ON THE INJECTIVE AND PROJECTIVE LIMIT OF COMPLEXES

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1. Introduction.

Let R be a commutative ring with unit. Let $(A_\alpha, \varphi_{\beta\alpha})$ $(\alpha \leq \beta)$ (resp. $(B_\alpha, \psi_{\alpha\beta})$ $(\alpha \leq \beta)$) be an injective (resp. projective) system of R-algebras indexed by a directed set I; let $((X_\alpha, d_\alpha), f_{\beta\alpha})$ $(\alpha \leq \beta)$ (resp. $((Y_\alpha, \delta_\alpha), g_{\alpha\beta})$ $(\alpha \leq \beta)$) be an injective (resp. projective) system of complexes, indexed by the same set I, such that for each $\alpha \in I$, (X_α, d_α) (resp. $(Y_\alpha, \delta_\alpha)$) is a complex over A_α (resp. over B_α). The purpose of this paper is to show that the covariant functor $\varinjlim$ from the category of all such injective systems of complexes and complex homomorphisms over the R-algebra $\varinjlim A_\alpha$ is such that it associates with an injective system $((U_\alpha, d_\alpha), h_{\beta\alpha})$ of universal complexes a universal complex over $\varinjlim A_\alpha$ whereas the same is not true of the covariant functor $\varprojlim$ on the category of all such projective systems of complexes and their maps.

The projective system of algebras considered here is (A_k, φ_{k1}) $(k \leq 1)$ where $A_k = K[y]/J_k$, $K[y]$ being the polynomial ring in one indeterminate y over a field K of characteristic p

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and J_k being the ideal generated by y^{p^k} in K[y]. The result that the projective limit of a projective system $((U_k, d_k), f_{k\ell})$ $(k \leq \ell)$, where (U_k, d_k) is a universal complex over A_k for all k, is not a universal complex over $\varprojlim A_k = K\{y\}$ which is the formal power series ring in one indeterminate, is obtained by proving that a universal complex over $K\{y\}$ is infinitely generated.

2. Preliminaries.

Let X and Y be two graded algebras over a commutative ring R with unit. Let $f: X \to Y$ be a graded R-algebra homomorphism. We recall that an R-linear mapping $d: X \to Y$ of X into Y is called an R-derivation of degree 1 if (i) d is a homogeneous R-linear mapping of degree 1; and (ii) for any x, x' in X, with x homogeneous of degree n, $d(xx') = dx \cdot f(x') + (-1)^n f(x)\, dx'$. In particular, if Y = X, then an R-derivation of degree 1 of X into itself is called an R-derivation of degree 1 of X. For any unitary commutative R-algebra A, a pair (X, d), where X is an anticommutative graded R-algebra such that $X_0 = A$ and where $d: X \to X$ is an R-derivation of degree 1 of X such that $d \cdot d = 0$, is called a complex over A. We call a complex (X, d) over A simple if X is generated by dA as an A-algebra. In this case we shall say that the complex (X, d) is generated by A. For any two complexes (X, d), (Y, δ) over A, a graded R-algebra homomorphism $f: X \to Y$ is called a complex homomorphism over A if (i) f maps A identically; and (ii) $f \circ d = \delta \circ f$. We write $f: (X, d) \to (Y, \delta)$. If $f: X \to Y$ is a graded algebra isomorphism then the complex homomorphism $f: (X, d) \to (Y, \delta)$ over A is called a complex isomorphism over A. If A and B are two unitary commutative R-algebras and $\varphi: A \to B$ an R-algebra homomorphism then, for a complex (X, d) over A and a complex (Y, δ) over B, $f: (X, d) \to (Y, \delta)$ is called a φ-complex homomorphism if (i) $f: X \to Y$ is a graded R-algebra homomorphism such that $f = \varphi$ on A; (ii) $f \circ d = \delta \circ f$. Finally, a homogeneous ideal $J \subseteq X$ is called a complex ideal iff $dJ \subseteq J$.

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3. Universal Complexes.

Definition 3.1: A complex (U, d) over an R-algebra A is called universal [5] if, given any other complex (Y, δ) over A, there exists a unique complex homomorphism $f:(u, d) \to (Y, \delta)$ over A.

