\phi\) satisfying \(\|\phi\|_{p'} = 1\). So the above supremum equals

\[\sup\{\|F\phi\|_2^2 : \|\psi\|_{2p'} = 1\} = \|F\phi\|_2^2\].

Since \((2p')' = p/(2-p) = \frac{2}{q}\), the supremum equals

\[\|F\phi\|_{\frac{2}{q}} = \|F\|_q\].

**Theorem 1.15.** Let \(G\) be a CAG, \(1 \leq p \leq 2\) and \(q = 2p/(2-p)\). Then

\[\|F\|_q \leq M_\phi \sup\{\|F\phi\|_p : f \in C(G), \|f\|_u \leq 1\}\]

for every \(F \in C(G)\). If \(F \in C(G)\) and \(F\phi \in l^p(\hat{G})\) for every \(f \in C(G)\), then \(F \in l^q(\hat{G})\). (Cf. [3], Corollary (2.3).)

**Proof.** By Lemma 1.14 and (b) of 1.1, we have
Lacunarity constants

\[ \|F\|_q = \sup \{ \|F\phi\|_2 : \|\phi\|_p, = 1 \} \]
\[ \leq M_G \sup \sup \|F\phi\|_1 \]
\[ = M_G \sup \sup \|F\phi\|_1 \]
\[ = M_G \sup \|F\phi\|_1 . \]

The rest follows from the closed graph theorem.

2. Estimates for \( h_n \) and \( h \)

**Theorem 2.1.** Let \( n \) be a positive integer, \( K \) any CAG and \( I \) any \( n \)-element strongly independent subset of \( \hat{K} \). Let \( G \) be any CAG and \( E \) a subset of \( \hat{G} \) having at least \( n \) elements. Then

\[ \Lambda_K(I) \leq \Lambda_G(E) . \]

Proof. Enumerate \( I \) as \( \{\xi_1, \ldots, \xi_n\} \) and choose \( n \) distinct elements \( \chi_1, \ldots, \chi_n \) of \( E \). Any \( f \in \mathcal{F}_I(K) \) can be written

\[ f = \sum_{k=1}^n c_k \xi_k , \]

the \( c_k \) being complex numbers. For \( y \in K \) let

\[ f_y : x \mapsto \sum_{k=1}^n c_k \xi_k(y) \chi_k(x) , \]

so that \( f_y \in \mathcal{F}_E(G) \). Then

\[ \|f\|_2 = \left( \sum_{k=1}^n |c_k|^2 \right)^{1/2} = \|f_y\|_2 \leq \Lambda_G(E) \|f_y\|_1 \]
\[ = \Lambda_G(E) \int_G \left| \sum_{k=1}^n c_k \xi_k(y) \chi_k(x) \right| d\lambda_G(x) , \]

and so also (using Fubini's Theorem)

\[ \|f\|_2 \leq \Lambda_G(E) \int_G \left( \sum_{k=1}^n |c_k \xi_k(y) \chi_k(x)| \right) d\lambda_G(y) d\lambda_G(x) . \]
By 1.11 (iii), the inner integral is equal to
\[
\int_K \left| \sum_{k=1}^n \alpha_k r_k(y) \right| d\lambda_K(y) = \|f\|_1,
\]
which is independent of \( x \in G \). Thus
\[
\|f\|_2 \leq \Lambda_G(E)\|f\|_1,
\]
showing that \( \Lambda_K(I) \leq \Lambda_G(E) \).

COROLLARY 2.2. Let \( K \) and \( I \) be as in Theorem 2.1. Then
\[
h = \min \{ \Lambda_G(E) : G a CAG, E \subseteq \hat{G}, \nu(E) = n \} = \Lambda_K(I).
\]
Proof. Let
\[
c = \inf \{ \Lambda_G(E) : G a CAG, E \subseteq \hat{G}, \nu(E) = n \}.
\]
The definitions in 1.5 show that \( c = h_n \). On the other hand, Theorem 2.1 shows that \( c \) is an assumed minimum equal to \( \Lambda_K(I) \).

REMARK 2.3. Corollary 2.2 shows that \( h_n \) can be computed in terms of \( \Lambda_2 \)-constants of \( n \)-element strongly independent sets of characters. Although there are no nontrivial independent subsets of \( \hat{T} \), Theorem 3.5 below shows that \( h_n \) can nevertheless be given in terms of \( \Lambda_2 \)-constants of \( n \)-element subsets of \( \hat{T} \).

COROLLARY 2.4. We have \( h_n \leq (2-1/n)^{1/2} \).
Proof. In view of Corollary 2.2, it suffices to show that
\[
\Lambda_K(P) \leq (2-1/n)^{1/2},
\]
where \( K = T^P \) and \( P = \{ \pi_1, \ldots, \pi_n \} \) is the set of all projections of \( K \). There exists \( f \in T_P(K) \) such that \( \|f\|_1 = 1 \) and
\[
\Lambda_K(P) = \|f\|_2.
\]
Write
Lacunarity constants

\[ f = \sum_{k=1}^{n} c_k^1, \]

where the \( c_k \) are certain complex numbers. Then

\[
\int_K |f|^2 d\lambda_K = \sum_{j,k=1}^{n} c_k \overline{c_j} \sum_{l,m=1}^{n} \int_K n_j n_k n_l n_m d\lambda_K,
\]

the integrals remaining being equal to 1 or 0 according as the integrand is or is not the character 1 of \( K \). It follows that

\[ \int_K |f|^2 d\lambda_K = \sum_{k=1}^{n} |c_k|^4 + 2 \sum_{j,k=1, j \neq k}^{n} |c_j|^2 |c_k|^2. \]

