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$\mathbb{A}_{\text{inf}}$ IS INFINITE DIMENSIONAL

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Abstract Given a perfect valuation ring R of characteristic p that is complete with respect to a rank-1 nondiscrete valuation, we show that the ring $\mathbb{A}_{\text{inf}}$ of Witt vectors of R has infinite Krull dimension.

Keywords: p-adic Hodge theory; Fargues-Fontaine curves; Krull dimension; Newton polygons

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1. Introduction

Fix a prime p. Let R be a perfect valuation ring of characteristic p and denote the valuation by v. Assume v is of rank 1 and nondiscrete and that R is complete with respect to v. Let $\mathbb{A} := \mathbb{A}_{\text{inf}} := W(R)$ be the ring of Witt vectors of R. This ring plays a central role in p-adic Hodge theory as it is the basic ring from which all of Fontaine's p-adic period rings are built. It is also central to the construction of the (adic) Fargues–Fontaine curve [4]. Recently, Bhatt, Morrow and Scholze constructed $\mathbb{A}_{\text{inf}}$-cohomology, a cohomology theory that specializes to étale, de Rham and crystalline cohomology [3]. In these works, there is a useful analogy between $\mathbb{A}$ and a two-dimensional regular local ring. In this paper, we prove the following theorem.

Theorem 1.1. *The ring* $\mathbb{A}$ *has infinite Krull dimension.*

Bhatt [2, Warning 2.24] and Kedlaya [5, Remark 1.6] note that the Krull dimension of $\mathbb{A}$ is at least 3. To see this, fix a pseudouniformizer $\varpi \in R$ and let κ denote the residue field of R. Let $W(\mathfrak{m})$ be the kernel of the natural map $W(R) \to W(\kappa)$ and $[-]\colon R \to W(R)$ the Teichmüller map. Then Bhatt and Kedlaya point out that $\mathbb{A}$ contains the following explicit chain of prime ideals:

$$(0) \subset \mathfrak{p} := \bigcup_{k=0}^{\infty} [\varpi^{1/p^k}]\mathbb{A} \subset W(\mathfrak{m}) \subset (p, W(\mathfrak{m})).$$

As suggested in [5, Remark 1.6], we use Newton polygons to find an infinite chain of prime ideals between $\mathfrak{p}$ and $W(\mathfrak{m})$.

The equal characteristic analogue of Theorem 1.1 is the statement that the power series ring $R[\![X]\!]$ has infinite Krull dimension. This was first proved by Arnold [1, Theorem 1], and the structure of our argument is very similar to his. We axiomatize Arnold's argument in Section 3.

Notation. We use the convention that the symbols $<, >, \subset, \supset$ denote strict inequalities and inclusions with the exception that we allow the statement "$\infty < \infty$" to be true. Otherwise, if equality is allowed, it will be explicitly reflected in the notation using the symbols $\leqslant, \geqslant, \subseteq, \supseteq$. An inequality between two $(\mathbb{R} \cup \{\pm\infty\})$-valued functions means that the inequality holds pointwise.

2. Review of Newton polygons

As above, let R be a perfect valuation ring of characteristic p that is complete with respect to a nondiscrete valuation v of rank 1. Let $\mathfrak{m}$ be the maximal ideal of R, and fix an element $\varpi \in \mathfrak{m}$ of valuation $v(\varpi) = 1$.

Let $\mathbb{A} := W(R)$ be the ring of Witt vectors of R. Write $[-]\colon R \to \mathbb{A}$ for the Teichmüller map, which is multiplicative. Recall that every element of $\mathbb{A}$ can be written uniquely in the form $\sum_{n \geqslant 0} [x_n] p^n$ with $x_n \in R$.