In the following we shall establish the existence of universal complex over A using the approach contained in Bourbaki's solution of the universal mapping problem [4]. For this we shall first define the product of a family of complexes over A. Let (X_α, d_α) $(\alpha \in I)$ be a family of complexes over A. We know that the product $\prod_\alpha X_\alpha$ is an R-algebra. Set $\bar{A} = \{(a_\alpha) \mid a_\alpha \in A$ and $a_\alpha = a$ for all $\alpha \in I\}$. Then $\bar{A} \subseteq \prod_\alpha X_{\alpha, o}$ and $\bar{A}$ is isomorphic to A under the natural isomorphism $(a_\alpha) \to a$. Form the direct sum $\bar{A} + \sum_{n \geq 1} \prod_\alpha X_{\alpha, n}$ inside $\prod_\alpha X_\alpha$. Here $X_{\alpha, n}$ denotes the homogeneous module of degree $n (n \geq 1)$ of X_α. One can easily verify that $\bar{A} + \sum_{n \geq 1} \prod_\alpha X_{\alpha, n}$ is an anticommutative graded R-algebra containing an isomorphic copy of A as the module of homogeneous elements of degree 0. Denote by X the anti-commutative graded R-algebra obtained from $\bar{A} + \sum_{n \geq 1} \prod_\alpha X_{\alpha, n}$ (dir) by identifying $\bar{A}$ with A under the natural isomorphism $(a_\alpha) \to a$. Now let d be the restriction of the product mapping $(d_\alpha)_{\alpha \in I} : \prod_\alpha X_\alpha \to \prod_\alpha X_\alpha$ to X. Since, for an arbitrary element $x = (a) + (x_{\alpha, 1})_{\alpha \in I} + \ldots + (x_{\alpha, n})_{\alpha \in I}$ in X,

$$dx = (d_\alpha)_{\alpha \in I} (a + (x_{\alpha, 1})_{\alpha \in I} + \ldots + (x_{\alpha, n})_{\alpha \in I})$$

$$= (d_\alpha a)_{\alpha \in I} + (d_\alpha (x_{\alpha, n}))_{\alpha \in I} + \ldots + (d_\alpha (x_{\alpha, n}))_{\alpha \in I},$$

it follows that d maps X into X and is a derivation of degree 1 of X such that $d \, d = 0$. Hence (X, d) is a complex over A. We call (X, d) the product over A of the family (X_α, d_α) $(\alpha \in I)$ of complexes over A.

Remark: Let φ_α be the restriction of the natural projection $\varphi: \prod_\alpha X_\alpha \to X_\alpha$ to $X = A^\alpha + \sum_{n \geq 1} \prod_\alpha X_{\alpha, n}$ (dir). Then $\varphi_\alpha : (X, d) \to (X_\alpha, d_\alpha)$ is a complex homomorphism over A. It is called the natural projection over A.

Now note that for any simple complex (X, d) over A, $|X| \leq |A| \aleph_0$. Therefore, we can choose a representative family (X_α, d_α) $(\alpha \in I)$ of simple complexes over A such that every simple complex over A is isomorphic to some (X_α, d_α) $(\alpha \in I)$. Now form the product (X, d) over A of the family (X_α, d_α) $(\alpha \in I)$ of complexes over A. Denote by u the A-subalgebra of X generated by dA which is a set of homogeneous elements of degree 0. Since u is an anticommutative graded R-algebra such that $u_0 = A$, the complex (u, δ) where δ denotes the restriction of d to u, is a complex over A. Clearly (u, δ) is a simple complex over A. We claim that (u, δ) is a universal complex over A. For this let (Y, ∂) be any simple complex over A. Then (Y, ∂) is isomorphic to some (X_α, d_α) $(\alpha \in I)$. Denote this isomorphism by j. If π_α denotes the restriction of the natural projection $\varphi_\alpha : (X, d) \to (X_\alpha, d_\alpha)$ to (u, δ) then π_α is a complex homormorphism $(u, \delta) \to (X_\alpha, d_\alpha)$ over A. So, $j \circ \pi_\alpha = f$ is a complex homomorphism $(u, \delta) \to (Y, \partial)$ over A. Since u is generated by dA as an A-algebra, f is unique. Since every complex over A contains a simple complex over A, it follows that (u, δ) is a universal complex over A. (u, δ) is obviously unique up to isomorphism.