On the other hand,

\[ \left( \int_K |f|^2 d\lambda_K \right)^2 = \left( \sum_{k=1}^{n} |c_k|^2 \right)^2 = \sum_{k=1}^{n} |c_k|^4 + \sum_{j,k=1, j \neq k}^{n} |c_j|^2 |c_k|^2. \]

Write \( |c_k|^2 = A_{k-1} \) for \( k \in \{1, 2, \ldots, n\} \). We claim that

\[ \sum_{r=0}^{n-1} A_r^2 + 2 \sum_{r,s=0, r \neq s}^{n-1} A_r A_s \leq (2-1/n) \left( \sum_{r=0}^{n-1} A_r^2 + \sum_{r=0, s=0, r \neq s}^{n-1} A_r A_s \right), \]

that is, that

\[ \sum_{r,s=0, r \neq s}^{n-1} A_r A_s \leq (n-1) \sum_{r=0}^{n-1} A_r^2. \]

In fact, define \( \rho : \mathbb{Z} \to \{0, 1, \ldots, n-1\} \) by

\[ t = qn + \rho(t), \]

where \( q \in \mathbb{Z} \). Then

\[ \sum_{r,s=0, r \neq s}^{n-1} A_r A_s = \sum_{r=0}^{n-1} \sum_{s=0, s \neq r}^{n-1} A_r A_s \]

which, since \( m \mapsto \rho(m) \) maps \( \{1, 2, \ldots, n-1\} \) one-to-one onto \( \{0, 1, \ldots, n-1\} \setminus \{r\} \), equals
Since \( r \mapsto \rho(r+m) \) maps \( \{0, 1, \ldots, n-1\} \) one-to-one onto itself, this equals

\[
\sum_{m=1}^{n-1} \left( \sum_{r=0}^{m-1} A_r^2 \right) = (n-1) \sum_{r=0}^{n-1} A_r^2,
\]

which verifies (4). Collecting (2), (3) and (4), we see that

\[
\|f\|_4^4 \leq (2-1/n) \|f\|_2^4,
\]

and hence

\[
\|f\|_4 \leq (2-1/n)^{1/4} \|f\|_2.
\]

From (5) and Hölder's inequality it follows that

\[
\|f\|_2 \leq (2-1/n)^{\frac{1}{2}} \|f\|_1 = (2-1/n)^{1/2},
\]

and the proof is completed by reference to (1).

**COROLLARY 2.5.** We have \( \mathfrak{h} \leq \sqrt{2} \).

Proof. Lemma 1.6 and Corollary 2.4.

Corollaries 1.8 and 2.5 provide an alternative proof of Helgason's version of 1.1 (a).

**COROLLARY 2.6.** Let \( K \) be a CAG such that \( \hat{K} \) contains an infinite strongly independent set \( I \). Then

\[
\mathfrak{h} = \min \{ \Lambda_0(E) : G \text{ a CAG, } E \subseteq \hat{G}, E \text{ infinite} \} = \Lambda_K(I).
\]

In particular,

\[
\mathfrak{h} = \Lambda_{T^\infty}(\{\pi_1, \pi_2, \ldots\}),
\]

where \( T^\infty = T^N \) with \( N = \{1, 2, \ldots\} \) and \( \pi_n \) is the \( n \)-th projection of \( T^\infty \).
Proof. Let $G$ be a CAG and $E$ an infinite subset of $G$. Let $F$ be any finite subset of $I$. By Theorem 2.1 and Corollary 2.2, we have

$$\Lambda_G(E) \geq \Lambda_K(F) = h_n$$

where $n = v(F)$.

Hence $\Lambda_G(E) \geq h_n$ for all $n$ and so, by Lemma 1.6, $\Lambda_G(E) \geq h$. Using (1) and Lemma 1.6, we also have

$$\Lambda_K(I) = \sup\{\Lambda_K(F) : F \subseteq I, F \text{ finite}\}$$

$$= \sup\{h_n : n = 1, 2, \ldots\} = h,$$

and this completes the proof.

**COROLLARY 2.7.** If $n$ is a positive integer, then

$$h_n = \sup\left\{\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} / e_n(a_1, \ldots, a_n)\right\},$$

where

$$e_n(a_1, \ldots, a_n) = (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left|a_1 e_{1} + \cdots + a_n e_{n}\right| \, d\theta_1 \cdots d\theta_n$$

and the supremum is taken over all nonnegative numbers $a_1, \ldots, a_n$, not all zero.

Proof. Applying Corollary 2.2 with $K = T^n$ and $I$ the set of all projections of $T^n$, we see that

$$h_n = \sup\left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} / \left(\sum_{k=1}^{n} a_k^\pi k\right)^{1/2},$$

the $a_k$ being complex and not all zero. By 1.11 (iii),

$$\left\|\sum_{k=1}^{n} a_k^\pi k\right\|_1 = \left\|\sum_{k=1}^{n} |a_k|^\pi k\right\|_1,$$

and so we may assume all the $a_k$ to be real and nonnegative. Finally, since $\lambda_T^\pi = \lambda_T \otimes \cdots \otimes \lambda_T$ ($n$ factors),
\[ \left\| \sum_{k=1}^{n} c_k \pi_k \right\|_1 = E_n(c_1, \ldots, c_n) . \]

**Remark 2.8.** Corollary 2.7 indicates connections between the numbers \( h_n \) and the so-called Pearson random walk ([10], pp. 419-421; [9], pp. 496-500; [1], pp. 10-13), wherein the walker begins at the origin and walks in the plane for a distance \( c_1 \) at random angle \( \theta_1 \), then proceeds for a distance \( c_2 \) at a random angle \( \theta_2 \), and so on. The integral \( E_n(c_1, \ldots, c_n) \) plainly denotes the expected distance of the walker from the origin after completing the first \( n \) steps. A search of the literature indicates that the numbers \( E_n(c_1, \ldots, c_n) \) have not yet been computed or estimated by machine.

