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THE HILLE–YOSIDA THEOREM FOR LOCAL CONVOLUTED SEMIGROUPS

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Abstract The characterization theorem for the Banach-space-valued local Laplace transform established by Keyantuo, Müller and Vieten is used to obtain a real variable characterization of generators of local convoluted semigroups. The concept of local convoluted semigroups extends that of distribution as well as ultradistribution semigroups. Complete characterizations existed only for exponentially bounded semigroups integrated α times, whereas for the non-exponential case generation results had been obtained in terms of complex conditions only.

Keywords: local Laplace transform; local convoluted semigroup; integrated semigroup; distribution semigroup; ultradistribution semigroup.

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1. Introduction

In the study of the abstract Cauchy problem, the Laplace transform plays a crucial role. The resolvent of the generator of a semigroup is linked to the semigroup via the Laplace transform. The Hille–Yosida Theorem gives necessary and sufficient conditions under which the resolvent is a Laplace transform. However, there are many important problems that cannot be handled with the classical Laplace transform.

As early as 1960, Lions [**13**] introduced the notion of distribution semigroups and gave a complex condition for a densely defined linear operator A to be the generator of an exponential distribution semigroup. This concept received much attention, and in 1971 Chazarain [**5**] obtained a generation result involving not only distribution semigroups but also ultradistribution semigroups without growth condition. It turns out that the concept of exponentially bounded integrated semigroups is equivalent to that of exponential distribution semigroups (see [**1**]). Compared with the results of Lions, Arendt's approach has the advantage that it relates the characterization to the order of the distribution. We refer to [**1**] and to the systematic exposition given in [**3**]. Of importance is the relationship to the abstract Cauchy problem, which is one of the main motivations of the theory. Following [**1**], investigations were made on the Cauchy problem in cases where one does not have exponentially bounded solutions nor even solutions on $[0,\infty)$. One seeks instead solutions on $[0,T)$, where T is some fixed number in $(0,\infty]$. This led to the notions of local integrated semigroups [**2**, **16**] and local convoluted semigroups [**6**, **8**]. These concepts seem easier to handle than those of distribution and ultradistribution semigroups, to which they turn out to be equivalent. However, the Laplace transform is no longer applicable. Let A be a closed linear operator in the Banach space X. We consider the initial-value problem,

$$u'(t) = Au(t) + F(t)x, \quad 0 \leqslant t < T, \qquad u(0) = 0, \tag{1.1}$$

where $F(t) = \int_0^t E(s)\,\mathrm{d}s$. The conditions we put on E will be specified later. By definition, A is the generator of a local E-convoluted semigroup (on $[0,T)$) if, for every $x \in X$, (1.1) has a unique solution. The above-mentioned theories of distribution and ultradistribution semigroups are obtained by appropriate choice for the function E. We remark that if one takes $F(t) = 1$, so that E is the Dirac delta function, then one recovers the case of strongly continuous semigroups (see [**2**], where the result is attributed to van Casteren). The case of distribution semigroups corresponds to $F(t) = t^\alpha/\Gamma(\alpha+1)$, $0 \leqslant t < T$, $\alpha \geqslant 0$. Under this definition, we set $S(t)x = u'(t)$, $0 \leqslant t < T$, $x \in X$. Then $S : [0,T) \to \mathcal{L}(X)$ is strongly continuous. The family $(S(t))_{0\leqslant t<T}$ is called the local E-convoluted semigroup generated by A.

Our objective in this paper is to establish a real variable characterization of generators of local convoluted semigroups parallel to the one developed in [**1**] for exponentially bounded integrated semigroups. The need for real inversion results stems from the fact that the complex conditions obtained (although very useful in practice) always result in a loss of regularity even in the exponential case (see, for example, [**12**, Theorem 1.5, Chapter I], [**15**, § 1.7, especially Theorem 7.4 and its corollaries] and [**7**, Theorems 4.1 and 4.2]). Our main tool is a real variable characterization of the local Laplace transform obtained in [**11**].

We again stress that in the general situation that we consider, the classical Laplace transform is inoperative. For the application of the characterization of the vector-valued local Laplace transform to the Cauchy problem, we use the Phragmén–Doetsch inversion formula in a crucial way.

In § 2, we give some results on the abstract Cauchy problem and conditions on the function E as preparation for the main result established in § 3. In § 3, we combine the results of §§ 2 and 3 to establish the Hille–Yosida Theorem for local convoluted semigroups, Theorem 3.2. In the latter, the denseness of the domain is also clarified (much as in [**1**]). Finally, we make the connection between the local case and the global exponentially bounded convoluted semigroups. In our framework, we may restate the Hille–Yosida Theorem as follows.