4. Complexes over the homomorphic image of an algebra.

Let A and B be two unitary commutative R-algebras and let $\Phi : A \to B$ be an R-algebra epimorphism. Let F be the kernel of Φ. Then for any complex (X, d) over A, $(X/J, \bar{d})$, where $J = XF + XdF$ is the complex ideal generated by F in X and $\bar{d}$ is the unique derivation induced by d on X/J, is a complex over B. Also, the natural graded algebra homomorphism $\Phi_X : X \to X/J$ is a Φ-complex homomorphism and, so, extends Φ. Now let (Y, δ) be another complex over A and $(Y/P, \bar{\delta})$ be the

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corresponding complex over B. Then one readily sees that any complex homomorphism $f:(X,d)\to(Y,\delta)$ over A induces a unique complex homomorphism $\bar{f}:(X/J,\bar{d})\to(Y/P,\bar{\delta})$ over B such that $\Phi_Y \circ f = \bar{f} \circ \Phi_X$. Moreover, for any complex homomorphism f' from (Y,δ) over A, $\overline{f' \circ f} = \bar{f'} \circ \bar{f}$. Thus, we have the following proposition.

Proposition 4.1. Every R-algebra epimorphism $\Phi:A\to B$ induces a covariant functor T_Φ from the category of all complexes and complex homomorphisms over A into the category of all complexes and complex homomorphisms over B.

Proposition 4.2. T_Φ is onto and maps the universal complexes over A to the universal complexes over B.

Proof. Let (Y,δ) be any complex over B and ${}_\Phi Y$ be the anticommutative graded algebra obtained from Y by the change of the basic ring to A. Then $X = A \oplus \sum_{n\geq 1} {}_\Phi Y_n$ is an anticommutative graded R-algebra such that the module X_o of homogeneous elements of degree 0 is A. Moreover, the mapping $\partial : X\to X$ given by $\partial_o = \delta_o \circ \Phi$ on A and $\partial_n = \delta_n$ on ${}_\Phi Y_n$ $(n \geq 1)$ is a derivation of degree 1 of X such that $\partial \circ \partial = 0$. Thus (X,∂) is a complex over A. Since $J = XF + X\partial F$ $= (A \oplus \sum_{n\geq 1} {}_\Phi Y_n)F + (A \oplus \sum_{n\geq 1} {}_\Phi Y_n)\partial F = AF = F$, it follows that $X/J = B \oplus \sum_{n\geq 1} Y_n = Y$. One can also easily verify that $\Phi_X \circ \partial = \delta \circ \Phi_X$. Therefore, by the uniqueness of $\bar{\partial}$, it follows that $\delta = \bar{\partial}$. Hence $T_\Phi(X,\partial) = (Y,\delta)$. Next, let (Z,d) be any other complex over B and $(A \oplus \sum_{n\geq 1} {}_\Phi Z_n, D)$ be the corresponding complex over A. Then, for a complex homomorphism $f:(Y,\delta)\to(Z,d)$ over B, $g:(A \oplus \sum_{n\geq 1} {}_\Phi Y_n, \partial)\to(A \oplus \sum_{n\geq 1} {}_\Phi Z_n, D)$ given by g = identity on A and f on $\sum_{n\geq 1} {}_\Phi Y_n$ is a complex homomorphism over A. One immediately sees that $T_\Phi(g) = f$.

Thus T_{Φ} is onto. The second part of the proposition is an immediate consequence of the ontoness of T_{Φ}.

Remark. We know that the collection of all R-algebras together with their epimorphisms forms a category $\mathcal{A}(R)$. We also know that with every R-algebra A we can associate the category $\mathcal{C}(A)$ of all complexes over A. Moreover, Propositions 4.1, 4.2 say that with every epimorphism $\Phi: A \to B$ between two R-algebras A and B we can associate a covariant functor $T_{\Phi}: \mathcal{C}(A) \to \mathcal{C}(B)$ which is again onto. Thus, we get a correspondence $\Phi \to T_{\Phi}$ which has the following two properties:

(i) If Φ is the identity mapping $I: A \to A$, then $T_I: \mathcal{C}(A) \to \mathcal{C}(A)$ is the identity mapping.

(ii) If C is another R-algebra, and $\psi: B \to C$ an R-algebra epimorphism, then the functors $T_{\psi \circ \Phi}$ and $T_{\psi} \circ T_{\Phi}$ are naturally equivalent, the natural equivalence being given by the canonical isomorphism $X/M \to \bar{\bar{X}}$ where M is the complex ideal generated by $\ker(\psi \circ \Phi)$ in X; $\bar{\bar{X}} = (X/J)/N$, N being the complex ideal generated by the $\ker(\psi)$ in X/J.