**Lemma 2.9.** (i) Let \( G \) be a CAG and let \( x_1 \) and \( x_2 \) be elements of \( \hat{G} \) such that \( \phi = x_1 x_2^{-1} \) is of infinite order. Then

\[ \Lambda_G(\{x_1, x_2\}) = \frac{\pi}{2} / B \]

(ii) If \( G \) is a connected CAG, then \( \Lambda_G(E) = \frac{\pi}{2} / B \) for every two-element subset \( E \) of \( \hat{G} \).

**Proof.** (i) Let \( E = \{x_1, x_2\} \). We need to show that the maximum of \( \|g\|_2 / \|g\|_1 \), for \( g = c_1 x_1 + c_2 x_2 \) subject to \( (c_1, c_2) \neq (0, 0) \), is \( \pi / 2 \). In doing this we may plainly assume that \( |c_1| \leq |c_2| = 1 \) and also that \( c_2 = 1 \). Let \( r = |c_1| \) and select \( w \in T \) so that \( wr = c_1 \).

Then we have

\[ \|g\|_2 = \left(1 + r^2\right)^{1/2} \].

The character \( \phi \) is of infinite order if and only if \( \{\phi\} \) is strongly independent, and (by 1.11 (i)) this is so if and only if \( \phi(G) = T \). Also we have \( g = (f \circ \phi)x_2 \) where \( f(z) = c_1 z + 1 \) for \( z \in T \). Hence by 1.13 (3), we have
\[ \|g\|_1 = \|f^g\|_1 = \int_T |f(z)| d\lambda_T(z) = \int_T |z+1| d\lambda_T(z) \]
\[ = \int_T |rz+1| d\lambda_T(z) \]
\[ = \int_T |rz+1| d\lambda_T(z) \quad \text{(by invariance of } \lambda_T ) \]
\[ = (2\pi)^{-1} \int_0^\pi |re^{i\theta}+1| d\theta \]
\[ = \pi^{-1} \int_0^\pi (1+r^2+2r\cos\theta)^{\frac{1}{2}} d\theta . \]

Thus we have to show that the maximum of
\[ (1+r^2)^{\frac{1}{2}} \pi^{-1} \int_0^\pi (1+r^2+2r\cos\theta)^{\frac{1}{2}} d\theta , \]
subject to \( 0 \leq r \leq 1 \), is \( \pi\sqrt{2}/4 \), that is, that the minimum of
\[ (1+r^2)^{-\frac{1}{2}} \int_0^\pi (1+r^2+2r\cos\theta)^{\frac{1}{2}} d\theta , \]
subject to \( 0 \leq r \leq 1 \), is \( 2\sqrt{2} \). On putting \( a = (1+r^2)^{-1}2r \), it comes to the same thing to show that the minimum of
\[ I(a) = \int_0^\pi (1+a\cos\theta)^{\frac{1}{2}} d\theta , \]
subject to \( 0 \leq a \leq 1 \), is \( 2\sqrt{2} \). Now
\[ I(1) = \int_0^\pi (1+\cos\theta)^{\frac{1}{2}} d\theta = \sqrt{2} \int_0^\pi \cos^\frac{1}{2}\theta d\theta = 2\sqrt{2} \int_0^{\frac{\pi}{2}} \cos^a d\theta = 2\sqrt{2} , \]
and so it will suffice to show that \( I'(a) \leq 0 \) for \( 0 < a < 1 \). But
\[ I'(a) = \frac{1}{2} \int_0^\pi \cos\theta(1+a\cos\theta)^{-\frac{1}{2}} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos\theta(1+a\cos\theta)^{-\frac{1}{2}} d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos\phi(1-a\cos\phi)^{-\frac{1}{2}} d\phi \]
\[ = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos\theta[(1+a\cos\theta)^{-\frac{1}{2}}-(1-a\cos\theta)^{-\frac{1}{2}}] d\theta , \]
which is nonpositive since the integrand is nonpositive throughout the range of integration.

(ii) This statement follows from (i) because, if \( G \) is connected, \( \phi(G) \) is a closed connected subgroup of \( T \) and so coincides with \( T \) if and only if it has at least two elements, that is, if and only if \( \phi \) is not the constant character 1.

**Theorem 2.10.** We have \( h_2 = \pi \sqrt{2}/4 \).

**Proof.** By Lemma 2.9, we have

\[
\lambda_{T^2}(P) = \pi \sqrt{2}/4
\]

where \( P = \{\pi_1, \pi_2\} \) is the set of projections of \( T^2 \). Now apply Corollary 2.2.

**Remark.** It is evident from Lemma 1.6 and Theorem 2.10 that

\[
h \geq h_2 = \sqrt{2}/4 = 1.1107 \ldots .
\]

Here is a slight improvement on this estimate.

