Canad. Math. Bull. Vol. **40** (3), 1997 pp. 266–270

FINITE GROUPS WITH LARGE AUTOMIZERS FOR THEIR ABELIAN SUBGROUPS

H. BECHTELL, M. DEACONESCU AND GH. SILBERBERG

ABSTRACT. This note contains the classification of the finite groups G satisfying the condition $N_G(H)/C_G(H) \cong \mathrm{Aut}(H)$ for every abelian subgroup H of G

1. **Introduction.** The *automizer* of a subgroup H of a group G is $\mathrm{Aut}_G(H) = N_G(H)/C_G(H)$. Since $\mathrm{Aut}_G(H)$ can be regarded as a subgroup of $\mathrm{Aut}(H)$ and $\mathrm{Aut}_G(H)$ contains an isomorphic copy of $\mathrm{In}(H)$, we shall say that $\mathrm{Aut}_G(H)$ is *large* if $\mathrm{Aut}_G(H) \cong \mathrm{Aut}(H)$ and *small* if $\mathrm{Aut}_G(H) \cong \mathrm{In}(H)$.

H. Zassenhaus [7] observed that a finite group G is abelian if and only if $\mathrm{Aut}_G(H)$ is small for all abelian subgroups H of G. Lennox and Wiegold [6] studied groups in which the automizers of all subgroups are large, the so-called MD-groups. They proved—see also Deaconescu [1]—that a finite MD-group is isomorphic to one of the symmetric groups S_n, for $n \leq 3$.

Of interest is the fact that the finite MD-groups are precisely those finite groups in which all elements of the same order are conjugate—see Feit and Seitz [2]. In this paper attention is restricted to *finite* groups G satisfying the weaker condition that $\mathrm{Aut}_G(H)$ is large for all abelian subgroups H of G. Such groups will be referred to as LAAS-groups (Large Automizers for Abelian Subgroups).

By definition, every finite MD-group is an LAAS-group. As the quaternion group Q_8 shows, there exist LAAS-groups which are not MD-groups. Quite surprisingly, the quaternion group distinguishes the two classes. The main result is the following:

THEOREM. *An LAAS-group is isomorphic to either S_n, for $n \leq 3$ or to Q_8.*

2. **Preliminaries.** The following result is essential.

LEMMA 2.1. *Let G be a nontrivial LAAS-group.*

i) Every epimorphic image of G is a rational group.

ii) $|Z(P)| = p$ for every $P \in \mathrm{Syl}_p(G)$ and every $p \in \pi(G)$.

iii) The elements of order p are conjugate in G for every $p \in \pi(G)$.

iv) If $G' \neq G$, then G/G' is an elementary abelian 2-group.

The research of the second author was supported by Grant SM097 of the Kuwait University Research Administration.

Received by the editors February 6, 1996.

AMS subject classification: Primary: 20E34; Secondary: 20D45.

https://doi.org/10.4153/CMB-1997-032-2 Published online by Cambridge University Press

v) If $S \in \mathrm{Syl}_2(G)$ is nonabelian and has a unique involution, then $S \cong Q_8$.

PROOF. i) Let x be an element of order n of G and let $H = \langle x \rangle$. Then $\mathrm{Aut}(H) \cong N_G(H)/C_G(H)$ acts transitively on the set of generators of H. In particular for an integer k and $(k, n) = 1$, there exists an element g such that $x^k = x^g$. The result now follows from Satz V 13.7 of [4].

ii) Since $P \leq C_G\big(Z(P)\big)$ for $P \in \mathrm{Syl}_p(G)$, $\mathrm{Aut}\big(Z(P)\big)$ is a p'-group. Hence $|Z(P)| = p$.

iii) By i), ii) and Sylow's theorem, it suffices to prove that every subgroup U of order p of G is conjugate to $Z(P)$, where P is a fixed Sylow p-subgroup of G. If $U \leq P$, let $M = U \times Z(P)$. Since $\mathrm{Aut}(M) \cong \mathrm{GL}_2(p) \cong N_G(M)/C_G(M)$ acts transitively on the set of subgroups of order p of M, U is conjugate to $Z(P)$. If $U \not\leq P$, then $U \leq P^x$ for some $x \in G$ and the result follows from ii).

