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The Hausdorff and Packing Dimensions of Some Sets Related to Sierpiński Carpets

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Abstract. The Sierpiński carpets first considered by C. McMullen and later studied by Y. Peres are modified by insisting that the allowed digits in the expansions occur with prescribed frequencies. This paper (i) calculates the Hausdorff, box (or Minkowski), and packing dimensions of the modified Sierpiński carpets and (ii) shows that for these sets the Hausdorff and packing measures in their dimension are never zero and gives necessary and sufficient conditions for these measures to be infinite.

1 Introduction

Let m and n be two integers satisfying $n \geq m \geq 2$ and let $I = \{0, 1, \dots, n-1\}$ and $J = \{0, 1, \dots, m-1\}$. There is a function π from $(I \times J)^{\mathbf{N}}$ onto $[0,1]^2$ defined as follows: If x is a point in $(I \times J)^{\mathbf{N}}$ and if u and v are the points in $I^{\mathbf{N}}$ and $J^{\mathbf{N}}$, resp., such that $x_j = (u_j, v_j)$ for each $j \in \mathbf{N}$ then

$$\pi(x) = \Big(\sum_{j=1}^{\infty} u_j n^{-j}, \sum_{j=1}^{\infty} v_j m^{-j}\Big).$$

The sets to be studied in this paper are the images under this map of certain subsets of $(I \times J)^{\mathbf{N}}$.

Let D be a non-empty subset of $I \times J$ and let $p = (p_d)_{d \in D}$ be a probability vector on D. For any point x in $(I \times J)^{\mathbf{N}}$, any positive integer k, and any point $d \in I \times J$ let $N_k(x, d)$ denote the number of integers in the set

$$\{j : 1 \leq j \leq k \text{ and } x_j = d\}.$$

One can then consider the two subsets

$$L(D) = \{x \in (I \times J)^{\mathbf{N}} : x_j \in D \text{ for all } j \in \mathbf{N}\}$$

and

$$L(D, p) = \Big\{x \in L(D) : \lim_{k \to \infty} \frac{N_k(x, d)}{k} = p_d \text{ for all } d \in D\Big\}$$

of $(I \times J)^{\mathbf{N}}$ as well as their images

$$\Lambda(D) = \pi\big(L(D)\big) \quad \text{and} \quad \Lambda(D, p) = \pi\big(L(D, p)\big)$$

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under π. The set $\Lambda(D)$ is sometimes called a *Sierpiński carpet*. McMullen has calculated the Hausdorff and box dimensions of $\Lambda(D)$ [5] and Peres has calculated its packing dimension and investigated its Hausdorff and packing measures in the critical dimension [7], [8]. The object of the present paper is to carry out this analysis for the sets $\Lambda(D, p)$.

For any Borel subset A of $\mathbf{R}^2$ let $\mathcal{H}^\gamma(A)$ and $\mathcal{P}^\gamma(A)$ denote the γ-dimensional Hausdorff and packing measures of A, resp., and let $\dim_H A$, $\dim_P A$, and $\dim_B A$ denote the Hausdorff, packing, and box dimensions of A, resp. (These measures and dimensions are discussed in [1] and [4] and their definitions will not be reviewed here.) Let $|A|$ denote the cardinality of any finite set A, let $\theta = \log_n m$, let σ denote the projection of $\mathbf{R}^2$ onto its second coordinate, let $B = \sigma(D)$, and for each point $b \in B$ put $n_b = |D \cap (I \times \{b\})|$ and

$$q_b = \sum \{p_d : d \in D \cap (I \times \{b\})\}.$$

Then $0 < \theta \leq 1$ and $q = (q_b)_{b \in B}$ is a probability vector on B. Finally, $x \log_m x$ will be interpreted as being 0 in case $x = 0$.

Theorem 1

$$\dim_H \Lambda(D, p) = \dim_P \Lambda(D, p) = -\theta \sum_{d \in D} p_d \log_m p_d - (1 - \theta) \sum_{b \in B} q_b \log_m q_b \tag{1}$$

and

$$\dim_B \Lambda(D, p) = \log_m(|B|^{1-\theta} |D|^\theta). \tag{2}$$

The vector p is said to be *uniformly distributed* on D if $p_d = |D|^{-1}$ for all $d \in D$ and D is said to have *uniform horizontal fibers* if $n_b = n_{b'}$ for all $b, b' \in B$.

Corollary 2

(a) $\dim_H \Lambda(D, p) \leq \dim_H \Lambda(D)$ *with equality if and only if* $n_{\sigma(d)} p_d = q_{\sigma(d)}$ *for all* $d \in D$ *and* $(\sum_{b' \in B} n_{b'}^\theta) q_b = n_b^\theta$ *for all* $b \in B$.

(b) $\dim_P \Lambda(D, p) \leq \dim_P \Lambda(D)$ *with equality if and only if p is uniformly distributed on D and D has uniform horizontal fibers.*

The two conditions that part (a) of this corollary assert as being necessary and sufficient for the equality of $\dim_H \Lambda(D, p)$ and $\dim_H \Lambda(D)$ are implied by but do not imply that p is uniformly distributed on D and that D has uniform horizontal fibers. There is, in fact, a simple example with $|D| = 3$, $|B| = 2$, and p not uniformly distributed on D for which $\dim_H \Lambda(D, p) = \dim_H \Lambda(D)$.