Theorem 1.1. *A closed and densely defined linear operator is the generator of a strongly continuous semigroup if and only if its resolvent is a local Laplace transform.*

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2. The abstract Cauchy problem

In the sequel, X is a real or complex Banach space over $\boldsymbol{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let τ be a positive real number, and $F : [0, \tau] \to X$ a continuous function. We call $u : [0, \tau] \to X$ a solution of the local abstract Cauchy problem,

$$u'(t) = Au(t) + F(t), \quad 0 \leqslant t \leqslant \tau \quad \text{and} \quad u(0) = 0, \tag{2.1}$$

if u belongs to $C^1([0, \tau], X) \cap C([0, \tau], D(A))$ and fulfils (2.1).

The fundamental relation between local Laplace transforms and solutions of the local Cauchy problem is given by the following lemma. The main tool for its proof is the Phragmén–Doetsch inversion formula for Laplace transforms (for a proof see [**9**] or [**4**]).

Theorem 2.1. *Let $\phi : [0, \infty) \to X$ be Lipschitz continuous and $f(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\phi(t)$, $\lambda > 0$ the Laplace–Stieltjes transform of ϕ. Then*

$$\phi(t) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \mathrm{e}^{tnj} f(nj) \quad \textit{for all } t \geqslant 0.$$

Remark 2.2. If $(a_k)_k$ is a sequence in X with

$$\limsup_{k\to\infty} \frac{\ln \|a_k\|}{k} \leqslant -\tau \quad \text{for some } \tau > 0,$$

then

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \mathrm{e}^{tnj} a_{nj} = 0 \quad \text{for all } 0 \leqslant t < \tau.$$

This is easy to verify by showing that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{\mathrm{e}^{tnj}}{j!} \|a_{nj}\| = 0 \quad \text{for all } 0 \leqslant t < \tau.$$

Lemma 2.3. *Let $A : D(A) \to X$ be a closed operator in X. Then the following assertions are equivalent.*

(i) *There exists a solution of the abstract Cauchy problem (2.1).*

(ii) *There exists $v \in C([0, \tau], X)$ such that $\int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t \in D(A)$ for all $\lambda \in \boldsymbol{K}$, and*

$$(\lambda - A) \int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t = \lambda \int_0^\tau \mathrm{e}^{-\lambda t} F(t)\,\mathrm{d}t - \mathrm{e}^{-\lambda\tau}(v(\tau) - F(\tau))$$

for all $\lambda \in \boldsymbol{K}$.

(iii) *There exists $v \in C([0, \tau], X)$, $k_0 \in \mathbb{N}$ and a sequence $(a_k)_k$ in X with*

$$\limsup_{k\to\infty} \frac{\ln \|a_k\|}{k} \leqslant -\tau$$

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such that $\int_0^\tau \mathrm{e}^{-kt} v(t)\,\mathrm{d}t \in D(A)$ *for all* $k \geqslant k_0$, *and*

$$(k - A)\int_0^\tau \mathrm{e}^{-kt} v(t)\,\mathrm{d}t = k\int_0^\tau \mathrm{e}^{-kt} F(t)\,\mathrm{d}t - a_k \quad \textit{for all } k \geqslant k_0.$$

Proof. We first note that a finite Laplace transform $\int_0^\tau \mathrm{e}^{\lambda t}\,\mathrm{d}\phi(t)$ exists if ϕ is continuous or of bounded variation (see [**18**, Theorem 4b, p. 7]).

Suppose (i) holds. Let $v = u'$. Because the Riemann–Stieltjes integrals

$$\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}u(t) = \int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}Au(t) = \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}(v(t) - F(t))$$

exist we obtain $\int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t \in D(A)$, and

$$\begin{aligned} A&\int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t \\ &= \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}(v(t) - F(t)) \\ &= \mathrm{e}^{-\lambda\tau} v(\tau) - v(0) + \lambda\int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t - \biggl[\mathrm{e}^{-\lambda\tau} F(\tau) - F(0) + \lambda\int_0^\tau \mathrm{e}^{-\lambda t} F(t)\,\mathrm{d}t\biggr]. \end{aligned}$$

Hence, in view of $v(0) = F(0)$,

$$(\lambda - A)\int_0^\tau \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t = \lambda\int_0^\tau \mathrm{e}^{-\lambda t} F(t)\,\mathrm{d}t - \mathrm{e}^{-\lambda\tau}(v(\tau) - F(\tau)).$$

