SOME RESULTS FOR QUADRATIC ELEMENTS
OF A BANACH ALGEBRA

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Abstract. Several properties of some quadratic elements of a unitial Banach algebra are studied. Deddens subspaces are also introduced and discussed.

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1. Introduction. Let $A$ be a complex Banach algebra with unit $e$. An element $a \in A$ is called quadratic if it satisfies $a^2 + \lambda_1 a + \lambda_2 e = 0$ for some scalars $\lambda_1$ and $\lambda_2$. Observe that idempotents and nilpotent elements of order 2 are quadratic elements.

Our main goal in this paper is to study some properties of such elements. See [1, 2, 3, 4] for concrete applications.

2. Deddens subspaces. Let $A$ be a Banach algebra with a unit $e$. For any two invertible elements $a_1, a_2 \in A$ put

$$D_{a_1, a_2} \stackrel{\text{def}}{=} \left\{ x \in A : \sup_{n \geq 0} \| a_1^n x a_2^{-n} \| \stackrel{\text{def}}{=} c_x < \infty \right\}.$$ 

We call the subspaces $D_{a_1, a_2}$ and $D_{a_2, a_1}$ the Deddens subspaces. Note that, when $a_1 = a_2$ the notion of Deddens subspace coincides with the notion of Deddens algebra, introduced in [4].

Our main result in this section is the following theorem.

THEOREM 1. Let $A$ be a Banach algebra with unit $e$. Let $p$ be any idempotent and $q$ a nilpotent of order 2, respectively. We have

(a) $D_{e+p, e+q} = \{ x \in A : px = xq \}$,

(b) $D_{e+q, e+p} = \{ x \in A : qx = xp \}$.

Proof. (a) Let us denote $\text{Intertw} \{ p, q \} = \{ x \in A : px = xq \}$. The inclusion $\text{Intertw} \{ p, q \} \subset D_{e+p, e+q}$ is obvious. To prove the reverse inclusion $D_{e+p, e+q} \subset \text{Intertw} \{ p, q \}$, let $x \in D_{e+p, e+q}$ be any element. Putting

$$c_n = a_1^n x a_2^{-n} \quad (n \geq 0),$$

where $a_1 = e + p$, $a_2 = e + q$, we deduce that

$$\| c_n \| \leq c_x \quad (n \geq 0).$$

(1)
We have
\[ c_n a_2 = d_1^n x d_2^{-n} a_2 = d_1 \left( d_1^{n-1} x d_2^{-n+1} \right) = a_1 c_{n-1}; \]
that is
\[ c_n a_2 = a_1 c_{n-1} \quad (n \geq 1). \tag{2} \]
From (2) we obtain
\[ c_n a_2^2 = d_1^n c_0 \quad (n \geq 1) \]
or
\[ c_n (e + q)^n = (e + p)^n c_0 = (e + (2^n - 1)p) x \quad (n \geq 1); \]
that is
\[ c_n = (e + (2^n - 1)p) x (e - n q) \quad (n \geq 1). \]
From this equality, we deduce that
\[ c_n - x = (2^n - 1)p x - n xq - n(2^n - 1)p xq, \tag{3} \]
for all \( n \geq 1 \). By taking the equality (1) into account, it follows from (3) that
\[ \| p x q \| \leq \frac{\| c_n - x \|}{n(2^n - 1)} + \frac{\| p x \|}{n} + \frac{\| x q \|}{2^n - 1} \rightarrow 0 \]
as \( n \rightarrow \infty \). Hence \( p x q = 0 \), and therefore
\[ c_n - x = (2^n - 1)p x - n xq. \]
From this we obtain
\[ \| p x \| \leq \frac{\| c_n - x \|}{2^n - 1} + \frac{n}{2^n - 1} \| x q \| \rightarrow 0 \]
as \( n \rightarrow \infty \). Hence \( p x = 0 \). Therefore \( c_n - x = -n xq \), which implies that
\[ \| x q \| = \frac{\| c_n - x \|}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty \]
and so \( x q = 0 \). Hence \( c_n - x = 0 \) \( (n \geq 1) \). In particular \( c_1 = x \), so that \( x = (e + p) x (e - q) \). Hence, \( (e + p) x = x (e + q) \). Therefore \( p x = x q \), which means that \( x \in \text{Intert} \{ p, q \} \), and so \( D_{e+p, e+q} \subset \text{Intert} \{ p, q \} \), which completes the proof of (a).

