ON THE INTEGRAL PART OF A LINEAR FORM WITH PRIME VARIABLES

I. DANICIC

The object of this paper is to prove the following:

THEOREM. Suppose that \( \lambda, \mu \) are real non-zero numbers, not both negative, \( \lambda \) is irrational, and \( k \) is a positive integer. Then there exist infinitely many primes \( p \) and pairs of primes \( p_1, p_2 \) such that

\[
[\lambda p_1 + \mu p_2] = kp.
\]

In particular \([\lambda p_1 + \mu p_2]\) represents infinitely many primes.

Here \([x]\) denotes the greatest integer not exceeding \( x \). The proof is based on the following

PRINCIPAL LEMMA. Suppose \( \lambda_1, \lambda_2, \lambda_3, \gamma \) are real numbers, the \( \lambda \)'s being non-zero and not all of the same sign, and \( \lambda_1/\lambda_2 \) being irrational. Then for every \( \epsilon > 0 \) there exists a sequence of integers \( N \to \infty \) such that the number of solutions of the inequality

\[
\left\{ \begin{array}{l}
|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \gamma| < \epsilon \\
p_i \leq N \quad (p_i \text{ prime})
\end{array} \right.
\]

is greater than \( CN^2/(\log N)^3 \) where \( C \) is a positive number independent of \( N \).

We first deduce the theorem from the lemma. We apply the lemma with

\[
\lambda_1 = \lambda, \quad \lambda_2 = -k, \quad \lambda_3 = \mu, \quad \gamma = -\frac{1}{2}, \quad \epsilon = \frac{1}{2}.
\]

This gives for more than \( CN^2/(\log N)^3 \) triples of primes \( p_1, p_2, p_3 \)

\[
p_i \leq N, \quad kp_2 < \lambda p_1 + \mu p_3 < kp_2 + 1,
\]

that is

\[
(1) \quad [\lambda p_1 + \mu p_3] = kp_2.
\]

Suppose only finitely many primes \( p_2 \), say \( d \), occur in (1). Then for some fixed \( p_2 \) (1) has more than \( CN^2/d(\log N)^3 \) solutions \( p_1, p_2 \) \((p_i \leq N)\). For given \( p_1 \) there are at most \( |\mu|^{-1} + 1 \) primes \( p_3 \) satisfying (1) and for \( p_1 \) we have at most \( C_1 N/\log N \) choices.

Thus

\[
(|\mu|^{-1} + 1) \frac{C_1 N}{\log N} \geq \frac{CN^2}{d(\log N)^3},
\]

which gives a contradiction for sufficiently large \( N \).

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The proof of the principal lemma is based on an adaptation of Vinogradov's method for Goldbach's problem. By replacing the $\lambda_1$ and $\gamma$ by $\lambda_1 \epsilon^{-1}$ and $\gamma \epsilon^{-1}$ it suffices to prove the lemma for $\epsilon = 1$. Since $\lambda_1 \lambda_2^{-1}$ and $\lambda_2 \lambda_3^{-1}$ are not both rational, we may further suppose that $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

Notation. $p$ indicates primes, $e(\alpha) = e^{2\pi i \alpha}$.

\[ S(\alpha) = S(\alpha, N) = \sum_{p \leq N} e(\alpha p), \]
\[ I(\alpha) = I(\alpha, N) = \int_{1/2}^{N} e(\alpha x) \frac{dx}{\log x}, \]
\[ K(\alpha) = \left( \frac{\sin \pi \alpha}{\pi \alpha} \right)^2, \]
\[ L = \log N. \]

**Lemma 1.** For real $\theta$,

\[ 2 \operatorname{Re} \int_{0}^{\infty} e(\theta \alpha) K(\alpha) d\alpha = \begin{cases} 0 & \text{if } |\theta| > 1, \\ 1 - |\theta| & \text{if } |\theta| \leq 1. \end{cases} \]

A proof is given by Davenport and Heilbronn (1, Lemma 4).

**Lemma 2.** The number of solutions of the inequality (*) (with $\epsilon = 1$) is not less than

\[ \text{Re} \int_{0}^{\infty} S(\lambda_1 \alpha) S(\lambda_2 \alpha) S(\lambda_3 \alpha) e(\gamma \alpha) K(\alpha) d\alpha. \]

**Proof.** This follows from Lemma 1 in the usual way, by multiplying out the trigonometric sums in the integrand and then interchanging summation and integration.

