A SIMPLE PROOF OF THE SUM FORMULA

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In this note we present a simple, short proof of the sum formula for subdifferentials of convex functions.

1. INTRODUCTION

Let $X$ be a Banach space, $X^*$ be its dual endowed with the dual norm. Let $f, g : X \to R \cup \{+\infty\}$ be proper, lower semicontinuous, convex functions with (effective) domains $\text{dom } f$ and $\text{dom } g$ respectively, and let $\partial f, \partial g : X \to 2^{X^*}$ be their subdifferential operators respectively. It is straightforward to verify that $\partial f(x) + \partial g(x) \subseteq \partial (f + g)(x)$ for any $x \in X$. However, the converse inclusion is not always true (see for example the remark in [2, Theorem, 3.16]). It is therefore important to find conditions which assure that $\partial f(x) + \partial g(x) = \partial (f + g)(x)$ for any $x \in X$. If for example the domain of $f$ and the interior of the domain of $g$ have nonempty intersection, then the above equality is true (see for example [2, Theorem 3.16]). Attouch and Brezis proved in [1] the following more general result (see also [3] for a different proof):

THE SUM FORMULA. Let $f$ and $g$ be as above and assume that

$$\bigcup_{\lambda > 0} \lambda (\text{dom } f - \text{dom } g) = \text{lin} (\text{dom } f - \text{dom } g)$$

Then $\partial f + \partial g = \partial (f + g)$.

(In the above statement, “lin” stands for the “linear span of”.)

The aim of this note is to present a short and simple proof of the sum formula.

2. A PARTICULAR CASE

We begin by recalling the definition of the epi-sum (or inf-convolution) of two functions $f, g : R \to R \cup \{+\infty\}$:

$$(f \begin{array}{c}
\vee
\end{array} g)(x) = \inf_{u+v=x} \{f(u) + g(v)\}$$
From the definition it follows that \( \text{dom}(f + g) = \text{dom } f + \text{dom } g \). The epi-sum \( f + g \) is called exact at \((u, v)\) if \( (f + g)(u + v) = f(u) + g(v) \). If \( f \) and \( g \) are convex, then \( f + g \) is also convex. Finally, a direct computation shows that if \( f \) and \( g \) are convex and \( f + g \) is exact at \((u, v)\) then

\[
\partial(f + g)(u + v) = \partial f(u) \cap \partial g(v).
\]

**Lemma 1.** Let \( f : X \to R \cup \{+\infty\} \) be a proper lower semicontinuous convex function and let \( Y \) be a closed subspace of \( X \) such that \( Y + \text{dom } f \) is an absorbing subset of \( X \) and \( (f + I_Y)(0) > -\infty \). Then \( 0 \in \text{Int}(\text{dom}(f + I_Y)) \) and \( f + I_Y \) is locally Lipschitz at 0. In particular \( f + I_Y \) is subdifferentiable at 0, that is, \( \partial(f + I_Y)(0) \neq \emptyset \).

**Proof:** By [3, Theorem 3], there exist \( \varepsilon > 0 \) and \( \lambda > 0 \) such that

\[
\varepsilon B \subseteq \{x \in X; \|x\| \leq \lambda, f(x) \leq \lambda \} + Y.
\]

It follows immediately that \( (f + I_Y)(x) \leq \lambda \) for any \( x \in \varepsilon B \). Since \( (f + I_Y)(0) > -\infty \), it follows that \( f + I_Y \) is nowhere \(-\infty\). It is well known (see for example [2, Proposition 1.6] and the remark following it) that \( f \) is locally Lipschitz on \( \varepsilon B \) and therefore is subdifferentiable at 0 (see for example [2, Proposition 1.11]).

**Lemma 2.** Let \( f : X \to R \cup \{+\infty\} \) be a proper lower semicontinuous convex function and let \( Y \) be a closed subspace of \( X \) such that \( Y + \text{dom } f \) is an absorbing subset of \( X \). Then \( \partial(f + I_Y) = \partial f + \partial I_Y \).

**Proof:** It is well known and easy to check that \( \partial f(x) + \partial I_Y(x) \subseteq \partial(f + I_Y)(x) \) for any \( x \in X \). To prove the other inclusion, let \( x^* \in \partial(f + I_Y)(x) \). Define \( g : X \to R \cup \{+\infty\} \) by \( g(u) = f(x + u) - \langle x^*, x + u \rangle \). Then

(i) \( \text{dom } g = \text{dom } f - x \) and, since \( x \in Y \), \( Y + \text{dom } g = Y + \text{dom } f \) is absorbing;

(ii) \( u^* \in \partial f(x) \) if and only if \( u^* - x^* \in \partial g(0) \).

(iii) \( u^* \in \partial(f + I_Y)(x) \) if and only if \( u^* - x^* \in \partial(g + I_Y)(0) \).

Since \( x^* \in \partial(f + I_Y)(x) \), (iii) implies that \( 0 \in \partial(g + I_Y)(0) \) and thus

\[
g(0) + I_Y(0) = \inf_{u \in X} \{g(u) + I_Y(u)\}.
\]

It follows that

\[
(f + I_Y)(0) = \inf_{u \in X} \{g(u) + I_Y(-u)\} = \inf_{u \in X} \{g(u) + I_Y(u)\} = g(0) + I_Y(0) > -\infty.
\]
Thus $g + I_Y$ is exact at $(0,0)$. From (i) and Lemma 1 it follows that $g + I_Y$ is subdifferentiable at 0 and therefore, from (1),

$$0 \neq \partial (g + I_Y)(0) = \partial g(0) \cap \partial I_Y(0).$$

Thus, there exists $u^* \in X^*$ such that $u^* \in \partial g(0) \cap \partial I_Y(0)$. Clearly $-u^* \in \partial I_Y(x)$ and, by (ii), $x^* + u^* \in \partial f(x)$. Since $x^* = x^* + u^* + (-u^*) \in \partial f(x) + \partial I_Y(0)$, the lemma is proved.

3. Proof of the Sum Formula

The proof is now standard, but we shall sketch it for sake of completeness.

First, in view of [3, Lemma 25 (c)] we can assume without any loss of generality that $\text{lin(} \text{dom } f - \text{dom } g\text{)} = X$ and thus $\text{dom } f - \text{dom } g$ is absorbing in $X$. Define $h : X \times X \to R \cup \{+\infty\}$ by $h(x,y) = f(x) + g(y)$ and let $D = \{(x,x) ; x \in X\}$. Then $h$ is a proper lower semicontinuous convex function on $X \times X$, $D$ is a closed subspace of $X \times X$, and dom $h - D$ is an absorbing subset of $X \times X$. The Sum Formula follows from Lemma 2 and the following statements which, with the exception of (c), follow more or less directly from the definitions; (c) is a particular case of Lemma 2. As usual we shall identify $(X \times X)^*$ with $X^* \times X^*$.

(a) $\partial I_D(z,z) = \{(z^*,-z^*) ; z^* \in X^*\};$
(b) $z^* \in \partial (f+g)(z) \iff (z^*,0) \in \partial (h+I_D)(z,z)$
(c) $(z^*,0) \in \partial (h+I_D)(z,z) \iff (z^*,0) \in \partial h(z,z) + \partial I_D(z,z);$
(d) $(z^*,0) \in \partial h(z,z) + \partial I_D(z,z) \iff z^* = u^* + v^*, (u^*,v^*) \in \partial h(z,z);$  
(e) $(u^*,v^*) \in \partial h(z,z) \iff u^* \in \partial f(u)$ and $v^* \in \partial f(v).$

\[\square\]

REFERENCES


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