Theorem 3 *Let γ denote the common value of* $\dim_H \Lambda(D, p)$ *and* $\dim_P \Lambda(D, p)$ *(cf. Theorem 1).*

(a) If p is uniformly distributed on D and if D has uniform horizontal fibers then

$$0 < \mathcal{H}^\gamma\big(\Lambda(D, p)\big) \leq \mathcal{P}^\gamma\big(\Lambda(D, p)\big) < \infty.$$

(b) *If p is not uniformly distributed on D or if D does not have uniform horizontal fibers then*

$$\mathcal{H}^{\gamma}\big(\Lambda(D,p)\big) = \mathcal{P}^{\gamma}\big(\Lambda(D,p)\big) = \infty.$$

If $|B| = 1$ the set $\Lambda(D,p)$ may be regarded as a subset of $[0,1]$ and has been discussed by various authors. In particular, the calculation of its Hausdorff dimension is in [1, Section 10.1].

2 Preliminaries

For each point $x \in (I \times J)^{\mathbf{N}}$ and each positive integer k put

$$\begin{aligned} Q_k(x) = \{\pi(u) : u \in (I \times J)^{\mathbf{N}}, u_j = x_j \text{ for } 1 \le j \le [\theta k], \text{ and} \\ \sigma(u_j) = \sigma(x_j) \text{ for } [\theta k] + 1 \le j \le k\}, \end{aligned}$$

where, as usual, $[\cdot]$ denotes the greatest integer function. The sets $Q_k(x)$ are closed rectangles in $[0,1]^2$ whose sides have lengths $n^{-[\theta k]}$ and $n^{-\theta k}$ and whose diameters $\operatorname{diam} Q_k(x)$ satisfy

$$\sqrt{2}m^{-k} \le \operatorname{diam} Q_k(x) \le \sqrt{2}nm^{-k}. \tag{3}$$

The proofs of Theorems 1 and 3 necessarily involve estimating the Hausdorff and packing measures of $\Lambda(D,p)$ in various dimensions and this will accomplished by means of the following two density theorems. The first of these is a just a reformulation of the Rogers-Taylor density theorem as stated by Peres in [8, Section 2]. Note that the occurrence of $\log_m$ in these theorems is a consequence of (3).

Lemma 1 *Suppose that δ is a positive number, that λ is a finite Borel measure on $[0,1]^2$, and that E is a subset of $(I \times J)^{\mathbf{N}}$ such that $\pi(E)$ is a Borel subset of $[0,1]^2$ and $\lambda\big(\pi(E)\big) > 0$. Put*

$$A(x) = \limsup_{k\to\infty}\Big(k\delta + \log_m \lambda\big(Q_k(x)\big)\Big)$$

for each point $x \in E$.

(a) *If $A(x) = -\infty$ for all $x \in E$ then $\mathcal{H}^{\delta}\big(\pi(E)\big) = \infty$.*
(b) *If $A(x) = \infty$ for all $x \in E$ then $\mathcal{H}^{\delta}\big(\pi(E)\big) = 0$.*
(c) *If there are numbers a and b such that $a \le A(x) \le b$ for all $x \in E$ then $0 < \mathcal{H}^{\delta}\big(\pi(E)\big) < \infty$.*

Lemma 2 *Suppose that δ, λ, and E are as in Lemma 1 and put*

$$B(x) = \liminf_{k\to\infty}\Big(k\delta + \log_m \lambda\big(Q_k(x)\big)\Big)$$

for each point $x \in E$.

(a) *If $B(x) = -\infty$ for all $x \in E$ then $\mathcal{P}^{\delta}\big(\pi(E)\big) = \infty$.*

(b) *If* $B(x) = \infty$ *for all* $x \in E$ *then* $\mathcal{P}^\delta\big(\pi(E)\big) = 0$.
(c) *If there are numbers* a *and* b *such that* $a \le B(x) \le b$ *for all* $x \in E$ *then* $0 < \mathcal{P}^\delta\big(\pi(E)\big) < \infty$.

Proof Consider the rectangle $Q_k(x)$ for some $x \in (I \times J)^{\mathbf{N}}$ and $k \in \mathbf{N}$: This rectangle has the form $[u_1, u_2] \times [v_1, v_2]$ and one may associate with it the rectangle

$$Q_k(x)' = \begin{cases} [u_1,u_2] \times [v_1,v_2] & u_2 = v_2 = 1 \\ [u_1,u_2] \times [v_1,v_2) & u_2 = 1 \text{ and } v_2 < 1 \\ [u_1,u_2) \times [v_1,v_2] & u_2 < 1 \text{ and } v_2 = 1 \\ [u_1,u_2) \times [v_1,v_2) & u_2 < 1 \text{ and } v_2 < 1. \end{cases}$$

For any $k \in \mathbf{N}$ and any $x \in (I \times J)^{\mathbf{N}}$ the set $Q_k(x)$ determines k and it will be convenient to let $h\big(Q_k(x)\big) = k$. For each $k \in \mathbf{N}$ put

$$\mathcal{Q}_k = \{Q_j(x) : x \in (I \times J)^{\mathbf{N}} \text{ and } j \ge k\}$$

and define a *k-packing* of a non-empty subset F of $[0,1]^2$ to be a countable subset $\mathcal{R}$ of $\mathcal{Q}_k$ such that $Q' \cap F \ne \varnothing$ for each $Q \in \mathcal{R}$ and such that the Q', $Q \in \mathcal{R}$, are pairwise disjoint. Put

$$\widetilde{P}^s(F) = \lim_{k\to\infty} \sup\Big\{ \sum_{Q\in\mathcal{R}} m^{-h(Q)s} : \mathcal{R} \text{ is a } k\text{-packing of } F \Big\}$$

and

$$\widetilde{\mathcal{P}}^s(F) = \inf\Big\{ \sum_{j=1}^{\infty} \widetilde{P}^s(F_j) : F_j \subseteq [0,1]^2 \text{ for each } j \text{ and } F = \bigcup_{j=1}^{\infty} F_j \Big\}$$

for each positive number s and each subset F of $[0,1]^2$. In view of (3) the functions $\widetilde{P}^s$ and $\widetilde{\mathcal{P}}^s$ are reminiscent of the packing pre-measure and measure and, what is more to the point, are analogous to the functions denoted by $e^s - P^*$ and $e^s - p^*$ in [6]. In particular, the $\widetilde{\mathcal{P}}^s$, $s > 0$, are Borel measures on $[0,1]^2$. One can prove Lemma 2 by adapting some of the arguments given in [6, Sections 3 and 5].