As in [4, Section 1.5.2], given $f \in \mathbb{A}$ with $f = \sum_{n \geqslant 0} [x_n] p^n$, we define the *Newton polygon* $\mathcal{N}(f)$ of f as the largest decreasing convex polygon in $\mathbb{R}^2$ lying below the set of points $\{(n, v(x_n)) \colon n \geqslant 0\}$. We shall often view $\mathcal{N}(f)$ as the graph of a function $\mathcal{N}(f)\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. In particular, if n_f is the smallest integer such that $x_{n_f} \neq 0$, then $\mathcal{N}(f)(t) = +\infty$ for $t < n_f$ and $\mathcal{N}(f)(n_f) = v(x_{n_f})$. Furthermore, $\lim_{t \to \infty} \mathcal{N}(f)(t) = \inf_n v(x_n)$.

Following the conventions in [4, Section 1.5.2], for any integer $i \geqslant 0$ define

$$s_i(f) \coloneqq \mathcal{N}(f)(i) - \mathcal{N}(f)(i+1).$$

We call $s_i(f)$ the *slope* of $\mathcal{N}(f)$ on the interval $[i, i+1]$ even though one would typically call that slope $-s_i(f)$. With this convention, the slopes form a nonnegative decreasing sequence; that is, $s_i(f) \geqslant s_{i+1}(f) \geqslant 0$ for all i. We say that n is a *node* of $\mathcal{N}(f)$ if $\mathcal{N}(f)(n) = v(x_n)$.

We recall the theory of Legendre transforms from [4, Section 1.5.1]. Given a function $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ that is not identically equal to $+\infty$, define

$$\begin{aligned} \mathcal{L}(\varphi)\colon \mathbb{R} &\to \mathbb{R} \cup \{-\infty\} \\ \lambda &\mapsto \inf_{t \in \mathbb{R}} \{\varphi(t) + \lambda t\}. \end{aligned}$$

If φ is a convex function, then one can recover φ from $\mathcal{L}(\varphi)$ via the formula

$$\varphi(t) = \sup_{\lambda \in \mathbb{R}} \{\mathcal{L}(\varphi)(\lambda) - t\lambda\}.$$

From these definitions, it is easy to see that $\mathcal{N}(f) \leqslant \mathcal{N}(g)$ if and only if $\mathcal{L}(\mathcal{N}(f)) \leqslant \mathcal{L}(\mathcal{N}(g))$.

As explained in [4, Section 1.5], for any $f, g \in \mathbb{A}$, we have

$$\mathcal{L}(\mathcal{N}(fg)) = \mathcal{L}(\mathcal{N}(f)) + \mathcal{L}(\mathcal{N}(g)). \tag{1}$$

Motivated by this, one defines a convolution product on the set of $(\mathbb{R} \cup \{+\infty\})$-valued convex functions on $\mathbb{R}$ that are not identically $+\infty$ by

$$(\varphi * \psi)(t) := \sup_{\lambda \in \mathbb{R}} \{\mathcal{L}(\varphi)(\lambda) + \mathcal{L}(\psi)(\lambda) - t\lambda\}.$$

Thus we have $\mathcal{N}(fg) = \mathcal{N}(f) * \mathcal{N}(g)$. In particular, if $\mathcal{N}(f) > 0$, then $\mathcal{N}(f^m) < \mathcal{N}(f^{m+1})$ for all $m \geqslant 1$, and for any $t \in \mathbb{R}$ we have $\lim_{m \to \infty} \mathcal{N}(f^m)(t) = +\infty$.

There is another way of describing $\mathcal{N}(fg)$ in terms of $\mathcal{N}(f)$ and $\mathcal{N}(g)$ without explicitly using Legendre transforms. Write $f = \sum_{n \geqslant 0} [x_n] p^n$ and $g = \sum_{n \geqslant 0} [y_n] p^n$, and let n_f (respectively, n_g) be the smallest integer such that $x_n \neq 0$ (respectively, $y_n \neq 0$). Then $\mathcal{N}(fg)(t) = +\infty$ for all $t < n_f + n_g$, and $\mathcal{N}(fg)(n_f + n_g) = v(x_{n_f}) + v(y_{n_g})$. The slopes of $\mathcal{N}(fg)$ are given by interlacing the slopes of $\mathcal{N}(f)$ and $\mathcal{N}(g)$. That is, the slope sequence of $\mathcal{N}(fg)$ is given by combining the sequences $\{s_i(f) \colon i \geqslant 0\}$ and $\{s_i(g) \colon i \geqslant 0\}$ into a single decreasing sequence that incorporates all positive elements of both sequences. The relationship between this description and equation (1) is explained in [4, Section 1.5].

