Can. J. Math., Vol. XXXII, No. 4, 1980, pp. 937–944

ON EXTENSIONS OF WEAKLY PRIMITIVE RINGS

W. K. NICHOLSON, J. F. WATTERS AND J. M. ZELMANOWITZ

Introduction. If R is a ring an R-module M is called *compressible* when it can be embedded in each of its non-zero submodules; and M is called *monoform* if each partial endomorphism $N \to M$, $N \subseteq M$, is either zero or monic. The ring R is called (left) *weakly primitive* if it has a faithful monoform compressible left module. It is known that a version of the Jacobson density theorem holds for weakly primitive rings [**4**], and that weak primitivity is a Morita invariant and is inherited by a variety of subrings and matrix rings. The purpose of this paper is to show that weak primitivity is preserved under formation of polynomials, rings of quotients, and group rings of torsion-free abelian groups. The key result is that $R[x]$ is weakly primitive when R is (Theorem 1). In particular, a polynomial ring of a primitive ring is weakly primitive. Since a polynomial ring over a field is not primitive, this result clarifies the sense in which a polynomial ring of a primitive ring is a structurally well-behaved ring.

For an arbitrary ring R, the symbol R^1 will denote the union of R and the rational integers.

1. Semigroup rings. It is our intention to begin by showing that weak primitivity is inherited by certain semigroup rings. We will require the following useful criterion for a module to be monoform.

LEMMA 1. *Let ${}_RM$ be given. Assume that for each triple $m_1, m_2, m_3 \in M$ there exists a subring S of R and a monoform S-submodule ${}_SL \subseteq M$ such that $m_1, m_2, m_3 \in L$. Then ${}_RM$ is monoform.*

Proof. We treat the contrapositive statement. Let $f : N \to M$ be a partial R-homomorphism of M which is neither zero nor monic. Then there exists $m_1, m_2 \neq 0$ in N with $m_1 f \neq 0$, $m_2 f = 0$. By hypothesis there exists a subring S of R and a monoform S-submodule L of M with $m_1, m_2, m_1 f \in L$. But then f induces an S-homomorphism $S^1m_1 + S^1m_2 \to L$, which is neither zero nor monic.

Given a ring R and a multiplicative semigroup G, we let RG denote the semigroup ring. For any left R-module M we set $MG = R^1G \otimes_R M$. Observe that there is an R-embedding $M \to MG$ via $m \to g \otimes m$, for g

Received November 7, 1978 and in revised form March 14, 1979. The research of the first author was partially supported by NRC Grant A 8075 and that of the second author by NRC Grants A 7259, A 8225 and A 8075.

fixed in G. So we may regard $M \subseteq MG$, and we may identify the elements of MG uniquely as sums $\sum_{g \in G} m_g g$, where $m_g = 0$ for almost all $g \in G$; the RG-action being given by

$$\left(\sum_{g \in G} r_g g\right)\left(\sum_{g \in G} m_g g\right) = \sum_{g \in G}\left(\sum_{hk=g} r_h m_k\right) g.$$

A semigroup G is called a UP-*semigroup* (for "unique product") if given finite subsets $\{g_1, \ldots, g_m\}, \{h_1, \ldots, h_n\} \subseteq G$, at least one of the $g_i h_j$ has a unique representation as such a product. The UP-semigroups thus provide a generalization of the class of totally ordered semigroups.

PROPOSITION 1. *Let G be a* UP-*semigroup and M a left R-module.*

(1) *If $_RM$ is faithful or compressible, the same is true of the RG-module MG.*

(2) *If $_RM$ is monoform and MH is a monoform RH-module for each finitely generated subsemigroup H of G, then MG is a monoform RG-module.*