The first step is to modify the proof of [6, Corollary 5.9] to show that for any subset F of $[0,1]^2$ the three numbers

(i) $\dim_P(F)$,
(ii) $\sup\{s > 0 : \widetilde{\mathcal{P}}^s(F) = \infty\}$, and
(iii) $\inf\{s > 0 : \widetilde{\mathcal{P}}^s(F) = 0\}$

are equal.

Now suppose that $s > 0$ and that E and λ are as in the statement of the lemma and put

$$C(x) = \limsup_{k\to\infty} \frac{m^{-ks}}{\lambda\big(Q_k(x)\big)}$$

for each $x \in E$. The second step is to show that

$$\lambda(E) \inf_{x\in E} C(x) \le \widetilde{\mathcal{P}}^s(E) \le \lambda([0,1]^2) \sup_{x\in F} C(x)$$

and this can be done by a modification of the proof of [6, Theorem 5.4]. The point of introducing the rectangles $Q_k(x)'$ is that they resemble the dyadic cubes considered in [6] and are amenable to a Vitali-type argument. Namely, suppose that $G \subseteq (I \times J)^{\mathbf{N}}$ and $\mathcal{A} \subseteq \mathcal{Q}_1$ are sets with the property that the set $\{k : Q_k(x) \in \mathcal{A}\}$ is infinite for each point $x \in G$. Then there are (possibly finite) sequences (x_j) in G and (k_j) in $\mathbf{N}$ satisfying (i) $Q_{k_j}(x_j) \in \mathcal{A}$ for all j, (ii) $G = \bigcup_j Q_{k_j}(x_j)$, and (iii) the $Q_{k_j}(x_j)'$ are pairwise disjoint. The proof of this depends on the fact if Q_1 and Q_2 are two sets in $\mathcal{Q}_1$ such that $Q_1' \cap Q_2' \neq \varnothing$ and $h(Q_1) \leq h(Q_2)$ then it follows that $Q_2 \subseteq Q_1$.

The lemma follows easily from these two steps. ■

The measures on $[0,1]^2$ that will be used in the application of these two lemmas are constructed as follows. Suppose that $p^{(j)} = (p_d^{(j)})_{d\in D}$ is a probability vector on D for each $j \in \mathbf{N}$. Then $p^{(j)}$ determines, in an obvious manner, a probability measure on the power set of D and hence on the j-th factor of $L(D) = D^{\mathbf{N}}$. Let μ denote the Borel measure on $L(D)$ that is the infinite product of these measures (see [2, Section 38], for example) and let $\widetilde{\mu}$ denote the Borel measure on $\Lambda(D)$ that is the image of μ by π. One may, of course, regard $\widetilde{\mu}$ as a Borel measure on $[0,1]^2$ and this will be done in the applications of Lemmas 1 and 2. In order to apply these lemmas with the measure $\widetilde{\mu}$ it is necessary to have a formula for $\widetilde{\mu}\big(Q_k(x)\big)$ and to know that $\widetilde{\mu}\big(\Lambda(D,p)\big) > 0$. It is clear from the definition of $\widetilde{\mu}$ and from the product structure of $Q_k(x)$ that

$$\widetilde{\mu}\big(Q_k(x)\big) = \prod_{j=1}^{[\theta k]} p_{x_j}^{(j)} \prod_{j=[\theta k]+1}^{k} q_{\sigma(x_j)}^{(j)}. \tag{4}$$

Lemma 3 *If* $\lim_{j\to\infty} p_d^{(j)} = p_d$ *for each* $d \in D$ *then* $\widetilde{\mu}\big(\Lambda(D,p)\big) = 1$.

Proof If

$$K_d = \left\{ x \in L(D) : \lim_{k\to\infty} \frac{N_k(x,d)}{k} = p_d \right\}$$

for each point $d \in D$ then $L(D,p) = \bigcap_{d\in D} K_d$ and so it is sufficient to show that $\mu(K_p) = 1$ for each $d \in D$.

Fix a point $d \in D$ and put

$$X_j(x) = \begin{cases} 1 & x_j = d \\ 0 & x_j \neq d \end{cases}$$

for $j \in \mathbf{N}$ and $x \in L(D)$. Then (with respect to μ) $X_1, X_2, \dots$ are independent (but not necessarily identically distributed) random variables on $L(D)$ which are uniformly bounded and hence satisfy $\sum_{j=1}^{\infty} \operatorname{var}(X_j)\, j^{-2} < \infty$. Now

$$p_d = \lim_{k\to\infty} k^{-1} \sum_{j=1}^{k} p_d^{(j)}$$

(this is well-known—see [3, Lemma 2.3.1], for example), hence

$$\lim_{k\to\infty} \frac{N_k(x,d)}{k} - p_d = \lim_{k\to\infty} k^{-1} \sum_{j=1}^{k} \big(X_j(x) - p_d^{(j)}\big)$$
$$= \lim_{k\to\infty} k^{-1} \sum_{j=1}^{k} \left(X_j(x) - \int_{L(D)} X_j \, d\mu\right)$$

for all $x \in L(D)$, and therefore $\mu(K_d) = 1$ by the strong law of large numbers (see [3, Proposition 2.3.7]). ■

3 Proof of Theorem 1

Since $\Lambda(D,p)$ is dense in $\Lambda(D)$ it follows directly from the definition of the box dimension that the box dimensions of these two sets are equal. But McMullen has shown that the right side of (2) is equal to $\dim_B \Lambda(D)$ and this proves (2).