The remaining lemmas are devoted to establishing a lower bound for the expression in Lemma 2. For ease of reference we shall refer to Prachar's book (3).

Throughout $\rho$ will denote a positive constant, which will be fixed later. $C, C_1, \ldots$ denote positive numbers independent of $N$.

**Lemma 3.** If $a$ and $q$ are integers, $\alpha$ is real, and

\[ |\alpha - a/q| < 1/q T_1 \]

where

\[ T_1 = C_1 NL^{1-\rho}, \quad (a, q) = 1, \quad L^\rho < q \leq T_1, \]

then

\[ |S(\alpha)| < CNL^{(3/2) - \frac{1}{2}}. \]

**Proof.** This follows at once from (3, Chapter VI, Theorem 6.1) (Vinogradov's estimate).
LEMMA 4. If \( a \) and \( q \) are integers, \((a, q) = 1, 1 \leq q \leq L^\rho, \alpha \) is real, and
\[
\alpha = (a/q) + \beta, \quad |\beta| < q^{-1}T_1^{-1}, \quad T_1 = C_1 NL^{-\rho},
\]
then
\[
|S(\alpha) - \{\mu(q)/\phi(q)\} I(\beta)| < CN \exp(-AL^\frac{1}{2}),
\]
where \( A \) is an absolute positive constant, and \( \mu(q) \) and \( \phi(q) \) are the Möbius and Euler functions respectively.

Proof. This follows at once from (3, Chapter VI, Theorem 3.3) on replacing
\[
\sum_{2 \leq n \leq N} e\left(n\frac{\beta}{\log n}\right)
\]
by \( I(\beta) \). The error introduced by this is \( O(1 + L^{\rho-1}) \), which is negligible.

LEMMA 5.
\[
|I(\alpha)| < \begin{cases} CNL^{-1} & \text{all } \alpha, \\ CN^4L^{-1} & |\alpha| \geq N^{-\frac{1}{2}}, \\ |\alpha|^{-1}L^{-1} & |\alpha| < N^{-\frac{1}{2}}. \end{cases}
\]

Proof. The first estimate follows from \( |I(\alpha)| \leq |I(0)| \); the second and third are proved by integrating by parts.

LEMMA 6. The volume of the 3-dimensional body \( B \) defined by
\[
|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3| < \frac{1}{2}, \quad 2 \leq x_i \leq N (i = 1, 2, 3)
\]
is greater than \( CN^2 \).

Proof. We chose a positive constant \( \delta \) which satisfies
\[
(1 + \frac{1}{4})\lambda_1|\lambda_2|^{-1}\delta < 1, \quad \frac{1}{2}\lambda_1|\lambda_3|^{-1}\delta < 1, \quad (1 + \frac{1}{2} + \frac{1}{4})\delta < \frac{1}{2}.
\]
The three intervals
\[
\frac{1}{2}\delta N - \frac{1}{2}\lambda_1^{-1} < x_1 < (1 + \frac{1}{4} + \frac{1}{2})\delta N + \frac{1}{2}\lambda_1^{-1}, \quad \frac{2}{3}\lambda_1|\lambda_2|^{-1}\delta N < x_2 < (1 + \frac{1}{4})\lambda_1|\lambda_2|^{-1}\delta N, \quad \frac{1}{3}\lambda_1|\lambda_3|^{-1}\delta N < x_3 < \frac{1}{2}\lambda_1|\lambda_3|^{-1}\delta N
\]
lie in \((2, N)\) if \( N \) is sufficiently large, and have lengths at least \( \lambda_1^{-1}, \lambda_1 N, \) and \( \lambda_2 N \) respectively. These inequalities therefore define a box of volume greater than \( CN^2 \). Remembering that \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0, \) it is at once verified that for such \( x_i \) we also have
\[
|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3| < \frac{1}{2},
\]
which proves the lemma.

LEMMA 7. Let
\[
J_1 = \text{Re} \int_0^{T_1^{-1}} S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)e(\gamma\alpha)K(\alpha) \, d\alpha,
\]
where $T_1 = C_1 NL^{-\beta}$. Then $J_1 > CN^2L^{-3}$.