5. The Injective Limit of Complexes.

Let (X_α) $(\alpha \in I)$ be a family of anticommutative graded R-algebras indexed by I. For each $\alpha \leq \beta$, let $f_{\beta\alpha}: X_\alpha \to X_\beta$ be a graded algebra homomorphism such that $(X_\alpha, f_{\beta\alpha})$ $(\alpha \leq \beta)$ is an injective system. In [2] it is shown that $X = \sum_{n \geq 0} \varinjlim_{\alpha} X_{\alpha, n}$ (dir) is an anticommutative graded R-algebra and is the injective limit of the system $(X_\alpha, f_{\beta\alpha})$ $(\alpha \leq \beta)$.

Remarks. (1) Let $(A_\alpha, \varphi_{\beta\alpha})$ be an injective system of R-algebras indexed by a directed set I. If for each $\alpha \in I$, X_α is such that the module $X_{\alpha, 0}$ of homogeneous elements of degree 0 of X_α is equal to A_α then $X_0 = \varinjlim A_\alpha$ and X is an anticommutative graded algebra over $\varinjlim A_\alpha$.

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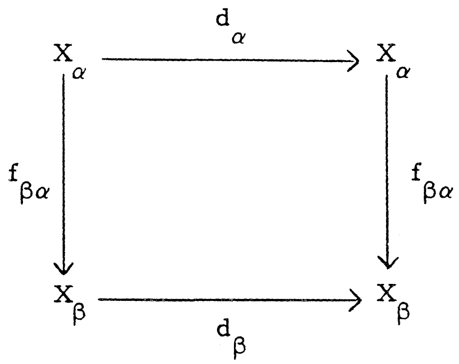
(2) For each $\alpha \in I$, the natural mapping $\varphi_\alpha : X_\alpha \to X$ is a graded R-algebra homomorphism.

Let I be a directed set and let $(A_\alpha, \varphi_{\beta\alpha})$ $(\alpha \leq \beta)$ be an injective system of R-algebras indexed by the set I. Let $((X_\alpha, d_\alpha), f_{\beta\alpha})$ $(\alpha \leq \beta)$ be an injective system of complexes such that for each $\alpha \in I$, (X_α, d_α) is a complex over A_α. Then $(X_\alpha, f_{\beta\alpha})$ is an injective system of anticommutative graded algebras. Set $X = \varinjlim X_\alpha$. By Remark 1, $X_o = \varinjlim A_\alpha$ and X is an anticommutative graded algebra over $\varinjlim A_\alpha$. Also, for each $\alpha \leq \beta$ in I, the following diagram commutes:



(since $f_{\beta\alpha}$ are complex homomorphisms). So (d_α) $(\alpha \in I)$ is an injective system of mappings. Now, for each $\alpha \in I$, set $\lambda_\alpha = \varphi_\alpha \circ d_\alpha$, where $\varphi_\alpha : X_\alpha \to X$ is the natural graded algebra homomorphism. Then, for $\alpha \leq \beta$ in I,

$$\lambda_\beta \circ f_{\beta\alpha} = \varphi_\beta \circ d_\beta \circ f_{\beta\alpha} = \varphi_\beta \circ f_{\beta\alpha} \circ d_\alpha = \varphi_\alpha \circ d_\alpha = \lambda_\alpha$$

implies the existence of the unique mapping $\varinjlim d_\alpha = d$ of X into itself such that $\varphi_\alpha \circ d_\alpha = d \circ \varphi_\alpha$ for all $\alpha \in I$. One can easily verify that d is an R-derivation of degree 1 of X such that

$d \circ d = 0$. Therefore, (X, d) is a complex over $\varinjlim A_\alpha$.

Definition 5.1. We call (X, d) the injective limit of the injective system $((X_\alpha, d_\alpha), f_{\beta\alpha})$ $(\alpha \leq \beta)$, and we shall write $(X, d) = \varinjlim (X_\alpha, d_\alpha)$.

Remark. If each (X_α, d_α) is a complex over A, then (X, d) is a complex over A.

For an injective system $(A_\alpha, \varphi_{\beta\alpha})$ $(\alpha \leq \beta)$ of R-algebras indexed by a set I, let $\mathcal{I}$ be the category whose objects are the injective systems $((X_\alpha, d_\alpha), f_{\beta\alpha})$ $(\alpha \leq \beta)$ of complexes (indexed by I) such that for each α in I (X_α, d_α) is a complex over A_α, and whose maps are the maps between these injective systems.