Let $p^{(j)} = p$ for all $j \in \mathbf{N}$ and let μ and $\widetilde{\mu}$ be the Borel probability measures on $L(D)$ and $\Lambda(D)$, resp., corresponding to the sequence $(p^{(j)})$ of probability vectors on D. For any point $x \in (I \times J)^{\mathbf{N}}$, any integer $k \in \mathbf{N}$, and any point $b \in B$ let

$$M_k(x,b) = |\{ j : 1 \le j \le k \text{ and } \sigma(x_j) = b\}|.$$

Then

$$M_k(x,b) = \sum \{N_k(x,d) : d \in D \cap (I \times \{b\})\}$$

and so

$$\lim_{k\to\infty} \frac{M_k(x,b)}{k} = \sum \{p_d : d \in D \cap (I \times \{b\})\} = q_b$$

for all $x \in L(D,p)$ and $b \in B$.

Now let γ denote the right side of (1) and consider a point x in $L(D,p)$. Then

$$\log_m \widetilde{\mu}\big(Q_k(x)\big) = \sum_{j=1}^{[\theta k]} \log_m p_{x_j} + \sum_{j=[\theta k]+1}^{k} \log_m q_{\sigma(x_j)}$$
$$= \sum_{d\in D} N_{[\theta k]}(x,d) \log_m p_d + \sum_{b \in B} \big(M_k(x,b) - M_{[\theta k]}(x,b)\big) \log_m q_b$$

by (4) and hence

$$\lim_{k\to\infty} \frac{\log_m \widetilde{\mu}\big(Q_k(x)\big)}{k} = \sum_{d\in D} \theta p_d \log_m p_d + \sum_{b\in B} (q_b - \theta q_b) \log_m q_b$$
$$= -\gamma.$$

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For any positive number δ one can write

$$k\delta + \log_m \widetilde{\mu}\big(Q_k(x)\big) = k\left(\delta + \frac{\log_m \widetilde{\mu}\big(Q_k(x)\big)}{k}\right)$$

and therefore

$$\lim_{k\to\infty}\Big(k\delta + \log_m \widetilde{\mu}\big(Q_k(x)\big)\Big) = \begin{cases}\infty & \delta > \gamma\\ -\infty & \delta < \gamma.\end{cases}$$

Since this holds for all points $x \in L(D, p)$ it follows from Lemmas 1 and 3 that

$$\mathcal{H}^\delta\big(\Lambda(D, p)\big) = \begin{cases}0 & \delta > \gamma\\ \infty & \delta < \gamma,\end{cases}$$

and therefore $\dim_H\big(\Lambda(D, p)\big) = \gamma$. A similar argument using Lemma 2 in place of Lemma 1 will show that $\dim_P\big(\Lambda(D, p)\big) = \gamma$.

4 Proof of Corollary 2

McMullen has shown that

$$\dim_H \Lambda(D) = \log_m\Big(\sum_{b\in B} n_b^\theta\Big) \tag{5}$$

and Peres that

$$\dim_P \Lambda(D) = \log_m(|B|^{1-\theta}|D|^\theta) \tag{6}$$

([5, p. 1] and [7, Corollary 2.3(i)], resp.).

Since $\Lambda(D, p) \subseteq \Lambda(D)$ there is no doubt that the inequalities in parts (a) and (b) must hold, and straightforward calculations using Theorem 1 will show that the stated conditions are sufficient for equality in the inequalities.

The proofs that the stated conditions are also necessary for equality depend on two inequalities. The first of these is that

$$\sum_{d\in E} p_d \log_m p_d \geq \Big(\sum_{d\in E} p_d\Big) \log_m\Big(|E|^{-1}\sum_{d\in E} p_d\Big) \tag{7}$$

for any non-empty subset E of D with equality if and only if the p_d, $d \in E$, are all equal and the second, that

$$\sum_{b\in B} q_b(\log_m n_b^\theta - \log_m q_b) \leq \log_m\Big(\sum_{b\in B} n_b^\theta\Big) \tag{8}$$

with equality if and only if $(\sum_{b'\in B} n_{b'}^\theta)q_b = n_b^\theta$ for all $b \in B$.

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It now follows from Theorem 1 and (7), (8), and (5) that

$$\begin{aligned}
\dim_H \Lambda(D,p) &= -\theta \sum_{d\in D} p_d \log_m p_d - (1-\theta)\sum_{b\in B} q_b \log_m q_b \\
&= -\theta \sum_{b\in B} \sum_{\substack{d\in D\\ \sigma(d)=b}} p_d \log_m p_d - (1-\theta)\sum_{b\in B} q_b \log_m q_b \\
&\le -\theta \sum_{b\in B} q_b \log_m (q_b/n_b) - (1-\theta)\sum_{b\in B} q_b \log_m q_b \\
&= \sum_{b\in B} q_b(\log_m n_b^\theta - \log_m q_b) \\
&\le \log_m \Big(\sum_{b\in B} n_b^\theta\Big) \\
&= \dim_H \Lambda(D).
\end{aligned}$$

Now suppose that $\dim_H \Lambda(D,p) = \dim_H \Lambda(D)$. Then (7) with $E = \sigma^{-1}(b)$ must be an equality for each $b \in B$ and (8) must be an equality, and this means that the two conditions in the statement of part (a) are necessary for the equality of the Hausdorff dimensions.