Proof. Let $\lambda = \max_i |\lambda_i|$. In the interval of the integral we have

$$|\lambda_i\alpha| \leq |\lambda_i|T_1^{-1} \leq \lambda CN^2N^{-1}L^\delta.$$ 

Hence, by Lemma 4 with $a = 0, q = 1$,

$$|S(\lambda_i\alpha) - I(\lambda_i\alpha)| < CN \exp(-AL^i) \quad (i = 1, 2, 3).$$

Since $0 < K(\alpha) < 1$, we may replace each $S(\lambda_i\alpha)$ in $J_1$ by $I(\lambda_i\alpha)$ with an error at most

$$3N^2T_1^{-1}CN \exp(-AL^\delta) < C_3 N^2L^\delta \exp(-AL^\delta),$$

since trivially $|S(\lambda_i\alpha)| \leq N, |I(\lambda_i\alpha)| \leq N$.

In the resulting expression

$$\text{Re} \int_0^{T_1^{-1}} I(\lambda_1\alpha)I(\lambda_2\alpha)I(\lambda_3\alpha)e(\gamma\alpha)K(\alpha) \, d\alpha$$

we replace $e(\gamma\alpha)$ by 1 with an error less than

$$\int_0^{T_1^{-1}} N^2|e(\gamma\alpha) - 1| \, d\alpha < |\gamma|N^2 \int_0^{T_1^{-1}} \alpha \, d\alpha < CN^3T_1^{-2} = C_2 NL^{2\delta}.$$ 

In the resulting expression

$$J_2 = \text{Re} \int_0^{T_1^{-1}} I(\lambda_1\alpha)I(\lambda_2\alpha)I(\lambda_3\alpha)K(\alpha) \, d\alpha$$

we replace the upper limit of integration by $\infty$ with an error

$$\int_{T_1^{-1}}^{\infty} |I(\lambda_1\alpha)I(\lambda_2\alpha)I(\lambda_3\alpha)|K(\alpha) \, d\alpha.$$

By Lemma 5 this is less than

$$C\left\{ \int_{T_1^{-1}}^{N^{-1}} L^{-3}N^{-3/2} \, d\alpha + \int_{N^{-1}}^{1} N^{3/2}L^{-3} \, d\alpha + \int_{1}^{\infty} N^{3/2}L^{-3}K(\alpha) \, d\alpha \right\}$$

$$< C[ L^{-3}T_1^2 + L^{-3}N^{3/2} + L^{-3}N^{3/2}] < CN^2L^{-3-2\delta}.$$

Thus

$$(2) \quad |J_1 - J_3| < CN^2L^{-3-2\delta},$$

where

$$J_3 = \text{Re} \int_0^{\infty} I(\lambda_1\alpha)I(\lambda_2\alpha)I(\lambda_3\alpha)K(\alpha) \, d\alpha.$$
Now
\begin{align*}
J_3 &= \text{Re} \int_0^\infty K(\alpha) \left\{ \int_1^\infty \int_1^\infty \int_1^\infty \frac{e(\alpha(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3))}{\log x_1 \log x_2 \log x_3} \, dx_1 \, dx_2 \, dx_3 \right\} \, d\alpha \\
&= \text{Re} \int_2^\infty \int_2^\infty \int_2^\infty (\log x_1 \log x_2 \log x_3)^{-1} \\
&\quad \times \int_0^\infty e(\alpha(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)) K(\alpha) \, d\alpha \, dx_1 \, dx_2 \, dx_3 \\
&= \int_2^\infty \int_2^\infty \int_2^\infty (\log x_1 \log x_2 \log x_3)^{-1} \\
&\quad \times \frac{1}{2} \max(0, 1 - |\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3|) \, dx_1 \, dx_2 \, dx_3
\end{align*}
by Lemma 1. Hence

\[ J_3 \geq \frac{1}{4} L^{-3} \int_2^\infty \int_2^\infty \int_2^\infty f(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3 > CL^{-3}N^2. \]

This together with (2) gives

\[ J_1 = J_2 + (J_1 - J_2) > C_2 N^2 L^{-2}. \]

The next lemma is similar to Lemma 13 in the paper (2) by Davenport and Roth.