If (iii) holds, then we have

$$A\biggl(\frac{1}{k}\int_0^\tau \mathrm{e}^{-kt} v(t)\,\mathrm{d}t\biggr) = \int_0^\tau \mathrm{e}^{-kt}(v(t) - F(t))\,\mathrm{d}t + \frac{a_k}{k}. \tag{2.2}$$

We let

$$v^{[1]} := \int_0^t v(s)\,\mathrm{d}s \quad \text{and} \quad v^{[k+1]} := \int_0^t v^{[k]}(s).$$

Because of

$$\frac{1}{\lambda}\int_0^\tau \mathrm{e}^{-\lambda s} v(s)\,\mathrm{d}s = \int_0^\tau \mathrm{e}^{-\lambda s}\,\mathrm{d}v^{[2]}(s) + \frac{1}{\lambda}\mathrm{e}^{-\lambda\tau} v^{[1]}(\tau),$$

the Phragmén–Doetsch inversion formula yields

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j!}\mathrm{e}^{tnj}\biggl(\frac{1}{nj}\int_0^\tau \mathrm{e}^{-njs} v(s)\,\mathrm{d}s\biggr) = v^{[2]}(t) \quad \text{for all } t \in [0, \tau).$$

On the other hand, applying the Phragmén–Doetsch inversion formula to equation (2.2) gives

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j!}\mathrm{e}^{tnj}\biggl(\frac{A}{nj}\int_0^\tau \mathrm{e}^{-njs} v(s)\,\mathrm{d}s\biggr) = v^{[1]}(t) - F^{[1]}(t) \quad \text{for all } t \in [0, \tau).$$

We obtain, using the closedness of A, that $v^{[2]}(t) \in D(A)$ and

$$Av^{[2]}(t) = v^{[1]}(t) - F^{[1]}(t) \quad \text{for all } t \in [0, \tau).$$

Again because A is closed the last equation holds for $t = \tau$. Differentiating this equation once and using the closedness of A yields the desired result:

$$Av^{[1]}(t) = v(t) - F(t) \quad \text{for all } t \in [0, \tau].$$

□

We recall the Ljubich Uniqueness Theorem for the abstract Cauchy problem which will be needed in the sequel.

Proposition 2.4. *Let $A : D(A) \to X$ be a closed operator with $(\omega, \infty) \subseteq \rho(A)$ for some $\omega > 0$, and let*

$$r_A = \limsup_{\lambda \to \infty} \frac{\ln \|R(\lambda, A)\|}{\lambda}.$$

If $r_A \leqslant 0$, then the abstract Cauchy problem

$$u'(t) = Au(t) + F(t), \quad 0 \leqslant t < T \quad \text{and} \quad u(0) = 0$$

has at most one solution.

For a proof, see [**15**, Theorem 1.2, Chapter 4] or [**12**, Theorem 3.1, Chapter 1]. We remark that the Ljubich Uniqueness Theorem can easily be proved using the preceding results of this section.

Now let $F : [0, T) \to \boldsymbol{K}$ be a locally integrable function. For $x \in X$ we call $u : [0, T) \to X$ a solution of the abstract Cauchy problem

$$u'(t) = Au(t) + F^{[1]}(t)x, \quad 0 \leqslant t < T \quad \text{and} \quad u(0) = 0, \tag{2.3}$$

if u belongs to $C^1([0, T), X) \cap C([0, T), D(A))$ and fulfils (2.3).

Definition 2.5. Suppose for every $x \in X$ the abstract Cauchy problem (2.3) has a unique solution u_x. Then we define the family $(S(t))_{0 \leqslant t < T}$ of bounded operators on X by $S(t)x = u_x'(t)$. This family is called the local F-convoluted semigroup on $[0, T)$ with generator A.

The fact that $S(t)$ is bounded follows from the Closed Graph Theorem.

To motivate this definition we consider the case where A generates a C_0-semigroup $(T(t))_{t \geqslant 0}$. The unique solution of (2.3) is given by the variation-of-constant formula,

$$u(t) = \int_0^t F^{[1]}(t - s) T(s) x \, \mathrm{d}s,$$

so A generates a local F-convoluted semigroup on $[0, T)$ for every $T > 0$, namely

$$S(t)x = \int_0^t F(t - s) T(s) x \, \mathrm{d}s.$$

If $F(t) = t^{\alpha-1}/\Gamma(\alpha), \alpha > 0$, then

$$u(t) = \int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)} T(s)x\,\mathrm{d}s \quad \text{and} \quad S(t)x = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T(s)x\,\mathrm{d}s,$$

the α-times integrated semigroup.