The proof of part (b) depends on (7) with $E = D$ and is even simpler, and it will therefore be left to the reader.

5 Proof of Theorem 3

The second inequality in part (a) is a well-known inequality relating the Hausdorff and packing measures (see [4, Theorem 5.12], for example) and means that in proving part (b) it is enough to consider the Hausdorff measure.

The proofs of the first and third inequalities in part (a) are easy. Let μ and $\widetilde{\mu}$ be as in the proof of Theorem 1, let γ denote the right side of (1), and suppose that $|B| = r$ and $|D| = rs$. Then $q_b = r^{-1}$ for all $b \in B$ and so (by (4))

$$\begin{aligned}
k\gamma + \log_m \widetilde{\mu}\big(Q_k(x)\big) &= \theta k \log_m rs + (1-\theta)k\log_m r \\
&\qquad - [\theta k]\log_m rs - (k - [\theta k])\log_m r \\
&= (\theta k - [\theta k])\log_m s
\end{aligned}$$

for all $k \in \mathbf{N}$ and all $x \in L(D,p)$. The first and third inequalities in Theorem 3(a) now follow from Lemmas 1(c) and 2(c).

The proof of part (b) is somewhat more complicated and it will be convenient to divide it into four cases depending in part on whether or not $p_d > 0$ for all $d \in D$. Accordingly, consider the set

$$D' = \{d \in D : p_d > 0\}.$$

The restriction p' of p to D' is a probability vector on D' and it is obvious that

$$\Lambda(D', p') \subseteq \Lambda(D, p) \tag{9}$$

and it follows from Theorem 1 that

$$\dim_H \Lambda(D', p') = \dim_H \Lambda(D, p). \tag{10}$$

The four cases into which the proof of Theorem 3(b) will be divided are as follows:

(i) p is uniformly distributed on D and D does not have uniform horizontal fibers,
(ii) $D' = D$ and p is not uniformly distributed on D,
(iii) $D' \neq D$ and either p' is not uniformly distributed on D' or else D' does not have uniform horizontal fibers, and
(iv) $D' \neq D$, p' is uniformly distributed on D', and D' has uniform horizontal fibers.

The last two of these four cases are not difficult. In fact, case (iii) follows from (i) and (ii) and (10). Now consider case (iv). Then $0 < \mathcal{H}^\gamma\big(\Lambda(D', p')\big) < \infty$ by Theorem 3(a). Put $T(u, v) = (u/n, v/m)$ for all points $(u, v) \in \mathbf{R}^2$ and consider a positive integer k. Then T is a linear transformation of $\mathbf{R}^2$ and the sets

$$T(d_1) + \cdots + T^k(d_k) + T^k\big(\Lambda(D', p')\big)$$

for $d_1, \ldots, d_k \in D$ are pairwise disjoint subsets of $\Lambda(D, p)$ with the property that

$$\Lambda(D', p') = \bigcup\{T(d_1) + \cdots + T^k(d_k) + T^k\big(\Lambda(D', p')\big) : d_1, \ldots, d_k \in D'\}.$$

Since $\mathcal{H}^\gamma$ is translation-invariant this implies that

$$\mathcal{H}^\gamma\big(\Lambda(D', p')\big) = |D'|^k \mathcal{H}^\gamma\Big(T^k\big(\Lambda(D', p')\big)\Big)$$

and hence

$$\begin{aligned}\mathcal{H}^\gamma\big(\Lambda(D, p)\big) &\geq |D|^k \mathcal{H}^\gamma\Big(T^k\big(\Lambda(D', p')\big)\Big)\\ &= (|D|/|D'|)^k \mathcal{H}^\gamma\big(\Lambda(D', p')\big).\end{aligned}$$

Since this inequality holds for all $k \in \mathbf{N}$ it follows that $\mathcal{H}^\gamma\big(\Lambda(D, p)\big) = \infty$.

The proofs of cases (i) and (ii) are similar to one another and somewhat more complicated than those of (iii) and (iv). Let $r = (r_b)_{b\in B}$ be a vector on B satisfying $r_b > 0$ for all $b \in B$ and $\sum_{b\in B} n_b r_b = 1$, let ω be a number in the interval $(0, 1/4)$, let $t_j = j^{-\omega}$ for $j \in \mathbf{N}$, and put

$$p_d^{(j)} = (1 - t_j)p_d + t_j r_{\sigma(d)}$$

and

$$q_b^{(j)} = (1 - t_j)q_b + t_j n_b r_b$$

for all $j \in \mathbf{N}$, $d \in D$, and $b \in B$. (The actual value of ω will be unimportant except in one argument near the end of the proof.) Notice that $(p^{(j)})$ is a sequence of probability vectors on D and

$$q_b^{(j)} = \sum\{p_d^{(j)} : d \in D \cap (I \times \{b\})\}$$

for all $b \in B$; let μ and $\widetilde{\mu}$ denote the corresponding Borel probability measures on $L(D)$ and $\Lambda(D)$, resp., (*cf.* Section 2). Now put

$$A_k(x) = k\gamma + \log_m \widetilde{\mu}\big(Q_k(x)\big)$$

for $k \in \mathbf{N}$ and $x \in L(D)$.

To prove cases (i) and (ii) of Theorem 3(b) it is sufficient, in view of Lemmas 1(a) and 3, to show the one can select ω and r in such a manner that if

$$K = \{x \in L(D, p) : \lim_{k\to\infty} A_k(x) = -\infty\}$$

then $\mu(K) > 0$.