(b) The proof is very similar to that of (a) and is omitted.

**Corollary 2.** Let \( A \) be a complex Banach algebra with unit \( e \). Let \( p \) be any idempotent and \( q \) a nilpotent of order 2, respectively. We have

\[ (D_{e+p, e+q} \cap D_{e+q, e+p}) \cap \{ p \}' = (D_{e+p, e+q} \cap D_{e+q, e+p}) \cap \{ q \}'. \]

Here \( \{ t \}' \) stands for the commutant of \( t \).
Let \( A \) be a Banach algebra with the idempotent \( p \) and with a unit \( e \). Define the set \( S_p \) as follows:

\[
S_p = \{ x \in A : px(e - p) = 0 \}.
\]

By analogy with the proof of Theorem 1, we can state directly the following theorem.

**Theorem 3.** Let \( A \) be a Banach algebra with an idempotent \( p \) and with a unit \( e \). Then \( D_{e+p,e+p} \) is an algebra and \( D_{e+p,e+p} = S_p \); thus the Deddens algebra \( D_{e+p,e+p} \) coincides with the algebra \( S_p \).

**Proof.** It follows from the definition of Deddens subspaces that \( D_{e+p,e+p} \) is an algebra. For the second statement of the theorem, it is easy to check that \((e + p)^{-1} = e - \frac{1}{2}p\). Therefore

\[
(e + p)^n = e + (2^n - 1)p \quad (n \geq 0)
\]

and

\[
(e + p)^{-n} = e + \left(\frac{1}{2^n} - 1\right)p \quad (n \geq 0).
\]

By setting

\[
c_n = (e + p)^n x (e + p)^{-n} \quad (n \geq 0),
\]

where \( x \) is any element of \( A \), we obtain

\[
c_n(e + p) = (e + p) \left[ (e + p)^{n-1} x \left( e - \frac{1}{2^n}p \right)^{n-1} \right] = (e + p)c_{n-1},
\]

for every \( n \geq 1 \), which implies that

\[
c_n(e + p)^n = (e + p)^n c_0 \quad (n \geq 0).
\]

By applying equalities (4), (5) we have

\[
c_n = x + \left( \frac{1}{2^n} - 1 \right) xp + (2^n - 1)p x + (2^n - 1) \left( \frac{1}{2^n} - 1 \right) pxp,
\]

\( n = 0, 1, 2, \ldots \). Then, for every \( x \in D_{e+p,e+p} \), it follows that

\[
0 = \lim_{n \to \infty} \frac{1}{(2^n - 1)\left( \frac{1}{2^n} - 1 \right)} (c_n - x)
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{2^n - 1} px + \frac{1}{2^n - 1} xp + pxp \right)
\]

\[
= pxp - px,
\]
or equivalently
\[ px(e - p) = 0, \]
i.e., \( x \in S_p. \)

Conversely, if \( x \in S_p \), then again it is clear from the equality (6) that
\[
\|c_n\| = \left\| x + \left( \frac{1}{2^n - 1} \right) xp + \frac{2^n - 1}{2^n} pxp \right\|
\leq \|x\| + \|xp\| + \|pxp\|
= c_x < +\infty,
\]
for any \( n \geq 0 \). Hence \( x \in D_{e+p,e+p}. \)

Prior to stating two corollaries of Theorem 3, we require some terminology and notation.

Let \( A \) be a complex Banach algebra with unit \( e \). An element \( a \in A \), is said to be a regular von Neumann element if there exists \( b \in A \), such that \( a = aba. \)

It is obvious that the invertible elements of \( A \) are regular von Neumann elements. Also, in the special case in which \( A = B(H) \), the algebra of bounded linear operators on a complex Hilbert space \( H \), all isometries of \( B(H) \) are regular von Neumann elements of \( B(H) \).