**Theorem 2.11.** We have

\[
h \geq 2\pi^{-\frac{k}{2}} = 1.1284 \ldots .
\]

**Proof.** Our aim is to apply the two-dimensional central limit theorem; see, for example, [4], Section VIII.4, Theorem 2. The underlying probability space will be \((S, m)\), where \( S = T^N \), \( N = \{1, 2, \ldots \} \) and \( m \) is normalised Haar measure on \( S \). As before, if \( k \in N \), \( \pi_k \) denotes the \( k \)-th projection of \( T^N \). Let

\[
X_k = (\text{Re} \pi_k, \text{Im} \pi_k) = \left( x_k^{(1)}, x_k^{(2)} \right).
\]

Then \( X_1, X_2, \ldots \) are mutually independent two-dimensional real random variables with a common distribution. Moreover, \( E\left(x_k^{(\alpha)}\right) = 0 \) for all \( k \in N \) and \( \alpha \in \{1, 2\} \) and the common covariance matrix

\[
\begin{bmatrix}
E[X_k^{(\alpha)}X_k^{(\beta)}]
\end{bmatrix}_{\alpha,\beta=1,2}
\]

is equal to
where $\sigma_1^2 = \sigma_2^2 = \frac{1}{2}$ and $\rho = 0$. Consequently the central limit theorem asserts that the distributions $V_n$ of the random variables

$$S_n = n^{-\frac{1}{2}}(X_1 + \ldots + X_n)$$

converge (weakly) to the distribution

$$V = g\lambda_{R^2},$$

where $\lambda_{R^2}$ denotes Lebesgue measure on $R^2$ and

$$g(x_1, x_2) = \pi^{-1}\exp\left(-\left(x_1^2 + x_2^2\right)\right).$$

We now show that

\[ (1) \quad \int_{R^2} |x|^2 dV_n(x) = 1 \quad \text{for all } n, \]

where $|x| = |(x_1, x_2)| = \left(\frac{x_1^2 + x_2^2}{2}\right)^{\frac{1}{2}}$ for $x \in R^2$. In fact, by definition of $V_n$ we have

\[ \int_{R^2} |x|^2 dV_n(x) = \int_S |S_n|^2 dm = n^{-1} \int_S |X_1 + \ldots + X_n|^2 dm \]

\[ = n^{-1} \int_{T^N} \left(\sum_{k=1}^n \text{Re} \pi_k\right)^2 + \left(\sum_{k=1}^n \text{Im} \pi_k\right)^2 dm. \]

We also find that

\[ (2) \quad \int_{T^N} \left(\sum_{k=1}^n \text{Re} \pi_k\right)^2 dm = n \int_{T^N} (\text{Re} \pi_k)^2 dm \]

\[ = \sum_{k=1}^n (2\pi)^{-1} \int_{-\pi}^\pi \cos^2 \theta d\theta = \frac{1}{2} n, \]

and similarly
Equalities (2) and (3) lead directly to (1).

From (1) and Lemma 2.12 proved below (with $F(x) = |x|^2 + 1$ and $f(x) = |x|$), we obtain

$$\lim_{n \to \infty} \int_{R^2} |x| d\nu_n(x) = \int_{R^2} |x| d\nu(x) ,$$

and hence

$$\lim_{n \to \infty} \int_{T^N} n^{-k} |X_1 + \ldots + X_n| dm = \int_{R^2} |x| d\lambda R^2(x) .$$

In the notation introduced in Corollary 2.7, the left hand side of (4) is equal to

$$\lim_{n \to \infty} n^{-k} E_n(1, \ldots, 1) ,$$

while the right hand side of (4) is equal to

$$\pi^{-1} \int_0^{2\pi} \int_0^\infty re^{-r^2} r dr d\theta = 2 \int_0^\infty r^2 e^{-r^2} dr = \int_0^\infty e^{-s} s^{1/2} ds = \Gamma(3/2) = \frac{\sqrt{\pi}}{2} ,$$

so that

$$\lim_{n \to \infty} n^{-k} E_n(1, \ldots, 1) = \frac{\sqrt{\pi}}{2} .$$

Hence, by Lemma 1.6 and Corollary 2.7,

$$h = \sup_n h_n \geq \lim_{n \to \infty} n^{-k} E_n(1, \ldots, 1) = 2\pi^{-k} .$$

REMARK. It seems quite possible that the supremum appearing in Corollary 2.7 is attained when all the $c_k$ are equal, that is, that

$$h_n = n^{-k} E_n(1, \ldots, 1) .$$

If this is so, 2.11 (5) and Lemma 1.6 imply that $h = 2\pi^{-k}$. Note that
Corollary 2.7 and examination of the proof of Lemma 2.9 confirm that
\[ h_n = 2^k/E \left( 1, 1 \right) . \]

**Lemma 2.12.** Let \( \mu \) and \( \mu_n \ (n = 1, 2, \ldots) \) be positive Radon
measures on \( \mathbb{R}^m \). Let \( F \) be a positive continuous function and \( f \) a
complex-valued continuous function on \( \mathbb{R}^m \). Suppose that

(i) \( \mu_n + \mu \) weakly in the dual of \( C_{00}(\mathbb{R}^m) \);

(ii) \( M = \sup \int F d\mu_n < \infty \);

(iii) \( \lim_{|x| \to \infty} \frac{|f(x)|}{F(x)} = 0 \).

Then

(iv) \( \sup \int |f| d\mu_n < \infty \);

(v) \( \int F d\mu \leq M \), \( |f| d\mu < \infty \);

(vi) \( \lim_{n \to \infty} \int f d\mu_n = \int f d\mu \).