Proof. (1) The proof that MG is faithful is left for the reader. Now assume that $_RM$ is compressible, and let $L \neq 0$ be an arbitrary RG-submodule of MG. Choose

$$0 \neq x = m_1 g_1 + \ldots + m_k g_k \in L$$

where $0 \neq m_i \in M$, $g_i \in G$, and k is minimal among all such elements of L. By hypothesis there is an R-monomorphism $f : M \rightarrow Rm_k$. f extends in an obvious way to an RG-monomorphism

$$\bar{f} : MG \rightarrow (Rm_k)G.$$

So it suffices to find an RG-monomorphism

$$\theta : (Rm_k)G \rightarrow L.$$

We define θ by

$$\sum_{g \in G} r_g m_k g \overset{\theta}{\mapsto} \sum_{g \in G} r_g g x \in L.$$

In order to show that θ is well defined, observe that for $r \in R$, $rx = 0$ if and only if $rm_i = 0$ for all $1 \leqq i \leqq k$; equivalently, $rm_i = 0$ for some i, $1 \leqq i \leqq k$. And, using the fact that G is a UP-semigroup, $rx = 0$ if and only if $rgx = 0$ for any $g \in G$. Hence

$$\sum_{g \in G} r_g m_k g = 0 \Leftrightarrow$$

$$\text{each } r_g m_k = 0 \Leftrightarrow$$

$$\text{each } r_g g x = 0 \Leftrightarrow$$

$$\sum_{g \in G} r_g g x = 0,$$

using once again the fact that G is a UP-semigroup. Thus θ is well defined and monic, as required.

(2) We use Lemma 1. Let $x_1, x_2, x_3 \in MG$. The support of x_1, x_2 and x_3 lies in a finitely generated subsemigroup H of G, and $x_i \in MH$ for each i. Since MH is assumed to be a monoform RH-module by hypothesis, the proof is complete.

2. Polynomial rings. We are now going to show that weak primitivity is inherited by polynomial rings. Given a polynomial ring $R[t]$ and an R-module M, we let

$$M[t] = R^1[t] \otimes_R M.$$

Since $R^1[t] = R^1G$ where G is the monoid $G = \{t^i | \, i = 0, 1, 2, \ldots\}$, we have at our disposal the identifications that were indicated in § 1. In particular, the elements of $M[t]$ are of the form $\sum_{i=0}^{k} m_i t^i$ with the $m_i \in M$.

PROPOSITION 2. *If M is a monoform R-module then $M[t]$ has the same property as an $R[t]$-module.*

Before proving this proposition, we state a result which may be of some interest in its own right.

LEMMA 2. *Let M be a uniform R-module and $K \neq 0$ an $R[t]$-submodule of $M[t]$. Set $K_p = \{g \in K | \, g \neq 0$ is of minimal degree p in $K\}$. Then given $0 \neq f \in K$, there exist $a \in R^1$ and $g_i \in K_p$ such that*

$$0 \neq af = \sum_{i=0}^{k} t^i g_i.$$

Proof. We proceed by induction on q, the degree of f. If $q = p$ there is nothing to prove. So assume $q > p$, and let $0 \neq m \in M$ be the leading coefficient of f; $f = mt^q +$ terms of lower degree. Let L_p be the set of leading coefficients of elements of K_p together with 0; L_p is a nonzero R-submodule of M.

Since ${}_RM$ is uniform there exists $b \in R^1$ with $0 \neq bm \in L_p$. Choose $g \in K_p$ with leading coefficient bm. Then $bf - t^{q-p}g \in K$ and has degree $< q$. Hence by the induction hypothesis there exist $c \in R^1$ and $g_i \in K_p$ such that

$$0 \neq c(bf - t^{q-p}g) = \sum_{i=0}^{k} t^i g_i.$$

Thus

$$cbf = \sum_{i=0}^{k} t^i g_i + t^{q-p} cg,$$

and since $cg \in K_p \cup \{0\}$ we will be done provided only that $cbf \neq 0$.

If it were the case that $cbf = 0$, then $cbm = 0$ so that $cg \in K$ has degree $< p$, which means in turn that $cg = 0$. But this would contradict

$$c(bf - t^{q-p}g) \neq 0.$$

Proof of Proposition 2. Let $\theta : K \to M[t]$ be a nonzero $R[t]$-homomorphism, where K is an $R[t]$-submodule of $M[t]$. We consider $K_q = \{f \in K |\ \text{degree } f = q\}$, where q is the least integer such that $K_q\theta \neq 0$. Let L_q be the set of leading coefficients of elements of K_q together with 0; L_q is an R-submodule of M.