Proposition 5.1. The covariant functor $\varinjlim$ from $\mathcal{I}$ into $\mathcal{C}(A)$, the category of all complexes over $A = \varinjlim A_\alpha$, is onto.

Proof. Let (X, d) be a complex over $A = \varinjlim A_\alpha$, and let $\psi_\alpha : A_\alpha \to A$ be the natural mappings $(\alpha \in I)$. We know that for each $\alpha \in I$, X can be made into an A_α-module by defining the scalar multiplication by the elements of A_α as follows:

$$a_\alpha x = \psi_\alpha(a_\alpha)x \text{ for each } a_\alpha \text{ in } A_\alpha \text{ and } x \text{ in } X.$$

Thus, for each $\alpha \in I$, (X_α, d_α) with $X_{\alpha 0} = A_\alpha$; $X_{\alpha,n}$ = the A_α-module X_n $(n \geq 1)$; and $d_{\alpha,0} = d_0 \circ \psi_\alpha$ on A_α, $d_{\alpha,n} = d_n$ on $X_{\alpha,n}$ $(n \geq 1)$ is a complex over A_α. So we have a family (X_α, d_α) $(\alpha \in I)$ of complexes such that for each $\alpha \in I$, (X_α, d_α) is a complex over A_α. Now, for each $\alpha \leq \beta$ in I, $h_{\beta\alpha} : X_\alpha \to X_\beta$ be the mapping given by $h_{\beta\alpha} = \varphi_{\beta\alpha}$ on A_α and identity on $\sum_{n \geq 1} X_n$. Then, since each $x \in X_\alpha$ can be written as $a_\alpha + x_\alpha$ with

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$a_\alpha \in A_\alpha$, $x_\alpha \in \sum_{n\geq 1} X_n$, it follows from the definitions of d_α and $h_{\beta\alpha}$, that $h_{\beta\alpha}: (X_\alpha, d_\alpha) \to (X_\beta, d_\beta)$ is a $\varphi_{\beta\alpha}$-complex homomorphism $(\alpha \leq \beta)$. Moreover, $((X_\alpha, d_\alpha), h_{\beta\alpha})$ $(\alpha \leq \beta)$ is an injective system of complexes indexed by I; since $\alpha \leq \beta$ and I directed implies the existence a of $\delta \in I$ with $\delta \geq \alpha, \beta$ and for this δ, $h_{\delta\beta} h_{\beta\alpha}(x) = h_{\delta\beta} h_{\beta\alpha}(a_\alpha + x_\alpha) = \varphi_{\delta\beta} \varphi_{\beta\alpha}(a_\alpha) + x_\alpha = \varphi_{\delta\alpha}(a_\alpha) + x_\alpha$ $= h_{\delta\alpha}(a_\alpha + x_\alpha) = h_{\delta\alpha}(x)$. Thus $((X_\alpha, d_\alpha), h_{\beta\alpha})$ $(\alpha \leq \beta)$ belongs to $\mathcal{A}$. Now we claim that $\varinjlim (X_\alpha, d_\alpha) = (X, d)$. For this we recall that (i) $\varinjlim X_\alpha = \varinjlim A_\alpha + \sum_{n\geq 1} \varinjlim_{\alpha} X_{\alpha, n} = \varinjlim A_\alpha + \sum_{n\geq 1} \varinjlim_{\alpha} X_n$ $= A \oplus \sum_{n\geq 1} X_n = X$; and (ii) the natural graded R-algebra homomorphism $\varphi_\alpha : X_\alpha \to X$ is given by $\varphi_\alpha = \psi_\alpha$ on A_α and identity on $X_{\alpha, n}$ $(n \geq 1)$. From (ii) and the definition of d_α it follows that $\varphi_\alpha \circ d_\alpha = d \circ \varphi_\alpha$ $\alpha \in I$. Therefore, by the uniqueness of $\varinjlim d_\alpha$, one has $d = \varinjlim d_\alpha$ and, hence $\varinjlim (X_\alpha, d_\alpha) = (X, d)$. Thus, it only remains to show that if (Y, δ) is another complex over A and $f: (X, d) \to (Y, \delta)$ is a complex homomorphism over A then there exists a map (g_α) $(\alpha \in I)$ from the injective system $((X_\alpha, d_\alpha), h_{\beta\alpha})$ to the corresponding injective system $((Y_\alpha, \delta_\alpha), g_{\beta\alpha})$ such that $\varinjlim g_\alpha = f$. This however follows from the fact that for each α in I, the mapping g_α: $A_\alpha \oplus \sum_{n\geq 1} X_n \to A_\alpha \oplus \sum_{n\geq 1} Y_n$ given by g_α = identity on A_α and f on $\sum_{n\geq 1} X_n$ is a complex homomorphism $(X_\alpha, d_\alpha) \to (Y_\alpha, \delta_\alpha)$ over A_α, such that $g_{\beta\alpha} \circ g_\alpha = g_\beta \circ h_{\beta\alpha}$ for each $\alpha \leq \beta$ in I; and that $f \circ \varphi_\alpha = \psi'_\alpha \circ g_\alpha$ $(\alpha \in I)$ where $\psi'_\alpha : Y_\alpha \to Y$ is the natural graded algebra homomorphism. Hence $\varinjlim$ is onto.