**Proof.** By (iii), there is a nonnegative number \( C \) such that
\[ |f| \leq CF \]
and hence (iv) follows from (ii). For the rest of the proof we may assume
without loss of generality that \( f \) is real-valued and nonnegative. Let
\[ (f_k)_{k=1}^{\infty} \]
be an increasing sequence of functions in \( C_{00}(\mathbb{R}^m) \) such that

(2) \[ 0 \leq f_k \leq 1 , \ f_k(x) = 1 \text{ for } |x| \leq k . \]

By (i), (2) and (ii) we have
\[ \int f_k F d\mu = \lim_{n \to \infty} \int f_k F d\mu_n \leq \liminf_{n \to \infty} \int F d\mu_n \leq M \]
for every \( k \) and so monotone convergence shows that

(3) \[ \int F d\mu \leq M . \]
Now (1) and (3) entail \( \int fd\mu < \infty \). Thus (v) is true. Next, if we define
\[
e_k = \sup\{f(x)/P(x) : |x| \geq k\},
\]
(iii) shows that
\[
(4) \quad \lim_{k \to \infty} e_k = 0
\]
and (2) shows that \((1-f_k)^f \leq e_k f^\infty \). Thus
\[
f_k f^\infty \leq f = f_k f^\infty + (1-f_k)^f \leq f_k f^\infty + e_k f^\infty,
\]
and (ii) implies that
\[
\int f_k f d\mu_n \leq \int f d\mu_n \leq \int f_k f d\mu_n + e_k \int P d\mu_n \\
\leq \int f_k f d\mu_n + e_k P_n.
\]
Letting \( n \to \infty \), it follows from (i) that
\[
\int f_k f d\mu \leq \liminf_{n \to \infty} \int f d\mu_n \leq \limsup_{n \to \infty} \int f d\mu_n \leq \int f_k f d\mu + e_k P_n.
\]
Now we let \( k \to \infty \) and use (4) and monotone convergence to conclude that
\[
\int f d\mu \leq \liminf_{n \to \infty} \int f d\mu_n \leq \limsup_{n \to \infty} \int f d\mu_n \leq \int f d\mu,
\]
which completes the proof.

2.13. We consider briefly the "change-of-arguments" operators \( T_\omega \) introduced in [3]. This will lead to a slight improvement of Helgason's inequality 1.1 (a) and an alternative characterisation of \( \Omega \).

Let \( G \) be a CAG and write \( \Omega \) for \( \hat{\Omega} \). (The present \( \Omega \) is denoted by \( \Omega^4 \) in [3].) For \( \chi \in \hat{G} \), \( \pi_\chi \) denotes the \( \chi \)-th projection on \( \Omega \), so that \( \pi_\chi(\omega) = \omega(\chi) \) for every \( \omega \in \Omega \).

For \( \omega \in \Omega \), \( T_\omega \) denotes the unitary endomorphism of \( L^2(G) \) defined by
\[
T_\omega f = \sum_{\chi \in \hat{G}} \omega(\chi) f^\chi(\chi) \chi.
\]
THEOREM 2.14. Let $G$ be a CAG and let the notation be as in 2.13.

(i) We have

\[ \|T_{w_0} f\|_1 \leq \|f\|_2 \leq \frac{1}{h} \int_{\Omega} \|T_{w} f\|_1 d\lambda_{\Omega}(w) \]

for $w_0 \in \Omega$ and $f \in L^2(G)$.

(ii) If $E$ is an infinite subset of $\hat{G}$ and $k$ a real number such that

\[ \|f\|_2 \leq k \int_{\Omega} \|T_{w} f\|_1 d\lambda_{\Omega}(w) \]

for every $f \in \mathcal{M}(G)$, then $k \geq \frac{h}{k}$.

Proof. (i) The first inequality is trivial, since

\[ \|T_{w_0} f\|_1 \leq \|T_{w_0} f\|_2 = \|f\|_2 . \]

For the rest, it is sufficient to deal with the case in which $f \in \mathcal{M}(G)$, for then a simple approximation argument extends the inequality to a general element of $L^2(G)$. We then have, by Fubini's Theorem,

\[ \int_{\Omega} \|T_{w} f\|_1 d\lambda_{\Omega}(w) = \int_{\Omega} \left\{ \int_{G} \left| \sum_{\chi \in \hat{G}} \omega(\chi)\varphi(\chi)\chi(x) \right| d\lambda_{\hat{G}}(\chi) \right\} d\lambda_{\Omega}(w) \]

\[ = \int_{G} \left\{ \int_{\Omega} \left| \sum_{\chi \in \hat{G}} \varphi(\chi)\chi(x)\pi_{\chi}(w) \right| d\lambda_{\Omega}(w) \right\} d\lambda_{\hat{G}}(\chi) . \]

By Corollary 2.2 or Corollary 2.6, the $A_2$-constant of the set of all projections $\pi_{\chi}$ of $\Omega$ is at most $\frac{1}{h}$, so that the last-written inner integral is not less than

\[ \frac{1}{h} \left( \int_{\chi \in \hat{G}} |\varphi(\chi)|^2 \right)^{\frac{1}{2}} = \frac{1}{h} \|f\|_2 , \]

and the second inequality in (i) follows.