For each $i = 0, 1, 2, \ldots$ define $h_i : L_q \to M$ via

$$mh_i = i^{\text{th}} \text{ coefficient of } f\theta,$$

where $f \in K_q$ is chosen with leading coefficient m and $f = 0$ if $m = 0$. h_i is well defined because of the minimality of q, and is clearly an R-homomorphism. Since ${}_RM$ is monoform, each h_i is either zero or monic. This implies in particular that each $f\theta$, for $f \in K_q$, has the same degree d.

Now if $g \in K_p$ where K_p is defined as in the Lemma, then $t^{q-p}g \in K_q$, so that

$$t^{q-p}(g\theta) = (t^{q-p}g)\theta \neq 0.$$

Thus $g\theta \neq 0$ and this means that in fact $q = p$. Finally, let $0 \neq f \in K$ be arbitrary. Since a monoform module is uniform we may apply the Lemma to get

$$0 \neq af = \sum_{i=0}^{k} t^i g_i \quad \text{for some } a \in R^1, \quad g_i \in K_p.$$

Then

$$a(f\theta) = (af)\theta = \sum_{i=0}^{k} t^i(g_i\theta),$$

whence the degree of $a(f\theta)$ is $k + d$. In particular $f\theta \neq 0$, and so θ is monic. This completes the proof.

THEOREM 1. *If R is a weakly primitive ring, then the ring of polynomials over R in an arbitrary number of indeterminates is also weakly primitive.*

Proof. This follows from Proposition 1 provided we can show that if M is a monoform R-module then $M[t_1, \ldots, t_n]$ is a monoform $R[t_1, \ldots, t_n]$-module. But this is immediate from Proposition 2 and the fact that

$$M[t_1, \ldots, t_n] = (M[t_1, \ldots, t_{n-1}])[t_n].$$

3. Rings of quotients. In order to facilitate the treatment of group rings over weakly primitive rings we will first discuss the behaviour of weak primitivity under formation of a ring of quotients. Although the application we have in mind involves a classical localization, it is possible to treat generalized rings of quotients with no additional effort.

By a *Gabriel topology* $\mathscr{F}$ on a ring R, we mean a filter $\mathscr{F}$ of left ideals of R with $R \in \mathscr{F}$, $0 \notin \mathscr{F}$, and such that (i) $I \in \mathscr{F}$ and $a \in R$ implies that

$$(I : a) = \{r \in R | \, ra \in I\} \in \mathscr{F}, \quad \text{and}$$

(ii) $(J : a) \in \mathscr{F}$ for all $a \in I \in \mathscr{F}$ implies that $J \in \mathscr{F}$. An R-module M is called *torsion-free* if

$$T(M) = \{m \in M | \, (0 : m) \in \mathscr{F}\} = 0,$$

where $(0 : m) = \{r \in R | \, rm = 0\}$; and M is *torsion* if $T(M) = M$.

Given an R-module M, we let $\hat{M}$ denote the injective hull of M and define $M \subseteq \mathscr{F}(M) \subseteq \hat{M}$ by $\mathscr{F}(M)/M = T(\hat{M}/M)$. One then defines

$$M_{\mathscr{F}} = \mathscr{F}(M/T(M)), \quad R_{\mathscr{F}} = \mathscr{F}(R/T(R)),$$

and discovers that $R_{\mathscr{F}}$ is a ring with identity called the *ring of quotients* of R with respect to $\mathscr{F}$, and $M_{\mathscr{F}}$ is an $R_{\mathscr{F}}$-module under an action which naturally extends its R-multiplication. The assignment $M \to M_{\mathscr{F}}$ is in fact a functor. The Gabriel topology $\mathscr{F}$ is called *perfect* if the canonical map $M \to R_{\mathscr{F}} \otimes_R M$ induces a natural equivalence of functors $M_{\mathscr{F}} \cong R_{\mathscr{F}} \otimes_R M$. In particular, for $\mathscr{F}$ a perfect Gabriel topology and M a torsion-free R-module, $R_{\mathscr{F}}M = M_{\mathscr{F}}$. For a complete discussion of these notions see [**3**].