For any non invertible regular von Neumann element \( a \), define
\[
q_a = a(e - ab), \quad p_a = ab,
\]
where \( b \in A \) and \( a = aba. \) Clearly \( q_a^2 = 0 \) and \( p_a^2 = p_a. \)

**Corollary 4.** Let \( A \) be a complex Banach algebra with unit \( e \). Then \( D_{e+p_a,e+p_a} = S_{p_a}. \)

**Proof.** This follows at once from Theorem 3.

Finally, we give one more result on quadratic elements \( q_a \) and \( p_a^+ \) \( e - p_a. \)

**Proposition 5.** Let \( A \) be a Banach algebra with unit \( e \). Suppose that \( x, y, a \in A \) are elements such that \( a \) is a regular element and
\[
xa - ay = q_ax. \tag{7}
\]
If \( \sigma(x) = \{0\} \) (that is, \( x \) is a quasinilpotent element), then \( \sigma(p_a^+x) = \{0\}. \)

**Proof.** By induction on \( n \), we prove that
\[
(p_a^+x)^n = p_a^+x^n, \tag{8}
\]
for every \( n \geq 1 \). For \( n = 1 \), the assertion is obvious. Let \( n > 1 \) and let \( (p_a^+)x^{n-1} = p_a^+x^{n-1}. \) The regularity of element \( a \) implies that \( p_a^+a = 0. \) Then by using the condition (7) we have
\[
(p_a^+x)^n = p_a^+x(p_a^+)x^{n-1} = p_a^+x p_a^+x^{n-1} = p_a^+x(e - ab)x^{n-1}
= p_a^+x^n - p_a^+x(ay + q_ax)b x^{n-1}
= p_a^+x^n - p_a^+a(y + p_a^+x)bx^{n-1} = p_a^+x^n.
\]
Thus, the equality (8) is proved. From (8) the assertion of the proposition is obvious.
This completes the proof.
8. Remarks. (a) We recall that for any two elements \(x, a\) of the Banach algebra \(\mathcal{A}\), the well-known “Kleinecke-Shirokov” condition \([x, [x, a]] = 0\) implies the quasinilpotency of the commutator \([x, a] \defeq xa - ax\). See [5], [6]. In particular, the condition

\[
[x, a] = x
\]

implies that \([x, a]\) is a nilpotent element. It is known that (9) is not a necessary condition for the nilpotency of \([x, a]\). The condition (7) of Proposition 5, in particular, gives such an example. Indeed, the relation (7) implies that \((xa - ay)^2 = 0\). Therefore, when \(y = x\), \((xa - ax)^2 = 0\), but clearly \(xa - ax = qa \not\equiv x\).

(b) It should be mentioned that the statement of the type “\(\sigma(x) = \{0\} \Rightarrow \sigma(ax) = \{0\}\)” is of importance in many problems of Banach algebra theory and operator theory. The Shulman’s paper [6] is a good reference in this sense. In particular, in [6] the following question is raised.

**Question.** Let elements \(x, a\) of Banach algebra satisfy the conditions

\[
[x, [x, a]] = 0
\]

and \(\sigma(x) = \{0\}\). Is it true that \(\sigma(ax) = \{0\}\)?

3. Reducing subspaces. Let \(H\) be a complex Hilbert space and \(\mathcal{B}(H)\) the algebra of bounded linear operators on \(H\).

**Corollary 6.** \(\text{AlgLat}Q = \bigcap_{E \in \text{Lat}Q} \mathcal{D}_{I+P_E, I+P_E}\), where \(Q\) is a subset of \(\mathcal{B}(H)\), \(\text{Lat}Q\) the lattice of closed subspaces \(E\) of \(H\) invariant under \(Q\), \(P_E\) is the orthogonal projection of \(H\) onto \(E\) and \(\text{AlgLat}Q = \{T \in \mathcal{B}(H) : TE \subseteq E\} \text{ for all } E \in \text{Lat}Q\).

**Proof.** This follows at once from Theorem 3.

We recall that a reducing subspace of a bounded linear operator \(T\) on \(H\) is a common invariant subspace for \(T\) and \(T^*\). It is known that a subspace \(E \subseteq H\) is a reducing subspace for \(T\) if and only if \(TP_E = P_ET\), where \(P_E\) is the orthogonal projection of \(H\) onto \(E\).