(ii) By Corollary 2.6, applied with $\Omega$ in place of $G$ and $E = \{\pi_{\chi} : \chi \in E\}$ in place of $I$, it suffices to show that

\[ A_{\Omega}(E) \leq k . \]
This in turn will follow, if it be shown that
\[(3) \quad \Lambda_\Omega \left\{ \pi_{X_1}, \ldots, \pi_{X_n} \right\} \leq k \]
for arbitrary distinct $X_1, \ldots, X_n \in E$. To this end, let
\[F = \sum_{j=1}^{n} \sigma_j \pi_{X_j} \]
and
\[f = \sum_{j=1}^{n} \sigma_j \chi_j \]
where the $\sigma_j$ are complex numbers. By (2) we have
\[\|F\|_2 = \left( \sum_{j=1}^{n} |\sigma_j|^2 \right)^{1/2} = \|f\|_2 \leq k \int_{\Omega} \|T_\omega f\|_1 d\lambda_\Omega(\omega). \]
Using Fubini's Theorem, this gives
\[\|F\|_2 \leq k \int_{G} \left\{ \int_{\Omega} \left| \sum_{j=1}^{n} \sigma_j \chi_j(x) \pi_{X_j}(\omega) \right| d\lambda_\Omega(\omega) \right\} d\lambda_G(x). \]
By 1.11 (iii), the inner integral here is independent of $x \in G$ and equal to
\[\int_{\Omega} \left| \sum_{j=1}^{n} \sigma_j \pi_{X_j}(\omega) \right| d\lambda_\Omega(\omega) = \|F\|_1. \]
Thus, $\|F\|_2 \leq k\|F\|_1$, which verifies (3) and completes the proof.

**COROLLARY 2.15.** The notation is as in 2.13. Suppose also that $E \subseteq \hat{G}$ and let
\[\kappa = \sup \{ \|T_\omega f\|_1 : f \in T_E(G), \|f\|_1 = 1, \omega \in \Omega \}. \]
Then
\[\kappa \leq \Lambda_G(E) \leq 1 + \kappa. \]
In particular, $E$ is a $\Lambda_2$-set if and only if $\kappa < \infty$.

Proof. The inequality $\kappa \leq \Lambda_G(E)$ follows from the first inequality
in 2.14 (1), since \( \|f\|_2 \leq \Lambda_g(E) \|f\|_1 \) for every \( f \in \mathbb{C}_c(G) \). The inequality \( \Lambda_g(E) \leq \|X\| \) follows from the second inequality in 2.14 (1).

3. \( h \) in terms of subsets of \( \hat{T} \)

In this section we show that each of the numbers \( h_n \) can be given in terms of \( n \)-element subsets of \( \hat{T} \). In view of Lemma 1.6, \( h \) can therefore be given in terms of finite subsets of \( \hat{T} \).

**NOTATION 3.1.** Here we consider the (compact) circle group \( T \); \( n \) will denote a fixed positive integer. For integers \( k \geq 2 \), we write \( E_n,k \) for the set of characters \( z \mapsto z^{j-1} \) of \( T \) corresponding to \( j \in \{1, 2, \ldots, n\} \). In Theorem 3.5, we will prove that

\[
\text{h}_n = \lim_{k \to \infty} \Lambda_T(E_n,k).
\]

For each \( k \), \( \phi_k \) will denote the mapping of \( T \) into \( T^n \) defined by

\[
\phi_k(z) = \left\{ z, z^k, z^{k^2}, \ldots, z^{k^n-1} \right\};
\]

and \( H_k \) will denote the image \( \phi_k(T) \) of \( T \). It is evident that \( \phi_k \) is a topological isomorphism of \( T \) onto \( H_k \).

**DEFINITION 3.2.** Let \( H \) denote the set of all closed subgroups of the compact group \( G \). We endow \( H \) with the topology for which an open basis consists of sets of the form

\[
U(K; U_1, \ldots, U_m) = \{ H \in H : H \cap K = \emptyset \text{ and } H \cap U_j \neq \emptyset \text{ for all } j \};
\]

here \( K \) is a compact subset of \( G \) and \( U_1, \ldots, U_m \) are nonvoid open subsets of \( G \). A net \( \{H_\gamma\}_{\gamma \in \Gamma} \) in \( H \) is said to converge in the sense of Hausdorff to \( H_0 \) in \( H \) provided it converges to \( H_0 \) in this topology; in this case we write

\[
\lim_{\gamma} H_\gamma = H_0 \text{ [Hausdorff].}
\]

Since \( G \) belongs to \( U(K; U_1, \ldots, U_m) \) if and only if \( K = \emptyset \), it follows...
that

\[(i) \lim_{\gamma} H_\gamma = G [\text{Hausdorff}]\]

if and only if

\[(ii) \text{ whenever } U_1, \ldots, U_m \text{ are given nonvoid open subsets of } G, \text{ there exists a } \gamma_0 \in \Gamma \text{ such that } \gamma > \gamma_0 \text{ implies } H_\gamma \cap U_j \neq \emptyset \text{ for all } j \in \{1, 2, \ldots, m\}.\]

We need the following lemma due to Fell (see appendix to [6]) and to Bourbaki [2]; see also [5].

**Lemma 3.3.** If \( \{H_\gamma\} \) is a net of closed subgroups of a compact group \( G \), and if

\[(i) \lim_{\gamma} H_\gamma = G [\text{Hausdorff}],\]

then for all \( F \) in \( C(G) \) we have

\[(ii) \int_G Fd\lambda_G = \lim_{\gamma} \int_{H_\gamma} Fd\lambda_{H_\gamma}\]

where \( \lambda_G \) and \( \lambda_{H_\gamma} \) denote normalized Haar measure on \( G \) and \( H_\gamma \), respectively.

**Lemma 3.4.** Let \( \{H_k\}_{k=2}^\infty \) denote the sequence of closed subgroups of \( T^n \) defined in 3.1. Then

\[\lim_{k \to \infty} H_k = T^n [\text{Hausdorff}].\]

**Proof.** We establish some local terminology for this proof. By a \( k^n \)-sector of \( T \) we shall mean a subset of \( T \) of the form

\[\{\exp(2\pi i \theta) : jk^{-r} \leq \theta < (j+1)k^{-r}\}\]

where \( r \) denotes a nonnegative integer and \( j \) any integer. A subset \( E \) of \( T^n \) will be termed \( k \)-dense if for every choice of \( n \) \( k \)-sectors \( S_1, \ldots, S_n \) of \( T \), the set
Lacunarity constants

\[ E \cap \left(S_1 \times S_2 \times \ldots \times S_n\right) \]
is nonvoid. We first prove that

(1) \quad \text{each } H_k \text{ is } k\text{-dense in } T^n.