In our applications we will be considering the following classical example of a perfect topology. Namely, let $U \subseteq R$ be a multiplicative semigroup of regular elements of R such that given $a \in U$ and $r \in R$ there exist $b \in U$ and $s \in R$ with $sa = br$. Then

$$\mathscr{F} = \{\text{left ideals } I \text{ of } R | I \cap U \neq \emptyset\}$$

is a perfect topology, and the elements of $R_{\mathscr{F}} = R_U$ can be identified as fractions $a^{-1}r$ with $a \in U$, $r \in R$, with addition and multiplication defined in the classical manner.

PROPOSITION 3. *Let $\mathscr{F}$ be a perfect Gabriel topology and M a torsion-free R-module. If $_RM$ is faithful, or compressible, or monoform, then $M_{\mathscr{F}}$ inherits the same property as an $R_{\mathscr{F}}$-module.*

Proof. Assume that $_RM$ is faithful and that $qM_{\mathscr{F}} = 0$ for some $q \in R_{\mathscr{F}}$. Choose $I \in \mathscr{F}$ with $Iq \subseteq \bar{R} = R/T(R)$. Then $IqM_{\mathscr{F}} = 0$, so $IqM = 0$, which implies that $Iq = \bar{0}$. But then $q = \bar{0}$, which proves that $M_{\mathscr{F}}$ is a faithful $R_{\mathscr{F}}$-module.

Next, suppose that $_RM$ is compressible and let X be a nonzero $R_{\mathscr{F}}$-submodule of $M_{\mathscr{F}}$. Then since M is an essential R-submodule of $M_{\mathscr{F}}$, $X \cap M \neq 0$. So we have by hypothesis an R-monomorphism $f : M \to X \cap M$. Then f extends to an $R_{\mathscr{F}}$-homomorphism $f_{\mathscr{F}} : M_{\mathscr{F}} \to (X \cap M)_{\mathscr{F}}$. $f_{\mathscr{F}}$ is a monomorphism because f is and $_RM$ is essential in $M_{\mathscr{F}}$; and the proof of this part is completed by the observation that

$$(X \cap M)_{\mathscr{F}} = R_{\mathscr{F}}(X \cap M) \subseteq X.$$

(This is the only part of the proof where the hypothesis that $\mathscr{F}$ is perfect is used.)

Finally, assume that M is a monoform R-module and let $h : X \to M_{\mathscr{F}}$ be an $R_{\mathscr{F}}$-homomorphism with X a nonzero $R_{\mathscr{F}}$-submodule of $M_{\mathscr{F}}$. Let

$$h_1 : X \cap M \cap Mh^{-1} \to M$$

denote the restriction of h; h_1 must be either zero or monic.

If $h_1 = 0$, then

$$0 = \text{image } h_1 = M \cap (X \cap M)h,$$

so necessarily $(X \cap M)h = 0$. Now let $x \in X$ be arbitrary and set

$$J = \{r \in R | \, rx \in M\}.$$

Then $J \in \mathscr{F}$ and

$$J(xh) = (Jx)h \subseteq (X \cap M)h = 0,$$

so $xh \in T(M_{\mathscr{F}}) = 0$. It follows that $h = 0$. If, on the other hand, h_1 is monic, then

$$0 = \ker h_1 = \ker h \cap M \cap Mh^{-1}$$

so

$$\ker h \cap Mh^{-1} = 0.$$

Since Mh^{-1} is an essential R-submodule of X, $\ker h = 0$ and h is monic. This completes the proof that $M_{\mathscr{F}}$ is a monoform $R_{\mathscr{F}}$-module.

It is perhaps worth pointing out that in the case of central localization we have a converse to Proposition 3.