In order to study the functions A_k and the set K it will be convenient to introduce two sequences of numbers and two sequences of random variables by putting

$$S_j = \sum_{d\in D} p_d^{(j)} \log_m p_d^{(j)} - \sum_{d\in D} p_d \log_m p_d$$

$$T_j = \sum_{b\in B} q_b^{(j)} \log_m q_b^{(j)} - \sum_{b\in B} q_b \log_m q_b$$

$$X_j(x) = \log_m p_{x_j}^{(j)} - \sum_{d\in D} p_d^{(j)} \log_m p_d^{(j)}$$

and

$$Y_j(x) = \log_m q_{\sigma(x_j)}^{(j)} - \sum_{b\in B} q_b^{(j)} \log_m q_b^{(j)}$$

for any $j \in \mathbf{N}$ and $x \in L(D)$. Then

$$\int_{L(D)} X_j \, d\mu = \int_{L(D)} Y_j \, d\mu = 0 \tag{11}$$

for each $j \in \mathbf{N}$ and (by (4))

$$\begin{aligned} A_k(x) &= k\gamma + \sum_{j=1}^{[\theta k]} \log_m p_{x_j}^{(j)} + \sum_{j=[\theta k]+1}^{k} \log_m q_{\sigma(x_j)}^{(j)} \\ &= ([\theta k] - \theta k) \sum_{d\in D} p_d \log_m p_d + (\theta k - [\theta k]) \sum_{b\in B} q_b \log_m q_b \\ &\quad + \sum_{j=1}^{[\theta k]} S_j + \sum_{j=[\theta k]+1}^{k} T_j + \sum_{j=1}^{[\theta k]} X_j(x) + \sum_{j=[\theta k]+1}^{k} Y_j(x). \end{aligned}$$

The first two terms in this last expression for $A_k(x)$ are bounded functions of k and so play no role in the identification of the set K; the middle two terms will be estimated directly and the last two will be estimated by the law of the iterated logarithm. (The suggestion to use the law of the iterated logarithm and the measure just constructed instead of the more natural measure employed in the proof of Theorem 1 came from Peres' analysis of the Hausdorff measure of $\Lambda(D)$ in its Hausdorff dimension [8, pp. 519–520].) Now (11) is part of the hypothesis of the law of the iterated logarithm for the sequences (X_j) and (Y_j) and the rest of the hypothesis involves the variances $\mathrm{var}(X_j)$ and $\mathrm{var}(Y_j)$ of the X_j and the Y_j.

Consider the numbers

$$u_k = \Bigl(\sum_{j=1}^{k} \mathrm{var}(X_j)\Bigr)^{1/2} \quad \text{and} \quad v_k = \Bigl(\sum_{j=1}^{k} \mathrm{var}(Y_j)\Bigr)^{1/2}$$

for $k \in \mathbf{N}$ and the sets

$$K_X = \left\{ x \in L(D,p) : \limsup_{k\to\infty} \frac{|X_1(x) + \cdots + X_k(x)|}{u_k\sqrt{2\ln\ln u_k^2}} = 1 \right\}$$

and

$$K_Y = \left\{ x \in L(D,p) : \limsup_{k\to\infty} \frac{|Y_1(x) + \cdots + Y_k(x)|}{v_k\sqrt{2\ln\ln v_k^2}} = 1 \right\}.$$

It will be convenient to write $\alpha_k \approx \beta_k$ for any two sequences (α_k) and (β_k) of positive numbers satisfying $\alpha_k = \mathrm{O}(\beta_k)$ and $\beta_k = \mathrm{O}(\alpha_k)$. Since $1 - \omega > 3/4$ it is clear that if the four conditions

$$\lim_{k\to\infty} \Bigl(\sum_{j=1}^{[\theta k]} S_j + \sum_{j=[\theta k]+1}^{k} T_j\Bigr) = -\infty, \tag{12}$$

$$\Bigl|\sum_{j=1}^{[\theta k]} S_j + \sum_{j=[\theta k]+1}^{k} T_j\Bigr| \approx k^{1-\omega} \tag{13}$$

$$\mu(K_X) = 1 \quad \text{and} \quad \sum_{j=1}^{[\theta k]} X_j(x) = \mathrm{o}(k^{3/4}) \text{ for each } x \in K_X \tag{14}$$

and

$$\mu(K_Y) = 1 \quad \text{and} \quad \sum_{j=[\theta k]+1}^{k} Y_j(x) = \mathrm{o}(k^{3/4}) \text{ for each } x \in K_Y \tag{15}$$

are all satisfied then $K_X \cap K_Y \subseteq K$ and therefore $\mu(K) = 1$. So to complete the proof of Theorem 3(b) it is sufficient to show that these four conditions hold in cases (i) and (ii).

In order to do this it is necessary to estimate the numbers S_j and T_j and the variances $\mathrm{var}(X_j)$ and $\mathrm{var}(Y_j)$, and this will be done by means of Taylor's theorem with remainder. Indeed, since $D' = D$ in cases (i) and (ii) a tedious but straightforward calculation will show that there is a positive constant c (depending on ω and r) and, for each $j \in \mathbf{N}$, numbers ε_j, η_j, ξ_j, and ζ_j in the interval $[-c, c]$ and such that

$$S_j = c_1 t_j + \varepsilon_j t_j^2,$$

$$T_j = c_2 t_j + \eta_j t_j^2,$$

$$\operatorname{var}(X_j) = \begin{cases} c_3 t_j^2 + \xi_j t_j^4 & p_d = |D|^{-1} \text{ for all } d \in D \\ c_4 + \xi_j t_j & \text{otherwise,} \end{cases}$$

and

$$\operatorname{var}(Y_j) = \begin{cases} c_5 t_j^2 + \zeta_j t_j^4 & q_b = |B|^{-1} \text{ for all } b \in B \\ c_6 + \zeta_j t_j & \text{otherwise,} \end{cases}$$