iv) This is a consequence of i).

v) The hypothesis implies that S is a generalized quaternion group. Let $|S| = 2^n$ and let M be a cyclic maximal subgroup of S. Then $|M| = 2^{n-1}$, $M = C_S(M)$ and $|\mathrm{Aut}(M)| = 2^{n-2}$. But $SC_G(M)/C_G(M) \cong S/C_S(M) = S/M$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}(M)$. Hence $2^{n-2} = 2$ and $G \cong Q_8$.

The fact that any LAAS-group is a rational group reduces our search.

LEMMA 2.2. *i) If G is a nonabelian simple rational group, then $G \cong \mathrm{Sp}_6(2)$ or $G \cong O_2^+(2)'$.*

ii) If G is a nonabelian composition factor of a rational group, then G is isomorphic either to an alternating group A_n or to one of the following groups: $P\,\mathrm{Sp}_4(3)$, $\mathrm{Sp}_6(2)$, $O_8^+(2)'$, $\mathrm{PSL}_3(4)$ or $\mathrm{PSU}_4(3)$.

PROOF. See Theorem B of Feit and Seitz [2].

The next result will be used in conjunction with Lemma 2.1 iii):

LEMMA 2.3. *Let G be a solvable group and let $p \in \pi(G)$ be odd. If all elements of order p of G are conjugate, then the Sylow p-subgroups of G are abelian.*

PROOF. This is a consequence of a result of Gaschütz and Yen [3]—see also Theorem 8.7, p. 512 of [5].

3. **Proof of the Theorem.** Throughout this section G will denote a nontrivial LAAS-group. The proof is in two parts. In the first part we shall determine all solvable LAAS-groups, while in the second part we shall prove that there are no nonsolvable LAAS-groups.

To begin with assume that G is a nontrivial solvable LAAS-group.

LEMMA 3.1. $\pi(G) \subseteq \{2, 3\}$

PROOF. Suppose that $p \geq 5$ is a prime divisor of $|G|$. If a Sylow p-subgroup of G is cyclic, G has Z_p as a composition factor and hence has Z_{p-1} as a quotient. But this is impossible since G is rational.

If a Sylow p-subgroup of G is not cyclic, it has a subgroup of type $Z_p \times Z_p$ whose automorphism group is $\mathrm{GL}(2, p)$. Hence G is not solvable, another contradiction.

https://doi.org/10.4153/CMB-1997-032-2 Published online by Cambridge University Press

LEMMA 3.2. *If G is nilpotent, then $G \cong Z_2$ or $G \cong Q_8$.*

PROOF. If G is abelian, then $1 = N_G(G)/C_G(G) \cong \mathrm{Aut}(G)$ forces $G \cong Z_2$. If G is nonabelian, then G is a 2-group by Lemmas 3.1 and 2.1 iv). Since $S_3 \cong \mathrm{Aut}(Z_2 \times Z_2)$, G cannot have subgroups isomorphic to $Z_2 \times Z_2$. Thus G is a generalized quaternion group. Then $G \cong Q_8$ by Lemma 2.1 v).

LEMMA 3.3. *If G is nonnilpotent, then $G \cong S_3$.*

PROOF. By hypothesis and by Lemma 3.1, $\pi(G) = \{2, 3\}$. Let $S \in \mathrm{Syl}_2(G)$ and let $P \in \mathrm{Syl}_3(G)$. By Lemmas 2.1 iii) and 2.3, P is abelian. Hence $|P| = 3$ by Lemma 2.1 ii).

If now A is a minimal normal subgroup of G, then the solvability of G and $\mathrm{Aut}(A) \cong G/C_G(A)$ imply $|A| \in \{2, 3, 4\}$. Suppose first that $A \subseteq S$ and note that $A \neq S$ for otherwise $G/S \cong P \cong Z_3$, a contradiction with Lemma 2.1 iv). If $|A| = 2$, then S has a unique involution by Lemma 2.1 iii). So $S \cong Q_8$ by Lemma 2.1 v). Since $|G| = 24$ and since S is not normal in G, this yields a contradiction. If $|A| = 4$, then A is a four group. Since $G/C_G(A) \cong \mathrm{Aut}(A) \cong S_3$, one can prove easily that $G \cong S_4$. This is a contradiction since S_4 has two conjugacy classes of involutions and hence cannot be an LAAS-group by Lemma 2.1 iii).