**Lemma 8.** If \( M \) runs through those real numbers for which \( M(\log M)^{-(5/2)\delta} \) runs through the denominators of the continued fraction for \( \lambda_1/|\lambda_2|, N = \lfloor M \rfloor \), then for

\[ \lambda N^{-1} L^\delta < \alpha < L^3, \quad \lambda = \max(\lambda_1^{-1}, |\lambda_2|^{-1}), \]
we have \( \min(|S(\lambda_1 \alpha)|, |S(\lambda_2 \alpha)|) < CNL^{(9/2)\delta - \delta}. \)

**Proof.** If \( Q \) is a denominator of the continued fraction for \( \lambda_1/|\lambda_2| \), then by Legendre’s law of best approximation

\[ |(\lambda_1/\lambda_2)q - a| > \frac{1}{2} Q^{-1} \]
for all integers \( a, q \) with \( 1 \leq q < Q \); cf. footnote in (2, p. 94). For every \( \alpha \) there are integers \( a_i, q_i \) such that

\[ (a_i, q_i) = 1, \quad |\lambda_i \alpha - (a_i/q_i)| < q_i^{-1} L^\delta N^{-1}, \quad 1 \leq q_i \leq NL^{-\delta} \]
\((i = 1, 2)\).

Put

\[ \beta_i = \lambda_i \alpha - (a_i/q_i). \]
For the \( a \) in the interval under consideration we cannot have \( a_1 = 0 \) or \( a_2 = 0 \), for otherwise for \( i = 1 \) or \( 2 \)

\[
N^{-1}L^p < |\lambda_i|a < L^p N^{-1},
\]
a contradiction. Further, if \( q_i > L^p \) for \( i = 1 \) or \( 2 \), then the required estimate follows from Lemma 3. Hence we may suppose that \( q_i \leq L^p (i = 1, 2) \). Hence by Lemma 4

\[
|S(\lambda_i a)| < (1/\phi(q_i))|I(\beta_i)| + CN \exp(-AL^{1/2})
\]

\[
< |\beta_i|^{-1}L^{-1} + CN \exp(-AL^{1/2}) < C|\beta_i|^{-1}L^{-1},
\]
by Lemma 5 since \( |\beta_i| < q_i^{-1}L^p N^{-1} < N^{-1} \). If the result to be proved is false, it follows from this that

\[
(3) \quad NL^{(9/2) - 3p} < C|\beta_i|^{-1}L^{-1} \quad (i = 1, 2).
\]
Now

\[
(\lambda_1/\lambda_2) a_2 q_1 - a_1 q_2 = q_1 q_2 (\beta_1 + \beta_2).
\]
Since \( q_i \leq L^p \), (3) gives

\[
(4) \quad |(\lambda_1/\lambda_2) a_2 q_1 - a_1 q_2| \leq q_1 q_2 (|\beta_1| + |\beta_2|) \leq 2C q_1 q_2 N^{-1}L^{-(11/2) + 3/2} < CN^{-1}L^{-(11/2) + (5/2)p}.
\]
We put \( q = a_2 q_1 \), \( a = \pm a_1 q_2 \), the sign being chosen so that

\[
|\lambda_1/\lambda_2| q - a = |(\lambda_1/\lambda_2) a_2 q_1 - a_1 q_2|.
\]
By (4),

\[
|(\lambda_1/\lambda_2)| q - a| < CN^{-1}L^{-(11/2) + (5/2)p},
\]
and

\[
1 \leq q = (a_2/q_2) q_2 \leq C q_2 L^{2 + 2p}
\]
since \(|\lambda_2| \alpha| < |\lambda_2| L^3 \) and \( a_2/q_2 \) is near \( \lambda_2 \alpha \). Putting

\[
Q = M (\log M)^{-(5/2)p},
\]
we have in particular

\[
1 \leq q < Q, \quad |(\lambda_1/\lambda_2)| q - a| < 1/2Q.
\]
Since \( Q \) is a denominator of the continued fraction for \( \lambda_1/|\lambda_2| \), this contradicts Legendre's law of best approximation. Hence (3) is false and the lemma is proved.