The following functional equation can be established for $(S(t))_{0\leqslant t<T}$ (see, for example, [**7**, Theorem 1.2]):

$$S(t)S(s) = \int_0^{t+s} F(t+s-r)S(r)\,\mathrm{d}r - \int_0^t F(t+s-r)S(r)\,\mathrm{d}r - \int_0^s F(t+s-r)S(r)\,\mathrm{d}r \tag{2.4}$$

for $0 \leqslant s,\ t < T,\ s+t < T$.

It follows from the functional equation that $S(t)S(s) = S(s)S(t)$ for $0 \leqslant s,\ t < T$. Starting with a well-posed Cauchy problem on $[0, T)$, one can show (assuming that F can be extended) that the Cauchy problem with $(F * F)(t) = \int_0^t F(t-s)F(s)\,\mathrm{d}s$ replacing F is well-posed on $[0, 2T)$. This is essentially contained in the functional equation (2.4). For E defined on $[0,\infty)$ this remark is used to make the connection to distribution and ultradistribution semigroups (see [**2**]).

We make a remark concerning the generator. The family $(S(t))_{0\leqslant t<T}$ is non-degenerate in the sense that if $S(t)x = 0,\ 0 \leqslant t < T$, then $x = 0$. This follows from the uniqueness assumption. Following [**17**], we define the operator $\tilde{A}$ by

$$D(\tilde{A}) = \left\{ x \in X \mid \exists y \in X,\ S(t)x = \int_0^t S(s)y\,\mathrm{d}s + F(t)x,\ 0 \leqslant t < T \right\}$$

with $\tilde{A}x = y$ for $x \in D(\tilde{A})$.

The non-degeneracy of $(S(t))_{0\leqslant t<T}$ implies that $\tilde{A}$ is single valued. By strong continuity of $(S(t))_{0\leqslant t<T}$, $\tilde{A}$ is closed. It is easy to see that $A \subset \tilde{A}$. Using the fact that the resolvent set of A is non-void (see below), we conclude that $A = \tilde{A}$.

We want to find a condition on F such that the resolvent $R(\lambda, A)$ exists. To this end we apply the previous Lemma 2.3 to the Cauchy problem $u_x(t) = A\int_0^\tau u_x(s)\,\mathrm{d}s + F^{[2]}(t)x$ and obtain

$$(\lambda - A)\int_0^\tau \mathrm{e}^{-\lambda t} u_x(t)\,\mathrm{d}t = \int_0^\tau \mathrm{e}^{-\lambda t} F^{[1]}(t)x\,\mathrm{d}t - \mathrm{e}^{-\lambda t} u_x(t) \tag{2.5}$$

for all λ and all x. Denoting the antiderivative $S^{[1]}$ of S by Q we obtain

$$\frac{\lambda - A}{\int_0^\tau \mathrm{e}^{-\lambda t} F^{[1]}(t)\,\mathrm{d}t} \int_0^\tau \mathrm{e}^{-\lambda t} Q(t)x\,\mathrm{d}t = \left(\mathrm{Id} - \frac{\mathrm{e}^{-\lambda\tau} Q(\tau)}{\int_0^\tau \mathrm{e}^{-\lambda t} F^{[1]}(t)\,\mathrm{d}t} \right) x =: M(\lambda)x. \tag{2.6}$$

Id denotes the identity. Note that the mapping $Q : [0,\tau] \to \mathcal{L}(X)$ with $Q(t)x = u_x(t)$ is Lipschitz continuous since S is uniformly bounded on $[0,\tau]$.

If the operator $M(\lambda)$ is invertible, then $R(\lambda, A)$ exists, as the following proposition shows.

Recall that $\hat{F}_\tau(\lambda) = \int_0^\tau \mathrm{e}^{-\lambda t} F(t)\,\mathrm{d}t$ denotes the finite Laplace transform of F on $[0,\tau]$.

Proposition 2.6. *Suppose A generates a local F-convoluted semigroup on $[0,T)$ and F fulfils*

$$\liminf_{\lambda\to\infty} \frac{\ln|\hat{F}_\tau(\lambda)|}{\lambda} \geqslant 0 \quad \text{for one (and then for all) } \tau\in(0,T). \tag{2.7}$$

Then $(\omega,\infty)\subset\rho(A)$ for some $\omega>0$, and

$$\limsup_{\lambda\to\infty} \frac{\ln\|R(\lambda,A)\|}{\lambda} \leqslant 0.$$

Remark 2.7. The given condition (2.7) is equivalent to

$$\lim_{\lambda\to\infty} \mathrm{e}^{\lambda\delta}|\hat{F}_\tau(\lambda)| = \infty$$

for all $\delta>0$, and the estimate on the resolvent is equivalent to

$$\lim_{\lambda\to\infty} \mathrm{e}^{-\lambda\delta}\|R(\lambda,A)\| = 0$$

for all $\delta>0$.