Allan and Zemanek proved in [2, Corollary 9] that every quadratic operator on \(H\) has a reducing subspace. Our next theorem describes the reducing subspaces of a nilpotent operator on a Hilbert space \(H\) in terms of \(C_Q\) classes. We first recall the definition of the \(C_Q\) class. Let \(S\) be a positive linear operator on a Hilbert space \(H\). There are positive real numbers \(m\) and \(M\) and \(Q \in \mathcal{B}(H)\) such that \(0 < mI \leq S \leq MI\) and \(Q = S^{-1/2}\). Then

\[
C_Q = \{T \in \mathcal{B}(H) : QT^nQ = P_HU^n|H, n = 1, 2, \ldots\},
\]

where \(U\) is a unitary operator on some Hilbert space \(K \supset H\). Note that \(T \in C_Q\) if and only if \(T\) satisfies the condition:

\[
(Sh, h) + 2Re(z(I - S)Th, h) + |z|^2((S - 2I)Th, Th) \geq 0,
\]

for any \(h \in H\) and \(z \in \mathbb{C}\), \(|z| \leq 1\). The classes \(C_Q\) were defined by Langer. See [7, p. 55].

**Theorem 7.** Let \(N \in \mathcal{B}(H)\) be a nilpotent operator. The subspace \(E \subseteq H\) is a reducing subspace of the operator \(N\) if and only if \(E = T^{k-1}H\), for some operator.
$T$ belonging to some class $C_Q$ and for some integer $k \geq 2$ satisfying $T^k = (I + N)T(I + N)^{-1}$.

**Proof.** The first part of the theorem is obvious. Indeed, if $E \subset H$ is a reducing subspace of $N$, then $P_E N = NP_E$, where $P_E$ is the orthogonal projection of $H$ onto $E$, and hence, $E = P_E H$, $T = P_E$, $Q = I$ and $k = 2$.

We now prove the “only part” of the theorem.

From the condition $T^k = (I + N)T(I + N)^{-1}$ it is easy to see that

$$T^k(I + N) = (I + N)T^{k-1},$$

for all $n \geq 1$ Since $T \in C_Q$, for some $Q$, then we have that

$$\|T^k\| \leq \|Q^{-1}\|^2,$$

and hence, from (10) and (11) by the result of Deddens and Wong [8, Lemma 2] we assert that $T^k = T$, see also [4, Lemma 2]. Therefore $TN = NT$, and hence, $T^{k-1}N = NTK^{-1}$. On the other hand,

$$T^{2(k-1)} = T^k T^{k-2} = TT^{k-2} = T^{k-1};$$

that is, $T^{k-1}$ is a projection. $T^{k-1}$ is an orthogonal projection, by [9], since $T \in C_Q$ and so the equality $T^{k-1}N = NT^{k-1}$ means that the subspace $T^{k-1}H$ reduces $N$; that is, $E$ reduces $N$ which completes the proof of theorem.

Before passing to the next result, we recall the following definition.

**Definition (10).** The operator $T \in B(H)$ is called *quasidiagonal* if there exists a non-decreasing sequence $\{P_n\}_{n \geq 1}$ of finite-dimensional orthogonal projections, for which $P_n \to I$ (strongly) and $\|TP_n - P_n T\| \to 0$ as $n \to \infty$.

Herrero [11] defined the notion of *module of quasidiagonality*:

$$qd(T) = \liminf_{P \in \mathcal{P}} \inf_{P \prec P} \|TP - PT\|,$$

where $\mathcal{P}$ is an ordered (with respect to natural order) set of all finite-dimensional orthogonal projections in $H$. It is known [11] that $T$ is a quasidiagonal operator if and only if $qd(T) = 0$. The following theorem was proved by Arora and Sahdev in [12].

**Theorem 8.** Let $T \in B(H)$, $\ker T^* \neq \{0\}$ and $C = \inf_{\|x\|=1} \|Tx\| > 0$. Then $qd(T) \geq C$.

According to a result of Allan and Zemanek [2, Example 6] there is an operator $R$ on $H$, with $R^2 = 0$, but having no finite-dimensional reducing subspace. In the remainder of this section, as an illustration of Theorem 8, we give an example (see Example 10 below) of a family $\{T_n\}$ of operators on a Hilbert space $H$, with no finite-dimensional reducing subspace, converging to the nilpotent operator (with the index of nilpotency2) with finite-dimensional reducing subspace. However, we first prove the following proposition.

**Proposition 9.** Let $V, W \in B(H)$ be operators such that $V$ is a nonunitary isometry, $WV = WV(i.e., W \in \{V^\dagger \})$ and $\|W\| < 1$. Let us consider the operator $N_{V,W} \overset{\text{def}}{=} V(I - WVV^\dagger)$. Then $qd(N_{V,W}) \geq 1 - \|W\|$. 