Therefore one must have $|A| = 3$. Hence $A = P$ is normal in G and $S \cong G/P = G/C_G(P) \cong \mathrm{Aut}(P) \cong Z_2$. This implies $G \cong S_3$ and completes the proof.

The next objective is to prove that every LAAS-group is solvable. For the sake of contradiction assume that there exists a nonsolvable LAAS-group G. We shall use freely the fact that the elements of the same prime order are conjugate in G for every prime in $\pi(G)$. Since G is a rational group and since both groups in Lemma 2.2 i) have more than one conjugacy class of involutions, it follows that G cannot be simple.

LEMMA 3.4. *G has a unique abelian minimal normal subgroup $A \cong Z_2 \times Z_2$.*

PROOF. Let $F(G)$ denote the Fitting subgroup of G. If $F(G) = 1$, there exists a nonabelian minimal normal subgroup $K < G$. Then K is the direct product of isomorphic nonabelian simple groups K_i for $1 \leq i \leq s$. Since G permutes the set $\{K_i | 1 \leq i \leq s\}$ via conjugation, an involution in K_1 cannot be conjugate in G with an involution in the diagonal if $s \neq 1$. Hence K is a simple nonabelian group. If $C_G(K) \neq 1$, $C_G(K)$ contains a nonabelian minimal normal subgroup of G because $F(G) = 1$. The unique conjugacy class of involutions leads to a contradiction. So $C_G(K) = 1$ and G can be regarded as a subgroup of $\mathrm{Aut}(K)$.

Since K is one of the groups indicated in Lemma 2.2 ii), the argument in the proof of Corollary B of Feit and Seitz [2] eliminates all but one candidate, namely $K \cong A_6$. But if $K \cong A_6$, then $|G : K| = 2$ or $|G : K| = 4$. Hence G has a Sylow 3-subgroup of order 9 which contradicts Lemma 2.1 ii). Consequently $F(G) \neq 1$.

Since $F(G) \neq 1$, there exists a minimal normal elementary abelian subgroup A of G of p-power order. If A is cyclic, $|A| = p$. Hence $G/C_G(A) \cong \mathrm{Aut}(A)$ is cyclic of order $p - 1$. By Lemma 2.1 iv), $p = 3$ or $p = 2$.

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Suppose first that $p = 3$ and let $P \in \mathrm{Syl}_3(G)$. Then A is the unique subgroup of order 3 of P since all elements of order 3 of G lie in A. This forces P to be cyclic and then by Lemma 2.1 ii), $P = A$. Hence G/A is a rational $3'$-group. Since none of the possible nonabelian composition factors of G/A, which are indicated in Lemma 2.2 ii), is a $3'$-group, there is a contradiction.

Suppose now that $|A| = 2$ and let $S \in \mathrm{Syl}_2(G)$. Since the unique involution of G lies in A and A is normal in G, S is either cyclic or isomorphic to Q_8 by Lemma 2.1 v). If S is cyclic, then $|S| = 2$ by Lemma 2.1ii). Hence $S = A$ and therefore G/A has odd order, contradicting the fact that G/A is a rational group.

If $S \cong Q_8$, then a Sylow 2-subgroup of the rational group G/A has order 4. But G/A is nonsolvable. The only possible nonabelian composition factor of G/A is A_5 because the other simple groups in Lemma 2.2 ii) have larger Sylow 2-subgroups. Consider now a chief factor G/H of G, with $A < H$. If G/H is abelian, G/H is a 2-group by Lemma 2.1 iv). Since G is nonsolvable, H must contain a chief factor isomorphic to A_5 by Jordan-Hölder theorem. This contradicts $|S| = 8$. One must then have $G/H \cong A_5$ and consequently H/A must have odd order.