It is now easy to verify that if F satisfies (2.7) for one τ, then this holds automatically for all τ.

Proof. Set $G=F^{[1]}$. First we note that (2.7) is equivalent to

$$\liminf_{\lambda\to\infty} \frac{\ln|\hat{G}_\tau(\lambda)|}{\lambda} \geqslant 0. \tag{2.8}$$

This follows from

$$\mathrm{e}^{\lambda\delta}\hat{G}_\tau(\lambda) = \frac{\mathrm{e}^{\lambda\delta}}{\lambda}\hat{F}_\tau(\lambda) - \frac{\mathrm{e}^{-\lambda(\tau-\delta)}}{\lambda}G(\tau).$$

Hence, equation (2.7) implies $\hat{G}_\tau(\lambda)\neq 0$ for large λ. For these λ define

$$V(\lambda) := \mathrm{Id} - M(\lambda) = \frac{\mathrm{e}^{-\lambda\tau}}{\hat{G}_\tau(\lambda)}Q(\tau),$$

where the operators $Q(t)$ and $M(\lambda)$ are as in equation (2.6). If $\omega>0$ is so large that $\|V(\lambda)\|\leqslant 1/2$ for all $\lambda\geqslant\omega$, then $M(\lambda)=\mathrm{Id}-V(\lambda)$ is invertible, and

$$\|M(\lambda)^{-1}\| \leqslant \sum_{k=0}^{\infty}\|V(\lambda)\|^k = \frac{1}{1-\|V(\lambda)\|} \leqslant 2 \quad \text{for all } \lambda>\omega.$$

To show

$$R(\lambda,A) = \frac{1}{\hat{G}_\tau(\lambda)}\int_0^\tau \mathrm{e}^{-\lambda t}Q(t)\,\mathrm{d}t\cdot M(\lambda)^{-1} \quad \text{for all } \lambda>\omega \tag{2.9}$$

we use equation (2.6) and show that A and $M(\lambda)^{-1}$ both commute with every $Q(t)$ on $D(A)$.

To do the first we show $Au_x(t) = u_{Ax}(t)$ for $x \in D(A)$ and $t \in [0,T)$. This is true because the function $p(s) = \int_0^s (u_{Ax}(t) + G(t)x)\,\mathrm{d}t$ is the solution of the abstract Cauchy problem (2.3) which yields $u_x'(t) = p'(t) = u_{Ax}(t) + G(t)x$. Consequently, $u_{Ax}(t) = u_x'(t) - G(t)x = Au_x(t)$.

Secondly, we show that a continuous operator B on X commuting with A also commutes with every $Q(t)$. For if B commutes with A, we put $v(t) = Bu_x(t)$ for every $x \in X$ and obtain $Av(t) = BAu_x(t) = B(u_x'(t) - G(t)x) = v'(t) - G(t)Bx$. Because of uniqueness we have $v = u_{Bx}$. Consequently, $Bu_x(t) = v(t) = u_{Bx}(t)$. Replacing B by $Q(t)$ we obtain $Q(s)Q(t) = Q(t)Q(s)$ for all $s, t \in [0,\tau]$, and therefore $M(\lambda)$ commutes with $Q(t)$.

To prove the estimate on the resolvent we fix $\delta > 0$ and obtain

$$\|R(\lambda, A)\|\mathrm{e}^{-\lambda\delta} \leqslant \frac{2\mathrm{e}^{-\lambda\delta}}{|\hat{G}_\tau(\lambda)|}\left\| \int_0^\tau \mathrm{e}^{-\lambda t} Q(t)\,\mathrm{d}t \right\| \quad \text{for all } \lambda > \omega.$$

Now, estimate (2.8) implies that the right side tends to zero as λ goes to infinity. □

Many functions that have appeared so far in connection with abstract Cauchy problems fulfil the estimate (2.7), e.g. the function

$$F(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} \quad \text{for } \alpha > -1.$$

This is a consequence of the following lemma.