Lemma 2.1. *Let f be an element of $\mathbb{A}$ such that $\mathcal{N}(f) > 0$. If g is an element of $\mathbb{A}$ and $t_0 \geqslant 0$ is such that for all $t \geqslant t_0$ we have $\mathcal{N}(g)(t) \leqslant \mathcal{N}(f)(t)$, then for all m sufficiently large we have $\mathcal{N}(g) \leqslant \mathcal{N}(f^m)$.*

Proof. As noted above, since $\mathcal{N}(f) > 0$, the sequence $\{\mathcal{N}(f^m)\}_m$ converges to $+\infty$. This convergence is uniform on the compact interval $[0, t_0]$. Thus for m sufficiently large, it follows that $\mathcal{N}(g)(t) \leqslant \mathcal{N}(f^m)(t)$ for all $t \in [0, t_0]$. On the other hand, for all $t \geqslant t_0$ we have

$$\mathcal{N}(g)(t) \leqslant \mathcal{N}(f)(t) < \mathcal{N}(f^m)(t).$$

Thus $\mathcal{N}(g) \leqslant \mathcal{N}(f^m)$ for all m sufficiently large. □

Proposition 2.2. *The ideal $\mathfrak{p} := \bigcup_{k=0}^{\infty} [\varpi^{1/p^k}] \mathbb{A}$ is a prime ideal of $\mathbb{A}$.*

Proof. Note that an element f of $\mathbb{A}$ lies in $\mathfrak{p}$ if and only if $\lim_{t \to \infty} \mathcal{N}(f)(t) > 0$. If $g, g' \in \mathbb{A} \backslash \mathfrak{p}$, then $\lim_{t \to \infty} \mathcal{N}(gg')(t) = \lim_{t \to \infty} (\mathcal{N}(g) * \mathcal{N}(g'))(t) = 0$ and so $gg' \notin \mathfrak{p}$. □

3. The strategy

We define infinitely many sequences in R as follows. For all $i \geqslant 0$, define $a_{1,i} := \varpi^{1/p^i} \in R$. For $n > 1$ and $i \geqslant 0$, define $a_{n,i}$ recursively by

$$a_{n,i} := a_{n-1,i^2} \in R.$$

Thus $a_{n,i} = \varpi^{1/p^{n_i}}$, where $n_i := i^{2^{n-1}}$, and $v(a_{n,i}) = p^{-n_i}$. For each $n \geqslant 1$, define

$$h_n := \sum_{i=0}^{\infty} [a_{n,i}] p^i \in \mathbb{A}.$$

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Note that $\mathcal{N}(h_n) > 0$, for any n we have $\lim_{t\to\infty} \mathcal{N}(h_n)(t) = 0$, and $\mathcal{N}(h_n)$ has a node at every integer.

Finally, we define the following subsets of $\mathbb{A}$. For $n \geqslant 1$, let

$$\mathcal{S}_n := \{g \in \mathbb{A} : 0 < \mathcal{N}(g) \leqslant \mathcal{N}(h_n^m) \text{ for some } m \geqslant 1\}.$$

In particular, $h_n \in \mathcal{S}_n$.

Proposition 3.1. *The sets $\mathcal{S}_n$ satisfy the following three properties:*

(1) *for all $n \geqslant 1$ we have $\mathcal{S}_{n+1} \subset \mathcal{S}_n$;*

(2) *each $\mathcal{S}_n$ is multiplicatively closed;*

(3) *for any $g \in \mathcal{S}_{n+1}$ and $f \in \mathbb{A}$, we have that $g + fh_n \in \mathcal{S}_{n+1}$.*

We prove this proposition in Section 4.