We claim that $A = Z(G) = F(G)$. For if the claim is false, then $|F(G)| = 4$ and $F(G)$ is cyclic. But then $G/C_G\big(F(G)\big) \cong \mathrm{Aut}\big(F(G)\big) \cong Z_2$. This implies that if T is a subgroup of order 5 of G, then 4 divides $|C_G(T)|$. In particular, $|N_G(T)| = |N_G(T) : C_G(T)|\,|C_G(T)| = |\mathrm{Aut}(T)|\,|C_G(T)|$ would be divisible by 16, a contradiction.

Thus $F(G) = Z(G)$ and by Satz 4.2 b), p. 277 of [4], $G/F(G) = C_G\big(F(G)\big)/F(G)$ contains no nontrivial abelian normal subgroups of $G/F(G)$. By our preceeding discussion, this shows that $Z(G) = F(G) = H$ with $G/Z(G) \cong A_5$. In particular, $|G| = 120$. If $Q \in \mathrm{Syl}_5(G)$, then by Sylow's theorem $|G : N_G(Q)|$ equals 1 or 6. If $N_G(Q) = G$, then $|C_G(Q)| = 30$. But in this case $C_G(Q)$ is cyclic and since $|\mathrm{Aut}(Z_{30})| = 8$, one obtains the contradiction: 240 divides the order of G.

If $|G : N_G(Q)| = 6$, then $|N_G(Q)| = 20$ and $|C_G(Q)| = 5$, a contradiction because $|Z(G)| = 2$. Therefore A cannot be cyclic.

Suppose now that A is an elementary abelian p-group of rank $n \geq 2$. Then $\mathrm{GL}_n(p) \cong \mathrm{Aut}(A) \cong G/C_G(A)$ is a homomorphic image of G, hence a rational group. This can happen only if $(n, p) \in \{(1, 2), (2, 2), (1, 3)\}$ and since $n \geq 2$ we see that (n,p)=(2,2). But then $A \cong Z_2 \times Z_2$. The uniqueness of A is evident.

We are now in a position to show that there exist no nonsolvable LAAS-groups. Suppose that G is a nonsolvable LAAS-group and let A be its unique minimal normal abelian subgroup. By Lemma 3.4, A is a four group and by Lemma 2.1 iii) all involutions of G lie in A. Moreover $G/C_G(A) \cong \mathrm{Aut}(A) \cong S_3$. There exists a 3-element x of G which acts nontrivially on A. There exists a 2-element y which inverts x. Thus $y^2x = xy^2$ and y^{2^m} commutes with x for all $m > 0$. Since y^n is an involution for some n, it is in A. This contradicts the definition of x. The proof of the Theorem is now complete.

REMARK. One may feel that the above proof relies too heavily on deep results about simple groups and that the LAAS-property is so strong that an elementary proof should be given to the Theorem. However, one should keep in mind that the LAAS-property is

https://doi.org/10.4153/CMB-1997-032-2 Published online by Cambridge University Press

weaker than the MD-property mentioned in the Introduction and that the MD-property is equivalent to the property that all elements of the same order are conjugate. As far as we know, there is no elementary (*i.e.* CFSG-free) proof that S_n, $n \leq 3$, are the only finite groups in which all elements of the same order are conjugate. Such a proof could be obtained possibly by showing that the conjugation property implies the MD-property. One may ask how far is the LAAS-property from the property that all elements of the same prime order p are conjugate for every prime $p \in \pi(G)$.

ACKNOWLEDGEMENT. The first author wishes to express his gratitude to Kuwait University for financial support which made possible a visit there in January 1995.

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Department of Mathematics
University of New-Hampshire
Durham NH
USA 03824
e-mail: bechtell@christa.unh.edu

Department of Mathematics and Computer Science
University of Kuwait
P.O. Box 5969
Safat 13060, Kuwait
e-mail: deacon@sun490.sci.kuniv.edu.kw

Department of Mathematics
Western University of Timisoara
Bd. V. Parvan 4, 1900-Timisoara
Romania
e-mail: silber@tim1.uvt.ro