Lemma 2.8. *$F \in L_1^{\mathrm{loc}}([0,T), \boldsymbol{K})$ satisfies (2.7) if there exists $\delta \in (0,T)$ with $F^{[1]} \geqslant 0$ on $[0,\delta]$ and a monotonic null sequence $(\delta_n)_n$ with $F^{[1]}(\delta_n) > 0$ (or $F^{[1]} \leqslant 0$ on $[0,\delta]$ and a monotonic null sequence $(\delta_n)_n$ with $F^{[1]}(\delta_n) < 0$).*

Proof. Set $G = F^{[1]}$. We show that if $G \geqslant 0$ on $[0,\delta]$ and $G(\delta) > 0$ for some $\delta \in (0,T)$, then

$$\lim_{\lambda\to\infty} \mathrm{e}^{\lambda\delta}\left| \int_0^\tau \mathrm{e}^{-\lambda t} G(t)\,\mathrm{d}t \right| = \infty.$$

The desired result follows because equations (2.7) and (2.8) are equivalent.

Choose $\varepsilon \in (0,\delta)$ with $G(\varepsilon) > 0$. Then

$$\begin{aligned}
\mathrm{e}^{\lambda\delta}\left| \int_0^\tau \mathrm{e}^{-\lambda t} G(t)\,\mathrm{d}t \right| &\geqslant \mathrm{e}^{\lambda\delta}\left| \int_0^\delta \mathrm{e}^{-\lambda t} G(t)\,\mathrm{d}t \right| - \mathrm{e}^{\lambda\delta}\left| \int_\delta^\tau \mathrm{e}^{-\lambda t} G(t)\,\mathrm{d}t \right| \\
&\geqslant \mathrm{e}^{\lambda\delta} \int_0^\varepsilon \mathrm{e}^{-\lambda t} G(t)\,\mathrm{d}t - \|G\|_\infty \frac{1 - \mathrm{e}^{-\lambda(\tau-\delta)}}{\lambda} \\
&\geqslant \mathrm{e}^{\lambda(\delta-\varepsilon)} \int_0^\varepsilon G(t)\,\mathrm{d}t - \frac{\|G\|_\infty}{\lambda}.
\end{aligned}$$

□

Another condition on $F \in L_1^{\mathrm{loc}}([0,T), \boldsymbol{K})$ so that (2.7) is fulfilled is

$$\lim_{\lambda\to\infty} \mathrm{e}^{\varphi(\lambda)}|\hat{F}_\tau(\lambda)| = \infty \quad \text{for one (and then for all) } \tau \in (0,T),$$

where $\varphi : (0,\infty) \to [0,\infty)$ satisfies $\lim_{\lambda\to\infty} \varphi(\lambda)/\lambda = 0$. Because then for a given $\delta > 0$ one can find $\lambda_0 > 0$ with $\varphi(\lambda)/\lambda < \delta/2$ for all $\lambda > \lambda_0$. For these λ we have

$$\begin{aligned} \mathrm{e}^{\varphi(\lambda)}|\hat{F}(\lambda)| &= \mathrm{e}^{\varphi(\lambda)-\delta\lambda}\mathrm{e}^{\lambda\delta}|\hat{F}(\lambda)| \\ &= \mathrm{e}^{\lambda[\varphi(\lambda)/\lambda-\delta]}\mathrm{e}^{\lambda\delta}|\hat{F}(\lambda)| \\ &\leqslant \mathrm{e}^{-\lambda\delta/2}\mathrm{e}^{\lambda\delta}|\hat{F}(\lambda)| = \mathrm{e}^{\lambda\delta/2}|\hat{F}(\lambda)|. \end{aligned}$$

If $F \in L_1^{\mathrm{loc}}([0,\infty),\boldsymbol{K})$ is defined on the positive semi-axis with $|F(t)| \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0$ with constants M and $\omega \geqslant 0$, there is a condition on F so that (2.7) is fulfilled for any choice of numbers $0 < \tau < T$, namely

$$\lim_{\lambda\to\infty} \mathrm{e}^{\varphi(\lambda)}|\hat{F}(\lambda)| = \infty,$$

where $\varphi : (0,\infty) \to [0,\infty)$ satisfies $\lim_{\lambda\to\infty} \varphi(\lambda)/\lambda = 0$. Because then there is $\lambda_0 > \omega$ with $\varphi(\lambda)/\lambda < \tau/2$ for all $\lambda > \lambda_0$. For these λ we have

$$\mathrm{e}^{\varphi(\lambda)}\hat{F}_\tau(\lambda) = \mathrm{e}^{\varphi(\lambda)}\hat{F}(\lambda) - \mathrm{e}^{\varphi(\lambda)}\int_0^\tau \mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t,$$

and the last term in this equation tends to zero as λ reaches infinity since

$$\left|\mathrm{e}^{\varphi(\lambda)}\int_0^\tau \mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t\right| \leqslant \frac{M}{\lambda-\omega}\mathrm{e}^{\omega\tau}\mathrm{e}^{-\lambda\tau/2}.$$

A similar condition was considered by Cioranescu and Lumer [**8**] but their condition involves the Laplace transform of F (the kernel here) considered as defined on $(0,\infty)$. We do not assume that F is defined on $(0,\infty)$, much less that it is Laplace transformable.