Theorem 3.2. *The ring $\mathbb{A}$ has infinite Krull dimension.*

Proof. We follow Arnold's proof of [1, Theorem 1]. We prove that for any $n \geqslant 1$, there exists a chain of prime ideals of $\mathbb{A}$, say $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$, such that $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$.

For $n = 1$, let $\mathfrak{p}_1 = \mathfrak{p}$. To see that $\mathfrak{p} \cap \mathcal{S}_1 = \emptyset$, note that if $f \in \mathfrak{p}$, then $f \in [\varpi^{1/p^k}]\mathbb{A}$ for some $k \geqslant 0$, and so $\mathcal{N}(f) \geqslant 1/p^k$. On the other hand, if $f \in \mathcal{S}_1$, then for some $m \geqslant 1$ we have that $\lim_{t\to\infty} \mathcal{N}(f)(t) \leqslant \lim_{t\to\infty} \mathcal{N}(h_1^m)(t) = 0$.

Fix $n \geqslant 1$ and suppose for induction that there is a chain $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of $\mathbb{A}$ such that $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$. Consider the ideal $\mathfrak{a}_n := \mathfrak{p}_n + h_n\mathbb{A}$. Note that $\mathfrak{a}_n \neq \mathfrak{p}_n$ since $h_n \in \mathcal{S}_n$ and $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$. We claim that $\mathfrak{a}_n \cap \mathcal{S}_{n+1} = \emptyset$. Indeed, given $g \in \mathcal{S}_{n+1}$, we have that $g + h_n f \in \mathcal{S}_{n+1}$ for all $f \in \mathbb{A}$ by property (3) of the sets $\mathcal{S}_n$. By property (1), it follows that $g + h_n f \in \mathcal{S}_n$ for all $f \in \mathbb{A}$. If $g \in \mathfrak{a}_n$, then there is some $f \in \mathbb{A}$ such that $g + h_n f \in \mathfrak{p}_n$. But $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$, so it follows that $g \notin \mathfrak{a}_n$.

Since $\mathcal{S}_{n+1}$ is multiplicatively closed by property (2), there is a prime ideal $\mathfrak{p}_{n+1}$ of $\mathbb{A}$ such that $\mathfrak{p}_n \subset \mathfrak{a}_n \subseteq \mathfrak{p}_{n+1}$ and $\mathfrak{p}_{n+1} \cap \mathcal{S}_{n+1} = \emptyset$. By induction on n, it follows that $\mathbb{A}$ has infinite Krull dimension. □

Remark 3.3. (a) *Arnold has used an argument as above to show that the ring $R[\![X]\!]$ has infinite Krull dimension [1, Theorem 1]. In fact given any ring A, if one can exhibit elements h_n of A and sets $\mathcal{S}_n$ satisfying the properties in Proposition 3.1 together with a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap \mathcal{S}_1 = \emptyset$, then the above argument shows that A has infinite Krull dimension.*

(b) *There is a rigorous way to view the power series ring $R[\![X]\!]$ as an equal characteristic version of $\mathbb{A}$ (see [4, Section 1.3]). Our definitions make sense in this more general setting, and our arguments give another proof that $R[\![X]\!]$ has infinite Krull dimension.*

4. The proof of Proposition 3.1

In this section we prove Proposition 3.1. Recall that v is the valuation on R and $s_i(h_n) := v(a_{n,i-1}/a_{n,i})$ is the ith slope of $\mathcal{N}(h_n)$.