<u>Proposition 4.2.</u> Let $((U_\alpha, d_\alpha), f_{\beta\alpha})$ be an injective system of complexes such that for each α in I, (U_α, d_α) is a universal complex over A_α. Then $(U, d) = \varinjlim (U_\alpha, d_\alpha)$ is a universal complex over $\varinjlim A_\alpha = A$.

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Proof. Let (Y, δ) be any complex over A. Then by the ontoness of the covariant functor $\varinjlim$ there exists an injective system $((Y_\alpha, \delta_\alpha), g_{\beta\alpha})$ $(\alpha \leq \beta)$ such that $(Y, \delta) = \varinjlim (Y_\alpha, \delta_\alpha)$. Since for each $\alpha \in I$, (U_α, d_α) is a universal complex over A_α, there exists a unique complex homomorphism $f_\alpha : (U_\alpha, d_\alpha) \to (Y_\alpha, \delta_\alpha)$ $(\alpha \in I)$ over A_α. It can easily be checked that for each $\alpha \leq \beta$ in I, $g_{\beta\alpha} \circ f_\alpha = f_\beta \circ f_{\beta\alpha}$; thus (f_α) $(\alpha \in I)$ is an injective system of mappings. Set $f = \varinjlim f_\alpha$. Then, clearly, $f:(U, d) \to (Y, \delta)$ is a complex homomorphism over A. Uniqueness of f follows from the fact that U is generated by dA as an A-algebra.

6. The Projective Limit of Complexes.

Let (X_α) $(\alpha \in I)$ be a family of anticommutative graded R-algebras. It is known that their cartesian product $\prod_\alpha X_\alpha$ is again an R-algebra. Inside $\prod_\alpha X_\alpha$, we form the sum $\sum_{n \geq 0} \prod_\alpha X_{\alpha, n} = \mathcal{P}_\alpha X_\alpha$ of the R-modules $\prod_\alpha X_{\alpha, n}$, where $X_{\alpha, n}$ denotes the module of homogeneous elements of degree n of X_α $(n \geq 0)$. Clearly, this sum is direct and $\mathcal{P}_\alpha X_\alpha$ is an anticommutative graded R-algebra. We call $\mathcal{P}_\alpha X_\alpha$ the product of the family (X_α) $(\alpha \in I)$ of anticommutative graded R-algebras. If φ_α denotes the restriction of the natural projection $\prod_\alpha X_\alpha \to X_\alpha$ to $\mathcal{P}_\alpha X_\alpha$, then φ_α is a graded R-algebra epimorphism and we call it the natural projection.

Now, let $(X_\alpha, f_{\alpha\beta})$ $(\alpha \leq \beta)$ be a projective system of anticommutative graded R-algebras, and set $X = \{x \mid x \in \mathcal{P}_\alpha X_\alpha,\ \varphi_\alpha(x) = f_{\alpha\beta} \circ \varphi_\beta(x)\}$. Then, from a straightforward computation it follows that $X = \sum_{n \geq 0} \varprojlim_\alpha X_{\alpha, n}$ (dir) and X is an anticommutative graded R-algebra.

Definition 6.1. $X = \sum_{n \geq 0} \varprojlim_{\alpha} X_{\alpha, n}$ (dir) will be called projective limit of the projective system $(X_{\alpha}, f_{\alpha\beta})$.

Remark. Let $(A_{\alpha}, \varphi_{\alpha\beta})$ $(\alpha \leq \beta)$ be a projective system of R-algebras indexed by a directed set I. Let $(X_{\alpha}, f_{\alpha\beta})$ $(\alpha \leq \beta)$ be a projective system of anticommutative graded R-algebras also indexed by I. If each X_{α} is such that $X_{\alpha, o} = A_{\alpha}$, then $X_{o} - \varprojlim A_{\alpha}$.