Let $F(t) = t^n/n!$, $0 \leqslant t < T$, and let $\tau \in (0,T)$. Then $\hat{F}_\tau(\lambda)$ can be computed explicitly. In fact,

$$\hat{F}_\tau(\lambda) = \frac{1}{\lambda^{n+1}} - \mathrm{e}^{-\lambda\tau}\sum_{k=0}^{n}\frac{\tau^k}{k!\lambda^{n+1-k}}. \tag{2.10}$$

It is then clear that the above conditions are satisfied. These kernels correspond to distribution semigroups and will lead to polynomial estimates on the resolvent. We further remark that if $T = \infty$, then F is Laplace transformable and $\hat{F}(\lambda) = 1/\lambda^{n+1}$, $\lambda > 0$. For $F(t) = t^\alpha/\Gamma(\alpha+1)$, $\alpha > -1$, the conditions are satisfied with $\psi(\lambda) = 0$.

An important example is the following one offered by Cioranescu [**6**] (corresponding to ultradistributions in the Gevrey classes). Let $0 < a < 1$, $P(\lambda) = \sum_{n=0}^\infty a_n\lambda^n$, $\lambda \in \mathbb{C}$, where $|a_n| = O(L^n(n!)^{-1/a})$, $n \in \mathbb{N}_0$, for some $L > 0$. Then $P(\lambda)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t$, $\lambda > 0$, where $F(\cdot)$ is a bounded C^∞ function. In this case, there exists $l > 0$ such that

$$\mathrm{e}^{(l|\lambda|)^a} \leqslant |P(\lambda)| \leqslant \mathrm{e}^{(L|\lambda|)^a}, \quad \lambda > 0,$$

so that the conditions on $\hat{F}_\tau(\lambda)$ are satisfied.

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3. Generation of local convoluted semigroups

Recall that a function F fulfils the local Widder conditions on $[0,\tau]$ with constants M and ω, if there exist functions $\Phi \in C^\infty((\omega,\infty),X)$ and $\varepsilon : (\omega,\infty) \to X$ with

$$F(\lambda) = \Phi(\lambda) + \varepsilon(\lambda), \quad \lambda > \omega,$$

such that $\limsup_{\lambda\to\infty} \ln\|\varepsilon(\lambda)\|/\lambda \leqslant -\tau$,

$$\sup_{k\in\mathbb{N}_0} \sup_{\mu>\omega+k/\tau} \left\| \frac{(\mu-\omega)^{k+1}}{k!} \Phi^{(k)}(\mu) \right\| \leqslant M \tag{3.1}$$

and

$$\sup_{k\in\mathbb{N}} \sup_{\omega<\mu<\omega+k/\tau} \|\tau^{-k} \mathrm{e}^{\mu\tau} \Phi^{(k)}(\mu)\| < \infty. \tag{3.2}$$

By $\mathrm{Lip}_\omega([0,\tau],X)$ we denote the space of functions $\phi : [0,\tau] \to X$ which are representable as

$$\phi(t) = \int_0^t \mathrm{e}^{\omega s}\, \mathrm{d}\psi(s), \quad 0 \leqslant t \leqslant \tau, \tag{3.3}$$

for some $\psi \in \mathrm{Lip}([0,\tau],X)$. Furthermore, we define $\|\phi\|_{\mathrm{Lip}_\omega} = \|\psi\|_{\mathrm{Lip}}$ (see [**11**] for these definitions).

With these definitions, we state the following result from [**11**, Corollary 2.10] of which we shall make frequent use.

Theorem 3.1. *Let $\omega \in \mathbb{R}$ and $M \geqslant 0$. For every $F \in C^\infty((\omega,\infty),X)$ the following two assertions are equivalent.*

(i) *F is the local Laplace–Stieltjes transform of a function $\phi \in \mathrm{Lip}_\omega([0,\tau],X)$ with $\|\phi\|_{\mathrm{Lip}_w[0,\tau]} \leqslant M$.*

(ii) *F satisfies the local Widder conditions on $[0,\tau]$ with constants M and ω.*

We now state and prove the Hille–Yosida Theorem for local convoluted semigroups. Recall that the Riemann–Stieltjes integral

$$\int_0^\tau \mathrm{e}^{-\lambda t}\, \mathrm{d}F(t) = \mathrm{e}^{-\lambda t} F(\tau) - F(0) + \lambda \int_0^\tau \mathrm{e}^{-\lambda t} F(t)\, \mathrm{d}t$$

exists if $F : [0,\tau] \to X$ is of bounded variation or continuous.