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Proposition 4.1. *Fix $n, m \geqslant 1$. For $t > 2m^2$ we have that*

$$\mathcal{N}(h_{n+1}^m)(t) < \mathcal{N}(h_n)(t).$$

Proof. Let $\ell = km + r \in \mathbb{Z}$ with $k > 2m$ and $0 \leqslant r < m$. We have

$$\mathcal{N}(h_n)(\ell) = v(a_{n,\ell}) = v(a_{n,km+r})$$

and

$$\mathcal{N}(h_{n+1}^m)(\ell) = mv(a_{n+1,k}) - s_{k+1}(h_{n+1})r \leqslant mv(a_{n+1,k}) = mv(a_{n,k^2}).$$

To see that $mv(a_{k^2}) < v(a_{n,km+r})$, recall that $v(a_{n,i}) = p^{-i^{2^{n-1}}}$. Thus we must show that

$$m < p^{k^{2^n} - (km+r)^{2^{n-1}}}.$$

Since $r < m$, it suffices to show that $m < p^{k^{2^n} - ((k+1)m)^{2^{n-1}}}$. One checks this quickly using that $k > 2m$ and therefore $k^2 - (km + m) > m$. □

Corollary 4.2. *For all $n \geqslant 1$ we have $\mathcal{S}_{n+1} \subset \mathcal{S}_n$.*

Proof. If $g \in \mathcal{S}_{n+1}$ then for some $m \geqslant 1$ we have $0 < \mathcal{N}(g) \leqslant \mathcal{N}(h_{n+1}^m)$. By Proposition 4.1 and Lemma 2.1, it follows that for m' sufficiently large (depending on m and n), we have $\mathcal{N}(h_{n+1}^m) < \mathcal{N}(h_n^{m'})$, so $g \in \mathcal{S}_n$. To see that the inclusion is strict, note that Proposition 4.1 also implies that $h_n \notin \mathcal{S}_{n+1}$, but $h_n \in \mathcal{S}_n$. □

Proposition 4.3. *Let h be an element of $\mathbb{A}$ such that $\mathcal{N}(h) > 0$. Then for any $f \in \mathbb{A}$, $\mathcal{N}(fh) \geqslant \mathcal{N}(h)$.*

Proof. The Newton polygon $\mathcal{N}(fh)$ starts at $n_f + n_h$. Note that the slopes of $\mathcal{N}(fh)$ are all positive and form a monotone sequence converging to zero. Therefore all slopes $s_i(h)$ of h eventually occur as slopes of $\mathcal{N}(hf)$. It follows that for any $l \geqslant n_f + n_h$, $\mathcal{N}(fh)(l) \geqslant \sum_{i \geqslant l}^{\infty} s_i(h) = \mathcal{N}(h)(l)$. □

Proposition 4.4. *For each $n \geqslant 1$, the set $\mathcal{S}_n$ is multiplicatively closed.*

Proof. Let $f, g \in \mathcal{S}_n$. Then by Proposition 4.3, we have that $\mathcal{N}(fg) \geqslant \mathcal{N}(g) > 0$.

For m sufficiently large, we have $0 < \mathcal{N}(f), \mathcal{N}(g) \leqslant \mathcal{N}(h_n^m)$. Thus for any $\lambda, t \in \mathbb{R}$ we have

$$\mathcal{N}(f)(t) + \lambda t \leqslant \mathcal{N}(h_n^m)(t) + \lambda t.$$

Taking the infimum over $t \in \mathbb{R}$, it follows that $\mathcal{L}(\mathcal{N}(f))(\lambda) \leqslant \mathcal{L}(\mathcal{N}(h_n^m))(\lambda)$ for all $\lambda \in \mathbb{R}$. Similarly, $\mathcal{L}(\mathcal{N}(g)) \leqslant \mathcal{L}(\mathcal{N}(h_n^m))$. Therefore

$$\mathcal{L}(\mathcal{N}(fg)) = \mathcal{L}(\mathcal{N}(f)) + \mathcal{L}(\mathcal{N}(g)) \leqslant 2\mathcal{L}(\mathcal{N}(h_n^m)) = \mathcal{L}(\mathcal{N}(h_n^{2m})).$$