where the various constants are given by the formulae

$$c_1 = \sum_{d \in D} (r_{\sigma(d)} - p_d) \log_m p_d,$$

$$c_2 = \sum_{b \in B} (n_b r_b - q_b) \log_m q_b,$$

$$c_3 = \frac{|D|}{(\ln m)^2} \sum_{b \in B} n_b (r_b - |D|^{-1})^2,$$

$$c_4 = \sum_{d \in D} p_d (\log_m p_d)^2 - \Big(\sum_{d \in D} p_d \log_m p_d \Big)^2,$$

$$c_5 = \frac{|B|}{(\ln m)^2} \sum_{b \in B} (n_b r_b - |B|^{-1})^2,$$

and

$$c_6 = \sum_{b \in B} q_b (\log_m q_b)^2 - \Big(\sum_{b \in B} q_b \log_m q_b \Big)^2.$$

The following lemma effectively reduces the four conditions (12)–(15) to questions about the values of the constants $c_1, \dots, c_6$.

Lemma 4

(a) *If $c_1, c_2 \le 0$ and $c_1 + c_2 < 0$ then (12) and (13) hold.*
(b) *If p is uniformly distributed on D and $c_3 > 0$ or if p is not uniformly distributed on D then (14) holds.*
(c) *If q is uniformly distributed on B and $c_5 > 0$ or if q is not uniformly distributed on B then (15) holds.*

Proof There is no difficulty in verifying (a).

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Suppose that $u_k \approx k^\alpha$ for some $\alpha > 0$. Then $\lim_{k\to\infty} u_k^{-1} X_k \sqrt{\ln\ln u_k^2} = 0$ uniformly on $L(D)$ since the X_j are uniformly bounded and so the law of the iterated logarithm (see [3, p. 108, Corollary 2], for example) implies that $\mu(K_X) = 1$. Moreover, for each point $x \in K_X$ and for each $\varepsilon > 0$ one has

$$\Big|\sum_{j=1}^{[\theta k]} X_j(x)\Big| = \frac{|X_1(x) + \cdots + X_{[\theta k]}(x)|}{u_{[\theta k]}\sqrt{2\ln\ln u_{[\theta k]}^2}} u_{[\theta k]}\sqrt{2\ln\ln u_{[\theta k]}^2} = \mathrm{o}(k^{\alpha+\varepsilon}).$$

A similar argument will show that if $v_k \approx k^\beta$ for some $\beta > 0$ then

$$\Big|\sum_{j=[\theta k]+1}^{k} Y_j(x)\Big| \le \Big|\sum_{j=1}^{k} Y_j(x)\Big| + \Big|\sum_{j=1}^{[\theta k]} Y_j(x)\Big| = \mathrm{o}(k^{\beta+\varepsilon})$$

for each $x \in K_Y$ and any $\varepsilon > 0$.

If p is uniformly distributed on D and $c_3 > 0$ then $u_k^2 \approx k^{1-2\omega}$ and hence $\mu(K_X) = 1$ and $\sum_{j=1}^{[\theta k]} X_j(x) = \mathrm{o}(k^{1/2})$ for all $x \in K_X$. On the other hand, if p is not uniformly distributed on D then $c_4 > 0$ by the condition for equality in the Cauchy-Schwarz inequality and hence $u_k^2 \approx k$ and therefore $\mu(K_X) = 1$ and $\sum_{j=1}^{[\theta k]} X_j(x) = \mathrm{o}(k^{3/4})$ for all $x \in K_Y$.

This proves part (b) of the lemma and the proof of (c) is similar. ■

The question of when the constants c_1 and c_2 are negative is answered by the next lemma.

Lemma 5 *If $\alpha_1, \ldots, \alpha_k$ are positive numbers with $\sum_{j=1}^{k} \alpha_j = 1$ then*

$$\sum_{j=1}^{k} (k^{-1} - \alpha_j) \log_m \alpha_j \le 0$$

with equality if and only if $\alpha_1 = \cdots = \alpha_k$.

Proof It is only necessary to observe that

$$\sum_{j=1}^{k} (k^{-1} - \alpha_j) \log_m \alpha_j = \sum_{j=1}^{k} (k^{-1} - \alpha_j)(\log_m \alpha_j - \log_m k^{-1}). \qquad ■$$

With these generalities out of the way it is now possible to finally begin the analysis of cases (i) and (ii). Consider case (i) first and recall that $p_d = |D|^{-1}$ for all $d \in D$ and that the n_b, $b \in B$, are not all the same. Put $r_b = (|B| n_b)^{-1}$ for all $b \in B$. Then q is not uniformly distributed on B, $c_2 < 0$ by Lemma 5, and $c_1 = 0$ and $c_3 > 0$. Lemma 4 now implies that the four conditions (12)–(15) hold and this completes the proof of case (i).

Now consider case (ii) and observe that, by Lemma 4, (14) holds. In analyzing case (ii) further it will be convenient to consider the following three conditions:

(N) $n_b = |D| q_b$ for all $b \in B$
(P) $q_{\sigma(d)} = p_d n_{\sigma(b)}$ for all $d \in D$
(Q) $q_b = |B|^{-1}$ for all $b \in B$.

Suppose first that at least one of (N) and (Q) hold and put $r_b = |D|^{-1}$ for all $b \in B$. Then $c_1 < 0$ by Lemma 5 and $c_2 = 0$, and so (12) and (13) hold by Lemma 4(a). If (N) does not hold then clearly $c_5 > 0$, and so if either (N) or (Q) fails to hold then Lemma 4(c) implies that (15) holds. And if both (N) and (Q) hold then $q_b^{(j)} = |B|^{-1}$ for all $b \in B$ and hence $T_j = 0$ and $Y_j(x) = 0$ for all $x \in L(D)$ and all $j \in \mathbf{N}$, and so in identifying the set K it is not necessary to consider (15).