We begin with an observation. If \( r \) is a nonnegative integer, if \( R \) is a \( k^r \)-sector of \( T \), and if \( S \) is a \( k \)-sector of \( T \), then there is some \( k^{r+1} \)-sector \( R' \subseteq R \) such that \( z \mapsto z^{k^r} \) maps \( R' \) into \( S \). In fact, we can write

\[ R = \left\{ \exp(2\pi i \theta) : mk^{-r} \leq \theta < (m+1)k^{-r} \right\} \]

and

\[ S = \left\{ \exp(2\pi i \theta) : jk^{-1} \leq \theta < (j+1)k^{-1} \right\}, \]

where \( m \in \mathbb{Z} \) and \( j \in \{0, 1, \ldots, k-1\} \), and then set

\[ R' = \left\{ \exp(2\pi i \theta) : (mk+j)k^{-r-1} \leq \theta < (mk+j+1)k^{-r-1} \right\}. \]

Now let \( S_1, \ldots, S_n \) be given \( k \)-sectors of \( T \). The preceding observation allows us to choose by recurrence \( k^r \)-sectors \( R_r \) for \( r \in \{1, 2, \ldots, n\} \) such that \( R_n \subseteq R_{n-1} \subseteq \ldots \subseteq R_1 \) and \( z \mapsto z^{k^{-r}} \) maps \( R_r \) into \( S_r \) for \( r \in \{1, 2, \ldots, n\} \). Select any \( z \) from \( R_n \). Then \( z^{k^{-r}} \) belongs to \( S_r \) for \( r \in \{1, 2, \ldots, n\} \) and so \( \phi_k(z) \) lies in \( S_1 \times S_2 \times \ldots \times S_n \); thus

\[ H_k \cap \left(S_1 \times S_2 \times \ldots \times S_n\right) \neq \emptyset. \]

This proves (1).

To complete the proof of the lemma, we verify 3.2 (ii) in the present setting. So consider nonvoid open subsets \( U_1, \ldots, U_m \) of \( T^n \). A simple argument shows that for each \( j \in \{1, 2, \ldots, m\} \), there is an integer \( k_j \) such that \( E \cap U_j \neq \emptyset \) whenever \( E \) is a subset of \( T^n \) that is \( k \)-dense.
for some \( k \geq k_j \). Thus if \( k \geq \max(k_1, k_2, \ldots, k_m) \), then (1) shows that \( H_k \cap U_j \neq \emptyset \) for all \( j \in \{1, 2, \ldots, m\} \). This verifies 3.2 (ii) and so

\[
\lim_{k \to \infty} H_k = T
\]

in the sense of Hausdorff.

**THEOREM 3.5.** For the sequence \((E_{n,k})_{k=2}^\infty\) of \( n \)-element subsets of \( T \) defined in 3.1, we have

\[
h_n = \lim_{k \to \infty} \Lambda_T(E_{n,k})
\]

Proof. Since \( n \) is fixed throughout the argument, we will write \( E_k \) in place of \( E_{n,k} \). The definition of \( h_n \) in 1.5 shows that \( \Lambda_T(E_k) \geq h_n \) for all \( k \geq 2 \) and so

\[
\lim \inf_{k \to \infty} \Lambda_T(E_k) \geq h_n.
\]

It therefore suffices to prove that

\[
(1) \quad \lim_{k \to \infty} \sup \Lambda_T(E_k) \leq h_n.
\]

Assume that (1) fails. Then there is a subsequence \((k_j)\) of integers and a number \( \kappa > h_n \) so that \( \Lambda_T(E_{k_j}) > \kappa \) for all \( j \). Then for each \( j \) we have

\[
(2) \quad \left( \sum_{j=1}^n |a_j^{(r)}|^2 \right)^{\frac{1}{2}} \geq \kappa \int_T \left( \sum_{j=1}^n |a_j^{(r)}(z)| \right)^{j-1} d\nu(z)
\]

for suitable complex numbers \( a_j^{(r)} \), \( j \in \{1, 2, \ldots, n\} \). We may clearly suppose that

\[
(3) \quad \left( \sum_{j=1}^n |a_j^{(r)}|^2 \right)^{\frac{1}{2}} = 1 \text{ for all } r.
\]

Let \( \Phi_k \) and \( H_k \) be as in 3.1. Since \( \Phi_k \) is a continuous homomorphism of \( T \) onto \( H_k \), 1.13 (3) shows that
Lacunarity constants

(4) \[ \int_{H_k} f d\lambda_{H_k} = \int_T (f \circ \phi_k) d\lambda_T \]
for all functions \( f \) continuous on \( H_k \). Let \( \pi_1, \ldots, \pi_n \) denote the projections of \( T^n \). We apply (4) to the right hand side of (2), taking \( k = k_r \) and \( f = F_r \), where

\[ F_r = \left| \sum_{j=1}^n \sigma_j^{(r)} \right|, \]

and so obtain

(5) \[ \left( \sum_{j=1}^n |\sigma_j^{(r)}|^2 \right)^{k} \geq \kappa \int_{H_{k_r}} F_r d\lambda_r, \]

here we have written \( \lambda_r \) for normalised Haar measure on \( H_{k_r} \). In view of (3), we may suppose (by passing to further subsequences of \( \{k_r\} \) if necessary) that the limits \( \lim_{r \to \infty} \sigma_j^{(r)} \) exist. Let