Hence, we have that $\mathcal{L}(\mathcal{N}(fg))(\lambda) - t\lambda \leqslant \mathcal{L}(\mathcal{N}(h_n^{2m}))(\lambda) - t\lambda$ for all $t, \lambda \in \mathbb{R}$. It follows that

$$\mathcal{N}(fg)(t) = \sup_{\lambda}\{\mathcal{L}(\mathcal{N}(fg))(\lambda) - t\lambda\} \leqslant \sup_{\lambda}\{\mathcal{L}(\mathcal{N}(h_n^{2m}))(\lambda) - t\lambda\} = \mathcal{N}(h_n^{2m})(t)$$

for all $t \in \mathbb{R}$. Therefore $fg \in \mathcal{S}_n$. □

Let $f, g \in \mathbb{A}$, and write $f = \sum_{n=0}^{\infty}[x_n]p^n$ and $g = \sum_{n=0}^{\infty}[y_n]p^n$. In order to prove property (3) from Proposition 3.1 we need to understand the Newton polygon of $f + g$ in terms of those of f and g. For that, we show a property of Witt vector addition in Lemma 4.5 below. First, recall the translation between Teichmüller expansions and Witt coordinates:

$$\sum_{n=0}^{\infty}[x_n]p^n = (x_0, x_1^p, x_2^{p^2}, \ldots, x_n^{p^n}, \ldots).$$

Recall also that addition of Witt vectors is governed by the polynomials

$$S_n(X_0, \ldots, X_n; Y_0, \ldots, Y_n),$$

which are defined recursively by

$$S_0(X_0; Y_0) := X_0 + Y_0$$

and

$$\sum_{k=0}^{n} p^k S_k(X_0, \ldots, X_k; Y_0, \ldots, Y_k)^{p^{n-k}} = \sum_{k=0}^{n} p^k (X_k^{p^{n-k}} + Y_k^{p^{n-k}}).$$

Thus

$$\begin{aligned} f + g &= (S_0(x_0; y_0), \ldots, S_n(x_0, \ldots, x_n^{p^n}; y_0, \ldots, y_n^{p^n}), \ldots) \\ &= \sum_{n=0}^{\infty}[S_n(x_0, \ldots, x_n^{p^n}; y_0, \ldots, y_n^{p^n})^{p^{-n}}]p^n. \end{aligned}$$

Lemma 4.5. *For all $n \geqslant 0$ we have that*

$$S_n(x_0, \ldots, x_n^{p^n}; y_0, \ldots, y_n^{p^n}) = x_n^{p^n} + y_n^{p^n} + \Sigma_n,$$

where Σ_n is a sum of terms of the form $\prod_{k=0}^{n-1} x_k^{p^k i_k} y_k^{p^k j_k}$ such that $\sum_{k=0}^{n-1} p^k(i_k + j_k) = p^n$.

Proof. Note that if the lemma holds for some n, then S_n^p is a sum of terms of the form $\prod_{k=0}^{n} x_k^{p^k i_k} y_k^{p^k j_k}$ such that $\sum_{k=0}^{n} p^k(i_k + j_k) = p^{n+1}$. The lemma then follows from the definition of S_n and induction on n. □

Proposition 4.6. *Let $f = \sum_{n=0}^{\infty}[x_n]p^n$, $g = \sum_{n=0}^{\infty}[y_n]p^n \in \mathbb{A}$. Assume that $\mathcal{N}(g)$ is strictly decreasing. Suppose there exists a $t_0 \geqslant 0$ such that for all $t \geqslant t_0$ we have $\mathcal{N}(g)(t) < \mathcal{N}(f)(t)$. Then there exists $t_1 \geqslant t_0$ such that for all $t \geqslant t_1$, we have that $\mathcal{N}(g + f)(t) \leqslant \mathcal{N}(g)(t)$.*