**Lemma 9.** If \( \lambda \neq 0 \), then

\[
\int_0^\infty |S(\lambda a)|^2 K(a) \, da < CNL^{-1}.
\]

**Proof.**

\[
|S(\lambda a)|^2 = \sum_{p_1 \leq N} \sum_{p_2 \leq N} e\{\lambda a (p_1 - p_2)\}.
\]
Hence by Lemma 1, the integral is less than the number of solutions of
\[ |\lambda(p_1 - p_2)| < 1, \quad p_1, p_2 \leq N. \]
For given \( p_1 \) this gives at most \( 2|\lambda|^{-1} + 1 \) choices for \( p_2 \) and hence the number of solutions is at most
\[ \pi(N)(2|\lambda|^{-1} + 1) < CNL^{-1}. \]

**Lemma 10.** If \( N, \rho, \) and \( \lambda \) are the same as in Lemma 8, then
\[ \int_{\Lambda_{N}^{-1}}^{L^2} |S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)|K(\alpha) \, d\alpha < CN^2L^{(7/2) - \rho}. \]

**Proof.** By Lemma 8, the integral is less than
\[ CNL^{(9/2) - \rho} \left( \int_0^\infty |S(\lambda_2 \alpha)|S(\lambda_3 \alpha)|K(\alpha) \, d\alpha + \int_0^\infty |S(\lambda_1 \alpha)S(\lambda_3 \alpha)|K(\alpha) \, d\alpha \right). \]
By Cauchy’s inequality, the first of the two integrals is
\[ \leq \left\{ \int_0^\infty |S(\lambda_2 \alpha)|^2K(\alpha) \, d\alpha \int_0^\infty |S(\lambda_3 \alpha)|^2K(\alpha) \, d\alpha \right\}^{1/2} < CNL^{-1} \]
by Lemma 9. The same estimate holds for the second integral and the result follows.

**Lemma 11.** If \( A > 4, \) then
\[ \int_A^\infty |S(\lambda \alpha)|^2K(\alpha) \, d\alpha \leq \frac{16}{A} \int_0^\infty |S(\lambda \alpha)|^2K(\alpha) \, d\alpha. \]

**Proof.** This is a special case of (2, Lemma 2).

**Lemma 12.**
\[ \int_{L^3}^\infty |S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)|K(\alpha) \, d\alpha < CN^2L^{-5}. \]

**Proof.** By Hölder’s inequality the integral is \( \leq (L_1L_2L_3)^{1/3} \), where
\[ L_i = \int_{L^3}^\infty |S(\lambda_i \alpha)|^2K(\alpha) \, d\alpha. \]
Using the trivial estimate \( |S(\lambda_1 \alpha)| < CNL^{-1} \) and Lemma 11,
\[ L_i \leq CNL^{-1} \int_{L^3}^\infty |S(\lambda_1 \alpha)|^2K(\alpha) \, d\alpha \leq C_1NL^{-1} \int_0^\infty |S(\lambda_1 \alpha)|^2K(\alpha) \, d\alpha \]
and this by Lemma 9 is \( \leq CN^2L^{-5}. \)

**Proof of the Principal Lemma.** We shall prove that if \( N \) runs through the sequence specified in Lemma 8 and \( \rho = 15, \) then
\[ J = \Re \int_0^\infty S(\lambda_1 \alpha)S(\lambda_2 \alpha)S(\lambda_3 \alpha)e(\gamma \alpha)K(\alpha) \, d\alpha > CN^2L^{-3}. \]
We divide the interval of integration into three subintervals $E_1$, $E_2$, $E_3$ where

$E_1: 0 < \alpha < \lambda N^{-1} L^p,$

$E_2: \lambda N^{-1} L^p < \alpha < L^3,$

$E_3: L^3 < \alpha,$

where $\lambda$ is defined in Lemma 8. By Lemma 7, the integral over $E_1$ is $> CN^2 L^{-3}$. The integral over $E_2$ is in absolute value

$$< CN^2 L^{(7/3)-(15/2)} = CN^2 L^{-4},$$

by Lemma 10. By Lemma 12 the integral over $E_3$ is in absolute value less than $CN^3 L^{-5}$. Hence

$$J > CN^3 L^{-3}$$

and this with Lemma 2 proves the Principal Lemma.

One might conjecture that if $\lambda$ is positive and irrational, then $[\lambda p]$ represents infinitely many primes.

REFERENCES


Bedford College,
London University