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Proof. Since $\|W\| < 1$, the operators $I - WVV^*$ and $I - VV^*W^*$ are invertible. Then we have

$$\ker N_{V,W}^* = \ker(I - VV^*W^*)V^* = \ker V^* = (VH)^{\perp} \neq \{0\}$$

and

$$C = \inf_{\|x\|=1} \|N_{V,W}x\|$$
$$= \inf_{\|x\|=1} \|V(I - WVV^*)x\|$$
$$= \inf_{\|x\|=1} \|(I - WVV^*)x\|$$
$$\geq \inf_{\|x\|=1} \|(I - WVV^*)^{-1}\|$$
$$= \frac{1}{\|(I - WVV^*)^{-1}\|} > 0.$$ 

Hence, the conditions of Theorem 8 are valid for the operator $N_{V,W}$. Then, by applying Theorem 8, we have

$$qd(N_{V,W}) \geq \frac{1}{\|(I - WVV^*)^{-1}\|}$$
$$= \frac{1}{\|\sum_{n \geq 0} (WVV^*)^n\|}$$
$$= \frac{1}{\|I + (\sum_{n \geq 1} W^n V^*)\|}$$
$$= \frac{1}{\|I + (\sum_{n \geq 1} W^n) V^*\|}$$
$$\geq \frac{1}{\sum_{n \geq 0} \|W\|^n}$$
$$= 1 - \|W\|,$$

which completes the proof.

Example 10. Let $N_{V,\alpha} = V(I - \alpha VV^*)$, where $\alpha$ is a scalar, $|\alpha| < 1$ and $V \in \mathcal{B}(H)$ is an isometry.

(i) Each of the operators $N_{V,\alpha}$ $(|\alpha| < 1)$ does not have any finite-dimensional reducing subspace.

(ii) $N_{V,\alpha}$ converges to $N_V = V(I - VV^*)$ in the uniform operator topology as $\alpha \to 1^-$. 

(iii) $N_V$ has a finite-dimensional reducing subspace.

Proof. (i) Indeed, by Proposition 9 $qd(N_{V,\alpha}) \geq 1 - |\alpha|$, and hence, by the definition of the value $qd(N_{V,\alpha})$, each of the operators $N_{V,\alpha}$, where $|\alpha| < 1$, has no finite-dimensional reducing subspace.

(ii) From the equality

$$N_{V,\alpha} = \alpha N_V + (1 - \alpha)V$$
it follows that $N_{V,\alpha} \to N_V$ as $\alpha \to 1^-$ in the uniform operator topology. Evidently, $N^2_V = 0$.

(iii) For arbitrary fixed $0 \neq x \in \ker V^*$ let $E_x = \text{span} \{x, Vx\}$. Then it is easy to verify that $N^*_V E_x \subset E_x$ and $N^*_V E_x \subset E_x$. Indeed, $N^*_V x = V(I - VV^*)x = Vx \in E_x$, $N^*_V Vx = (I - VV^*)Vx = 0 \in E_x$, and hence, $N^*_V \text{span} \{x, Vx\} \subset \text{span} \{x, Vx\}$, i.e., $N^*_V E_x \subset E_x$. On the other hand, $N^*_V Vx = (I - VV^*)V^*x = 0 \in E_x$, $N^*_V Vx = (I - VV^*)V^*Vx = (I - VV^*)x = x \in E_x$, and so, $N^*_V E_x \subset E_x$. Hence $E_x$ reduces $N_V$ and dim $E_x = 2$, which completes the proof.

4. Other properties. In this section we collect some other properties of the operators $N_{V,W}(W \in \{V\}')$.

PROPOSITION 11. Let $\alpha$ be a scalar and $V, W \in \mathcal{B}(H)$ operators such that $V$ is an isometry and $W \in \{V\}'$. Then the following statements are true.

(a) $r(N_{V,W}) \leq r(I - W)$, where $r(T)$ stands for the spectral radius of the operator $T$.

(b) If $\|W\| < 1$, then $\ker N^n_{V,W} = \ker N_{V,W}$ and $\ker N^{n+1}_{V,W} = V^n \ker N_{V,W}$ for any integer $n \geq 1$.

(c) $r(N_{V,\alpha}) = |1 - \alpha|$.

(d) $\|N_{V,\alpha}\| \geq |1 - \alpha|$.