Next, let $((X_{\alpha}, d_{\alpha}), f_{\alpha\beta})$ $(\alpha \leq \beta)$ be a projective system of complexes (indexed by a directed set I) such that for each $\alpha \in I$ (X_{α}, d_{α}) is a complex over A_{α} where $(A_{\alpha}, \varphi_{\alpha\beta})$ $(\alpha \leq \beta)$ is a projective system of R-algebras indexed by I. We note that the restriction δ of the mapping $(d_{\alpha})_{\alpha \in I} : \Pi X_{\alpha} \to \Pi X_{\alpha}$ to $\mathbb{P} X_{\alpha}$ is a derivation of degree 1 of $\mathbb{P} X_{\alpha}$ such that $\delta \circ \delta = 0$; thus, $(\mathbb{P} X_{\alpha}, \delta)$ is a complex over ΠA_{α} and the natural projections $\mathcal{P}_{\alpha} : \mathbb{P} X_{\alpha} \to X_{\alpha}$ are such that $\mathcal{P}_{\alpha} \circ \delta = \delta_{\alpha} \circ \mathcal{P}_{\alpha}$ $(\alpha \in I)$. Denote by d the restriction of δ to X. Then, for any x in X,

$$\mathcal{P}_{\alpha}(dx) = d_{\alpha}(\mathcal{P}_{\alpha}(x)), \quad \text{(since } d \text{ is the restriction of } \delta)$$

$$= d_{\alpha} f_{\alpha\beta} \mathcal{P}_{\beta}(x)$$

$$= f_{\alpha\beta} d_{\beta} \mathcal{P}_{\beta}(x), \quad \text{(since } f_{\alpha\beta} \text{ is a } \varphi_{\alpha\beta}\text{-complex homomorphism)}$$

$$= f_{\alpha\beta} \mathcal{P}_{\beta}(dx);$$

that is, $dx \in X$. Thus d maps X into itself. Clearly, d is a derivation of degree 1 of X such that $d \circ d = o$. Therefore, in view of the Remark 1, (X, d) is a complex over $\varprojlim A_{\alpha}$.

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Definition 6.2. We call (X, d), the projective limit of the projective system $((X_\alpha, d_\alpha), f_{\alpha\beta})$.

We need the following lemma:

Lemma 6.1. Let k be a field and let K be a purely transcendental extension of k such that the degree of transcendence of K over k is infinite. Let (U, d) be a universal complex over K. Then the dimension of U_1, which is the module of homogeneous elements of degree 1 of U, over K is equal to the degree of transcendence of K over k.

Proof. K a purely transcendental extension of k implies K is isomorphic to the quotient field of a polynomial ring $K[X]$ where X is a set of indeterminates such that cardinality of X = degree of transcendence of K over k. We recall that a universal complex (V, δ) over $K[X]$ is such that V_1 is a free $K[X]$-module on the set $\{ \delta x | x \in X\}$. Therefore, $U_1 = K \underset{k[X]}{\otimes} V_1$ is a free K-module on the set $\{ 1 \otimes \delta x | x \in X\}$ [1]. Hence, dimension of U_1 over K = degree of transcendence of K over k.

Remark. This lemma generalises Kähler's result [3] for finitely generated fields to the fields of arbitrary infinite degree of transcendence.

Now we are in a position to give the desired examples. Let K be a field of characteristic p, and let $K[y]$ be a polynomial ring in one indeterminate y over K. Let J_k be the ideal generated by y^{p^k} in $K[y]$, and let $A_k = K[y]/J_k$ $(k = 1, 2, 3, \ldots)$. We know that for each $k \leq \ell$, $J_\ell \subseteq J_k$; therefore, there exists a natural K-algebra homomorphism $\varphi_{k\ell} : A_\ell \to A_k$ such that $\varphi_{k\ell}(a + J_\ell) = a + J_k$ for all a in $K[y]$. Clearly, $(A_k, \varphi_{k\ell})$ is a projective system of K-algebras. If $A = \varprojlim A_k$, then it is known that A is the ring of formal power series in one indeterminate y over K([5]); that is, $A = K\{y\}$. For each $k \geq 1$, let (U_k, d_k) be a universal complex over A_k. Since, for each $k \geq 1$, A_k is generated over K by the single

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element $y_k = y + J_k$, it follows that U_k is generated by the single element $w_k = d_k y_k$ as an A_k-algebra.