(6) \[ \sigma_j = \lim_{r \to \infty} \sigma_j^{(r)} \quad \text{for} \quad j \in \{1, 2, \ldots, n\}, \]

and define

\[ F = \left| \sum_{j=1}^n \sigma_j \pi_j \right|. \]

By Lemma 3.4, we have \( \lim_{k \to \infty} H_k = T^n \) in the sense of Hausdorff, and so Lemma 3.3 applies to show that

(7) \[ \int_{T^n} F d\lambda_T = \lim_{r \to \infty} \int_{H_{k_r}} F d\lambda_r. \]

From (6) and (3) it follows that

(8) \[ \lim_{r \to \infty} \sum_{j=1}^n |\sigma_j^{(r)}|^2 = \sum_{j=1}^n |\sigma_j|^2 = 1. \]
From (6) it also follows that $F_r$ converges uniformly to $F$ and so

$$\lim_{r \to \infty} \left| \int_{H_k} F_r d\lambda_r - \int_{H_k} F d\lambda \right| = 0 .$$

Relations (7), (8) and (9) together with (5) yield

$$\kappa \left( \sum_{j=1}^{n} |c_j|^2 \right)^{\frac{1}{2}} \geq \int_{\mathbb{T}^n} \left| \sum_{j=1}^{n} c_j \pi_j \right| d\lambda_n ,$$

that is,

$$\kappa \left( \sum_{j=1}^{n} |c_j|^2 \right)^{\frac{1}{2}} \geq \kappa \int_{\mathbb{T}^n} \left| \sum_{j=1}^{n} c_j \pi_j \right| d\lambda_n .$$

Since (8) shows that both sides of (10) are nonzero, we conclude that

$$\Lambda_{\mathbb{T}^n}(\{\pi_1, \ldots, \pi_n\}) \geq \kappa > h_n ,$$

which contradicts Corollary 2.2.

We end by using the sets $E_{n,k}$ to establish the following interesting extension of Corollary 1.12.

**THEOREM 3.6.** We have $M_T = h_T$.

**Proof.** In view of Corollary 1.10, it is enough to show that the Sidon constant of $E_{n,k}$ is at most $\sec(2\pi/k)$ for $k \geq 5$. To achieve this we will show that, if $a_1, \ldots, a_n$ are arbitrary complex numbers, then

$$\cos(2\pi/k) \sum_{j=1}^{n} |a_j| \leq \sup \left\{ \left| \sum_{j=1}^{n} a_j \omega_j^{n-j} \right| : \omega \in \mathbb{H}_k \right\} ,$$

where $H_k$ is as in 3.1. We will use 3.4 (1) and the terminology introduced thereabouts. For each $j$, $a_j = |a_j| \exp(2\pi i \theta_j)$, where $\theta_j$ belongs to the interval $[m_j k^{-1}, (m_j+1) k^{-1}]$ for some integer.
$m_j \in \{0, 1, \ldots, k-1\}$. Let $S_j$ denote the $k$-sector
\[
\left\{ \exp(2\pi i \theta) : (-m_j-1)k^{-1} \leq \theta < -m_jk^{-1} \right\}.
\]

By 3.4 (1), some $\omega$ in $H_k$ has the property that $\omega_j \in S_j$ for all $j \in \{1, 2, \ldots, n\}$. Then each $\omega_j \exp(2\pi i \theta) \in \{-k^{-1} \leq \theta < k^{-1}\}$.

\[
\text{and so } \Re(\omega_j \cdot a_j) \geq \cos(2\pi/k) |a_j|.
\]

and hence (1) holds.

**Remark 3.7.** It is clear from 3.6 (1) that the Sidon constant of the infinite set of characters $z \mapsto z^{k^{-1}}$ of $T$ corresponding to $j \in \{1, 2, 3, \ldots\}$ is at most $\sec(2\pi/k)$ when $k \geq 5$.

**Corollary 3.8.** Let $G$ be a CAG such that $G$ contains an element $\chi_0$ of infinite order. Let $n$ and $k$ be positive integers and
\[
F_{n,k} = \left\{ \chi_0^{k^{-1}} : j \in \{1, 2, \ldots, n\} \right\}.
\]

Then

(i) $\lim_{k \to \infty} \Lambda_G(F_{n,k}) = \Lambda_{\phi(G)}(E_{n,k})$;

(ii) $h = M_G$.

**Proof.** We apply the substance of 1.13 with $K = T$, $\phi = \chi_0$ and $E = E_{n,k}$; since $\chi_0$ is of infinite order, $\{\chi_0\}$ is strongly independent and $\phi(G) = T$ by 1.11 (i). Then $F_{n,k} = \phi^*E_{n,k}$ and so
\[
S_G(F_{n,k}) = S_T(E_{n,k}) \text{ and } \Lambda_G(F_{n,k}) = \Lambda_T(E_{n,k}).
\]

Statement (i) accordingly follows from Theorem 3.5, while (ii) follows from Corollary 1.10 and the fact (established in the proof of Theorem 3.6) that $S_T(E_{n,k})$ is at most $\sec(2\pi/k)$ for large $k$. 

https://doi.org/10.1017/S0004972700043100 Published online by Cambridge University Press
References


Department of Mathematics, Institute of Advanced Studies, Australian National University, Canberra, ACT; Department of Mathematics, University of Oregon, Eugene, Oregon, USA.