(e) If $|\alpha|^2 - 2\Re \alpha \geq 0$, then $\|N_{V,\alpha}\| = (|\alpha|^2 - 2\Re \alpha + 1)^{\frac{1}{2}}$.

(f) $\omega(N_{V,\alpha}) \leq \frac{|\alpha|}{2} + |1 - \alpha|$, where $\omega(T)$ stands for the numerical radius of the operator $T$.

(g) If $0 < \alpha < 1$, then $1 - \alpha \leq \omega(N_{V,\alpha}) \leq 1 - \frac{\alpha}{2}$.

(h) If $\alpha > 2$ is any real number, then $r(N_{V,\alpha}) = \omega(N_{V,\alpha}) = \|N_{V,\alpha}\| = \alpha - 1$.

Proof. (a) Since $N_{V,W} = WN_V + (I - W)V$, $WV - VW = 0$ and $N_V V = 0$, we have

$$
N^2_{V,W} = [WN_V + (I - W)V]^2
= (I - W)V\{WN_V + (I - W)V\}
= (I - W)VN_{V,W},
$$

so that $N^2_{V,W} = (I - W)VN_{V,W}$. After these simple calculations we conclude that

$$
N^n_{V,W} = (I - W)^{n-1}V^{n-1}N_{V,W}, \tag{12}
$$

for each $n \geq 1$. Since $V$ is an isometry, by the last equality, it follows that $r(N_{V,W}) \leq r(I - W)$.

(b) From (12), it is clear that

$$
N^n_{V,W} = (I - VV^*W^*)(I - W^{*})^{n-1}V^{*n}, \tag{13}
$$

for each $n \geq 1$. The condition $\|W\| < 1$ ensures invertibility of the operators $(I - W)^{n-1}$ and $(I - VV^*W^*)(I - W^{*})^{n-1}$, so that the equalities (12),(13) apply.

(c) If $W = \alpha I$, then (12) directly implies that $r(N_{V,\alpha}) = |1 - \alpha|$.

(d) $\Rightarrow$ (d).
(e) In fact, for every $x \in H$, $\|x\| = 1$, we have

$$\|N_{V,\alpha}x\|^2 = \|V(I - \alpha V\alpha)x\|^2$$

$$= \|(I - \alpha V\alpha)x\|^2$$

$$= ((I - \alpha V\alpha)x, (I - \alpha V\alpha)x)$$

$$= (I - \alpha V\alpha\overline{V\alpha} + |\alpha|^2 V\alpha x, x)$$

$$= ((I - 2\Re\alpha) V\alpha + |\alpha|^2 V\alpha x, x)$$

$$= 1 + (|\alpha|^2 - 2\Re\alpha)(V\alpha x, x).$$

Since $|\alpha|^2 - 2\Re\alpha \geq 0$, we obtain $\|N_{V,\alpha}\| = (|\alpha|^2 - 2\Re\alpha + 1)^{\frac{1}{2}}$ by taking the suprema of both sides over all unit vectors $x$ in $H$.

(f) It is known [3] that the numerical range of the nilpotent operator $N_V$ is a closed circular disc with center 0 and radius $\frac{1}{2}$. (Note that in the work of Tso and Wu [13] a more general theorem was proved, describing the numerical range of any quadratic operator on a complex Hilbert space.) Then, taking into account the equality $N_{V,\alpha} = \alpha N_V + (1 - \alpha)V$, we get the required inequality.

(c), (f) $\Rightarrow$ (g).

(h) Since $r(N_{V,\alpha}) \leq \omega(N_{V,\alpha}) \leq \|N_{V,\alpha}\|$, from (c) and (e) follows (h). The proof of Proposition 11 is completed.

Now we apply the operator $N_V$ to estimate the angle between subspaces of a Hilbert space $H$. The angle between subspaces $E_1 \subset H$ and $E_2 \subset H$ is determined as follows:

$$\langle E_1, E_2 \rangle \in \left[0, \frac{\pi}{2}\right], \quad \cos(\langle E_1, E_2 \rangle) = \sup \left\{ \frac{\|x\|}{\|x\|} : x \in E_1, y \in E_2 \right\}.$$

From the definition it immediately follows that

$$\cos(\langle E_1, E_2 \rangle) = \sup \left\{ \frac{\|P_{E_2}x\|}{\|x\|} : x \in E_1 \right\} = \left\| P_{E_2}P_{E_1} \right\|,$$

$$\sin(\langle E_1, E_2 \rangle) = \inf \left\{ \frac{\|(I - P_{E_1})x\|}{\|x\|} : x \in E_1 \right\} = \left\| P_{E_1\perp E_2} \right\|^{-1},$$

where $P_E (i = 1, 2)$ are orthogonal projections of $H$ onto $E_i (i = 1, 2)$ and $P_{E_1\perp E_2}$ is the projection onto $E_1$, parallel to $E_2$.