Proof. We first show the desired inequality when $t \geqslant t_0$ is a node of $\mathcal{N}(g)$; these exist since g is strictly decreasing. Let $n \geqslant t_0$ be a node of $\mathcal{N}(g)$. Since $\mathcal{N}(g)$ is strictly decreasing, we have that

$$v(y_n) = \mathcal{N}(g)(n) < v(y_m)$$

for all $m < n$. Since $n \geqslant t_0$ and $\mathcal{N}(f)$ is decreasing, for all $m \leqslant n$ we have that

$$v(y_n) = \mathcal{N}(g)(n) < \mathcal{N}(f)(n) \leqslant v(x_m).$$

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Thus $v(y_n^{p^n}) < v(x_n^{p^n})$ and for any $i_0, j_0, \ldots, i_{n-1}, j_{n-1}$ such that $\sum_{k=0}^{n-1} p^k(i_k + j_k) = p^n$, it follows that

$$v\left(\prod_{k=0}^{n-1} y_k^{p^k i_k} x_k^{p^k j_k}\right) > p^n v(y_n) = v(y_n^{p^n}).$$

By Lemma 4.5, it follows that

$$v(S_n(y_0, \ldots, y_n^{p^n}; x_0, \ldots, x_n^{p^n})^{p^{-n}}) = v(y_n).$$

Therefore

$$\mathcal{N}(g+f)(n) \leqslant v(S_n(y_0, \ldots, y_n^{p^n}; x_0, \ldots, x_n^{p^n})^{p^{-n}}) = v(y_n) = \mathcal{N}(g)(n),$$

and the inequality holds at all nodes of $\mathcal{N}(g)$ beyond t_0.

Let $t_1 \geqslant t_0$ be the first node of $\mathcal{N}(g)$. Given $t \geqslant t_1$, let n_1 and n_2 be two consecutive nodes such that $n_1 \leqslant t \leqslant n_2$. On this segment, $\mathcal{N}(g)$ is the straight line connecting $(n_1, v(y_{n_1}))$ and $(n_2, v(y_{n_2}))$. Since $\mathcal{N}(g+f)$ is a convex function lying below $\mathcal{N}(g)$ at the two end points n_1 and n_2, it follows that $\mathcal{N}(g+f)(t) \leqslant \mathcal{N}(g)(t)$, as desired. □

Corollary 4.7. *If $g \in \mathcal{S}_{n+1}$ and $f \in \mathbb{A}$, then $g + fh_n \in \mathcal{S}_{n+1}$.*

Proof. Since $g \in \mathcal{S}_{n+1}$, it follows that $\mathcal{N}(g)$ is strictly decreasing and there exists $m \geqslant 0$ such that $\mathcal{N}(g) \leqslant \mathcal{N}(h_{n+1}^m)$. By Propositions 4.1 and 4.3, for all $t > 2m^2$, we have

$$\mathcal{N}(g)(t) \leqslant \mathcal{N}(h_{n+1}^m)(t) < \mathcal{N}(h_n)(t) \leqslant \mathcal{N}(fh_n)(t).$$

By Proposition 4.6, it follows that for all t sufficiently large,

$$\mathcal{N}(g + fh_n)(t) \leqslant \mathcal{N}(g)(t) \leqslant \mathcal{N}(h_{n+1}^m)(t).$$

By Lemma 2.1, it follows that $g + fh_n \in \mathcal{S}_{n+1}$. □

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References

1. J. T. Arnold, Krull dimension in power series rings, *Trans. Amer. Math. Soc.* **177** (1973), 299–304.
2. B. Bhatt, Specializing varieties and their cohomology from characteristic 0 to characteristic p, in *Algebraic geometry: Salt Lake City 2015*, Proc. Sympos. Pure Math., Volume 97, pp. 43–88 (American Mathematical Society, Providence, RI, 2018).
3. B. Bhatt, M. Morrow and P. Scholze, Integral p-adic Hodge theory, *Publ. Math. Inst. Hautes Études Sci.* **128** (2018), 219–397.
4. L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique, *Astérisque* **406** (2018), xiii+382 pp.
5. Kiran Kedlaya, Some ring-theoretic properties of $\mathbb{A}_{\inf}$, preprint, 2018, arXiv:1602.09016v3.

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