PROPOSITION 4. *Let $U \subseteq R$ be a multiplicatively closed subset of regular central elements of R, and Q the classical ring of fractions with denominators from U. Then R is weakly primitive if and only if Q is.*

Proof. Suppose ${}_RM$ is a faithful, compressible, monoform module. For $a \in U$, set

$$M_a = \{m \in M | \, am = 0\}.$$

M_a is an R-submodule of M. If $M_a \neq 0$, choose a monomorphism $f : M \to M_a$. Then

$$(aM)f = a(Mf) \subseteq aM_a = 0,$$

so $aM = 0$, a contradiction. Hence $M_a = 0$ for all $a \in U$, which shows that M is torsion-free for the Gabriel topology determined by U. Hence $Q = R_U$ is weakly primitive by Proposition 3.

Conversely, suppose X is a faithful, monoform, compressible Q-module. Without loss of generality, we may assume that $X = Qx$ is cyclic. We show that $M = R^1x$ has these properties as an R-module.

${}_RM$ is clearly faithful. To check compressibility, let $0 \neq N \subseteq M$ be an R-submodule. Then there exists a Q-monomorphism $f : X \to QN$; and if $xf = a^{-1}n$, $a \in U$, $n \in N$, define $f_1 : M \to N$ by $(rx)f_1 = rn$. This is well defined and monic since a is central, and is clearly R-linear. So ${}_RM$ is compressible. Finally, suppose $g : N \to M$ is an R-homomorphism where $N \subseteq M$. Define $g_1 : QN \to X$ by

$$(u^{-1}n)g_1 = u^{-1}(ng).$$

It is easily verified that g_1 is a Q-homomorphism which extends g. Consequently, if g_1 is zero or monic the same is true of g and the proof that ${}_RM$ is monoform is complete.

4. Group rings. We can now employ the results of previous sections to get a definitive result for group rings of abelian groups.

THEOREM 2. *If R is weakly primitive and G is an abelian group then the group ring RG is weakly primitive if and only if G is torsion-free.*

Proof. If G is torsion-free then it can be totally ordered, as is well known. If M is a faithful, monoform, compressible R-module, then MG is faithful and compressible over RG by Proposition 1(1). If x_1, x_2, x_3 are elements of MG, their support H is a finitely generated subgroup of G (and so is free), and it suffices by Lemma 1 to show that MH is RH-monoform. If $\{z_1, \ldots, z_n\}$ is a $\mathbf{Z}$-basis of H then $M[z_1, \ldots, z_n]$ is a monoform $R[z_1, \ldots, z_n]$-module by Proposition 2. If we now localize at the submonoid U of $R[z_1, \ldots, z_n]$ generated by the z_i, the ring of fractions is RH and $MH = RH \otimes_R M$ is a monoform RH-module by Proposition 3.

Connell has proved that a group ring is prime if and only if R is prime and G has no nontrivial finite normal subgroup [**1**]. Now assume conversely that RG is left weakly primitive. Since weakly primitive rings are prime, G must be torsion-free by Connell's result.

A natural question to ask, for which we do not have the answer, is whether an analog to Connell's theorem holds for weakly primitive rings.

Finally we remark that any ring with a faithful compressible module is "weakly primitive" in the sense of [**2**]. It therefore follows that certain extension rings of a weakly primitive ring R are in fact primitive, among them the free algebra $R\{X\}$ in any set of indeterminates X with $|X| \geqq |R|$. See Theorem 5 of [**2**] for details.

The authors wish to acknowledge the assistance of Dan Farkas as well as an anonymous referee in perfecting the final version of this article.

References

1. I. G. Connell, *On the group ring*, Can. J. Math. *15* (1963), 650–685.

2. J. Lawrence, *The coefficient ring of a primitive group ring*, Can. J. Math. *27* (1975), 489–494.

3. B. Stenström, *Rings of quotients* (Springer-Verlag, New York, 1975).

4. J. Zelmanowitz, *Dense rings of linear transformations*, Ring Theory II, Proceedings of the Second Oklahoma Conference (Marcel Dekker, New York, 1977).

University of Calgary,
Calgary, Alberta;
University of Leicester,
Leicester, England;
University of California,
Santa Barbara, California