**Proposition 12.** Let $K$ be an arbitrary subspace of a Hilbert space $H$ and let $V_1, V_2 \in \mathcal{B}(H)$ be isometries. Then

$$|\cos(\langle R(V_1)^\perp, K \rangle) - \cos(\langle R(V_2)^\perp, K \rangle)| \leq \|N_{V_1} - N_{V_2}\|.$$  \hspace{1cm} (14)

**Proof.** We use the arguments of the reference [14]. (See also [15].) Indeed,

$$\cos(\langle R(V_2)^\perp, K \rangle) = \sup_{x \in K} \frac{\|P_{(V_2H)^\perp}x\|}{\|x\|}$$

$$\sup_{x \in K} \frac{\|V_2P_{(V_2H)^\perp}x\|}{\|x\|}$$

$$\sup_{x \in K} \frac{\|V_2P_{(V_2H)^\perp}x\|}{\|x\|}.$$
valid:

For every pair of commuting isometries \( V_1, V_2 \), we denote by \( \text{IFD} \) the set of isometries with finite defects. Put

\[
\text{IFD} \cap \{ V \} = \{ V' : V \in \text{IFD} \},
\]

which is a subset of the set of all finite-dimensional operators in \( H \).

**Proof.** The simple calculations show that

\[
\cos(R(V_1)^\perp, K) \leq \| N_{V_1} - N_{V_2} \| + \cos(R(V_2)^\perp, K).
\]

From these inequalities, we get (14), which completes the proof.

We say that an isometry \( V \in \mathcal{B}(H) \) has a finite defect if \( \dim((V^*)^\perp) < +\infty \). Let us denote by \( \text{IFD} \) the set of isometries, with finite defects. Put \( \mathcal{N}_{\text{IFD}} = \{ N_V : V \in \text{IFD} \} \), which is a subset of the set of all finite-dimensional operators in \( H \).

**Corollary 13.** Let \( V \in \mathcal{B}(H) \) be an isometry. Then the following inequalities are valid:

\[
\inf_{U \in \mathcal{N}_{\text{IFD}} \setminus \{ V \}} \cos(R(U)^\perp, R(V)) \leq \text{dist}(N_V, \mathcal{N}_{\text{IFD}} \cap \{ V \}) \leq 4\text{dist}(V, \text{IFD} \cap \{ V \}).
\] (15)

**Proof.** The simple calculations show that

\[
\| N_{V_1} - N_{V_2} \| \leq 4\| V_1 - V_2 \|,
\] (16)

for every pair of commuting isometries \( V_1 \) and \( V_2 \). In fact, \n
\[
\| N_{V_1} - N_{V_2} \| = \| V_1(I - V_1 V_1^*) - V_2(I - V_2 V_2^*) \|
\]

\[
= \| V_1 - V_2 + V_2^* V_2^* - V_1^* V_1^* \|
\]

\[
\leq \| V_1 - V_2 \| + \| V_2^* V_2^* - V_1^* V_1^* \|
\]

\[
\leq \| V_1 - V_2 \| + \| (V_2^* - V_1^*) V_2^* \| + \| V_2^* (V_2^* - V_1^*) \|
\]

\[
\leq \| V_1 - V_2 \| + \| (V_2 - V_1) (V_2 + V_1) \| + \| V_2^* - V_1^* \|
\]

\[
\leq \| V_1 - V_2 \| + 2\| V_1 - V_2 \| + \| V_1 - V_2 \|
\]

\[
= 4\| V_1 - V_2 \|.
\]
By inequality (14) we have
\[ |\cos\langle R(U)^{\perp}, K \rangle - \cos\langle R(V)^{\perp}, K \rangle| \leq \|N_V - N_U\|, \]
for any \( U \in IFD \cap \{V\}' \) and \( K \subset H \). Then, by choosing \( K = VH \) and using (16), from the last inequality we get (15). This completes the proof.

REFERENCES