Now, let (V, δ) be a universal complex over $K[y]$. Since $\delta(y^{p^k}) = 0$ for each k, the complex ideal generated by J_k in V is $J_k + J_k \delta y$; and, so by Propositions 4.1 and 4.2, (U_k, d_k) is isomorphic to $(V/J_k + J_k \delta y, \bar{\delta}_k) = (V_k, \bar{\delta}_k)$ which gives the linear independence of w_k over A_k $(k \geq 1)$. Moreover, for each $k \leq \ell$, there exists a $\varphi_{k\ell}$-complex homomorphism $f_{k\ell} : (V_\ell, \bar{\delta}_\ell) \to (V_k, \delta_k)$ such that $f_{k\ell}(\delta y + J_\ell + J_\ell \delta y) = \delta y + J_k + J_k \delta y$. So $((V_k, \bar{\delta}_k), f_{k\ell})$ $(k \leq \ell)$ is a projective system of complexes. Since for each ℓ, (U_ℓ, d_ℓ) is isomorphic to $(V_\ell, \bar{\delta}_\ell)$, it follows that for each $k \leq \ell$ there exists a $\varphi_{k\ell}$-complex homomorphism $g_{k\ell} : (U_\ell, d_\ell) \to (U_k, d_k)$ such that $g_{k\ell}(w_\ell) = w_k$ and $((U_k, d_k), g_{k\ell})$ is a projective system of complexes. Let (U, d) be the projective limit of the projective system $((U_k, d_k), f_{k\ell})$. Then U is generated by the family $(w_k)_k$ as $\varprojlim A_k = K\{y\}$ algebra. For, let $u \in U$ be arbitrary. Then $u = (u_k)_k$, $u_k \in U_k$ $(k \geq 1)$ and $\mathcal{P}_k(u) = g_{kj}\, \mathcal{P}_j(u)$ for all $k \leq j$. That is, $u_k = g_{kj}(u_j)$, and so $a_k w_k = g_{kj}(a_j w_j) = \varphi_{kj}(a_j) w_k$. Since w_k is linearly independent over A_k, it follows that $a_k = \varphi_{kj}(a_j)$; that is $\mathcal{P}_k((a_k)_k) = \varphi_{kj}\, \mathcal{P}_j((a_k)_k)$, and so $(a_k)_k \in \varprojlim A_k$. Thus $u = (a_k)_k (w_k)_k$ implies that U is generated by the single element $(w_k)_k$ over $K\{y\}$. We claim that a universal complex over $K\{y\}$ is infinitely generated. Since $K\{y\}$ is an integral domain we can form its field of quotients, which we denote by $K((y))$. Then $K((y))$ is a purely transcendental extension of K and the degree of transcendence of $K((y))$ over K is equal to the cardinality of K^N where N is the set of natural numbers (see [3], ex. 13, Sec. 5). Now let (W, D)

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be a universal complex over $K\{y\}$. If S denotes the set of all non-zero elements of $K\{y\}$, then (W_S, D_S) is a universal complex over $K((y))$ where $W_S = K((y)) \otimes_{K\{y\}} W$ and $D_S: W_S \to W_S$ is the derivation given as follows:

for any $\frac{1}{s} \otimes x$ in W_s, homogeneous of degree n,

$$D_S(\frac{1}{s} \otimes x) = \frac{1}{s} \otimes Dx - (-1)^n \frac{1}{s^2} \otimes D_S .$$

Since the degree of transcendence of $K((y))$ over K is infinite, by Lemma 6.1, the dimensions of the module $(W_S)_1$ of homogeneous elements of degree 1 of W_S over $K((y))$ is infinite. Thus, W_S is not finitely generated over $K((y))$. If W is finitely generated over $K\{y\}$, then $W_S = K((y)) \otimes_{K\{y\}} W$ will imply that W_S is finitely generated over $K((y))$; a contradiction. Hence W is not finitely generated over $K\{y\}$, and so (U, d) is not isomorphic to (W, D). Hence (U, d) is not a universal complex over $K\{y\}$.

Remark. This example also proves that if $((M_\alpha, d_\alpha), f_{\alpha\beta})$ $(\alpha \leq \beta)$ is a projective system of universal derivation modules such that for each α, M_α is an A_α-module then the projective limit $(\varprojlim M_\alpha, d)$ need not be a universal derivation module over $\varprojlim A_\alpha = A$.

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