Now suppose that none of (N), (P), and (Q) hold and let $r_b = q_b/n_b$ for $b \in B$. Then (14) and (15) hold by Lemma 4, $c_2 = 0$, and

$$\sum_{d \in D} (r_{\sigma(d)} - p_d) \log_m r_{\sigma(d)} = \sum_{b \in B} (q_b - q_b) \log_m r_b = 0,$$

and therefore

$$c_1 = \sum_{d \in D} (r_{\sigma(d)} - p_d)(\log_m p_d - \log_m r_{\sigma(d)}) < 0.$$

So (12) and (13) also hold and again $\mu(K) = 1$.

Now suppose, as absolutely the last case to be considered, that neither (N) nor (Q) hold but that (P) does hold. Then again Lemma 4 implies that (14) and (15) hold. If there is a choice of ω and r for which both c_1 and c_2 are negative then this same lemma would also imply that (12) and (13) hold and the proof of case (ii) would be complete. So one may as well assume that there are no such ω and r. Now

$$c_1 = \sum_{b \in B} (n_b r_b - q_b) \log_m (q_b/n_b)$$

and thus

$$c_2 + \theta^{1-\omega}(c_1 - c_2) = \sum_{b \in B} (n_b r_b - q_b)(\log_m q_b - \theta^{1-\omega} \log_m n_b).$$

Let b_0 be any point in B and put $B_0 = B \setminus \{b_0\}$ and observe that

$$n_{b_0} r_{b_0} - q_{b_0} = \sum_{b \in B_0} (q_b - n_b r_b)$$

and therefore that

$$\begin{aligned} c_2 + \theta^{1-\omega}(c_1 - c_2) &= \sum_{b \in B_0} (n_b r_b - q_b)(\log_m q_b - \theta^{1-\omega} \log_m n_b) \\ &\quad + \Big(\sum_{b \in B_0} (q_b - n_b r_b)\Big)(\log_m q_{b_0} - \theta^{1-\omega} \log_m n_{b_0}) \\ &= \sum_{b \in B_0} (q_b - n_b r_b)\big(\log_m (q_{b_0}/q_b) - \theta^{1-\omega} \log_m (n_{b_0}/n_b)\big). \end{aligned}$$

If $\theta = 1$ the assumption that (N) does not hold implies that

$$\log_m(q_{b_0}/q_b) - \theta^{1-\omega}\log_m(n_{b_0}/n_b) \neq 0 \tag{16}$$

for at least one point $b \in B_0$. On the other hand, if $\theta < 1$ there must be an $\omega \in (0, 1/4)$ such that (16) holds for all $b \in B_0$. Now since $c_2 + \theta^{1-\omega}(c_1 - c_2) = 0$ if $r_b = q_b/n_b$ for all $b \in B$ there will be positive numbers r_b, $b \in B_0$, such that $\sum_{b\in B_0} n_b r_b < 1$ and such that, for each $b \in B_0$ for which (16) holds, $q_b - n_b r_b$ and the left side of (16) have opposite signs. This means that there are positive numbers r_b, $b \in B$, such that $\sum_{b\in B} n_b r_b = 1$ and

$$c_2 + \theta^{1-\omega}(c_1 - c_2) < 0. \tag{17}$$

By changing the numbers r_b, $b \in B$, slightly if necessary one may assume that neither c_1 nor c_2 is zero. The inequality (17) then implies that $c_1 c_2 < 0$.

Now that ω and r have been selected it is possible to prove that (12) and (13) hold. This will be done by combining upper and lower bounds on sums of the t_j with the approximations of the S_j and the R_j by linear Taylor polynomials.

The first step is to observe that sums of the form $\sum_{j=r}^{s} t_j$, where r and s are positive integers with $r < s$, satisfy

$$\frac{(s+1)^{1-\omega} - r^{1-\omega}}{1-\omega} \leq \sum_{j=r}^{s} t_j \leq \frac{s^{1-\omega} - (r-1)^{1-\omega}}{1-\omega}. \tag{18}$$

Regardless of which one of c_1 and c_2 is positive (recall that one of them is positive and the other is negative), one can use (18) to find two sequences (C_k) and (C'_k) of numbers satisfying

$$(1-\omega)^{-1} C_k k^{1-\omega} \leq c_1 \sum_{j=1}^{[\theta k]} t_j + c_2 \sum_{j=[\theta k]+1}^{k} t_j \leq (1-\omega)^{-1} C'_k k^{1-\omega}$$

for all $k \in \mathbf{N}$ and

$$\lim_{k\to\infty} C_k = \lim_{k\to\infty} C'_k = c_2 + \theta^{1-\omega}(c_1 - c_2).$$

The approximations of the S_j and the T_j by linear Taylor polynomials given in the discussion preceding Lemma 4 imply that

$$\begin{aligned}
\Big|\sum_{j=1}^{k} S_j + \sum_{j=[\theta k]+1}^{k} T_j - c_1 \sum_{j=1}^{k} t_j - c_2 \sum_{j=[\theta k]+1}^{k} t_j\Big| &= \Big|\sum_{j=1}^{k} \varepsilon_j t_j^2 + \sum_{j=[\theta k]+1}^{k} \eta_j t_j^2\Big| \\
&\leq c \sum_{j=1}^{k} t_j^2 \\
&\leq \frac{ck^{1-2\omega}}{1-2\omega}.
\end{aligned}$$

It follows easily from from these estimates that (12) and (13) hold and this completes the proof of case (ii) of Theorem 3(b).

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