Theorem 3.2. *Let $A : D(A) \to X$ be a closed linear operator on a Banach space X, let $T > 0$, and for each $\tau \in [0,T)$ let M_τ be a positive number. Furthermore, assume that F is a scalar-valued continuous function on $[0,T)$ with $F(0) = 0$ and*

$$\liminf_{\lambda\to\infty} \frac{\ln|\hat{F}_\sigma(\lambda)|}{\lambda} \geqslant 0$$

for at least one (and then for all) $\sigma \in (0,T)$. Then the following statements are equivalent.

(1) *A generates a local F-convoluted semigroup $(S(t))_{0\leqslant t<T}$ on $[0,T)$, and there is $\mu\in\mathbb{R}$ with $\|S\|_{\mathrm{Lip}_\mu[0,\tau]}\leqslant M_\tau$ for all $\tau\in[0,T)$.*

(2) *For all $\tau\in[0,T)$ there are $\omega_1,\mu\in\mathbb{R}$ with $(\omega_1,\infty)\subset\rho(A)$ and the function,*

$$\begin{aligned}(\omega_1,\infty)&\to\mathcal{L}(X),\\ \lambda&\mapsto\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)\cdot R(\lambda,A),\end{aligned}$$

is the local Laplace–Stieltjes transform on $[0,\tau]$ of a function $S:[0,T)\to\mathcal{L}(X)$ with $\|S\|_{\mathrm{Lip}_\mu[0,\tau]}\leqslant M_\tau$.

(3) *For all $\tau\in[0,T)$ there is $\omega\in\mathbb{R}$ with $(\omega,\infty)\subset\rho(A)$, and the function,*

$$\begin{aligned}(\omega,\infty)&\to\mathcal{L}(X),\\ \lambda&\mapsto\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)\cdot R(\lambda,A),\end{aligned}$$

fulfils the local Widder conditions on $[0,\tau]$ with constant M_τ.

Remark 3.3. In (1) and (2) the appearing constants μ can be chosen equal; from (2) to (3) one can take $\omega=\max\{\mu,\omega_1\}$; from (3) to (2) one can take $\omega_1=\mu=\omega$.

Proof. (1) $\Rightarrow$ (2). In view of Proposition 2.6 the resolvent exists on a right half-line (ω_1,∞). Using Lemma 2.5 we obtain for $\lambda\in\mathbb{C}$ and $x\in X$

$$(\lambda-A)\int_0^\tau \mathrm{e}^{-\lambda t}S(t)x\,\mathrm{d}t=\int_0^\tau \mathrm{e}^{-\lambda t}F(t)x\,\mathrm{d}t-\mathrm{e}^{-\lambda\tau}S(\tau)x.$$

Multiplying this equation by λ, using $F(0)=0$ and partial integration yields

$$(\lambda-A)\biggl(\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t)x-\mathrm{e}^{-\lambda\tau}S(\tau)x\biggr)=\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)x-\mathrm{e}^{-\lambda\tau}F(\tau)x-\lambda\mathrm{e}^{-\lambda\tau}S(\tau)x.$$

Defining $\varepsilon:(\omega_1,\infty)\to\mathcal{L}(X)$ by

$$\varepsilon(\lambda):=\lambda\mathrm{e}^{-\lambda\tau}R(\lambda,A)S(\tau)+\mathrm{e}^{-\lambda\tau}F(\tau)R(\lambda,A)-\mathrm{e}^{-\lambda\tau}S(\tau)$$

we obtain

$$\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)R(\lambda,A)=\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t)+\varepsilon(\lambda)\quad\text{for }\lambda>\omega_1$$

and

$$\limsup_{\lambda\to\infty}\frac{\ln\|\varepsilon(\lambda)\|}{\lambda}\leqslant-\tau.$$

(2) $\Rightarrow$ (3). Let $\omega=\max\{\mu,\omega_1\}$ and apply Theorem 3.1.
(3) $\Rightarrow$ (2). Let $\omega_1=\mu=\omega$ and again apply Theorem 3.1.

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(2) ⇒ (1). For all $\tau \in (0,T)$ we have

$$\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)\cdot R(\lambda,A) = \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t) + \varepsilon_\tau(\lambda) \quad \text{for all } \lambda > \omega_1$$

with

$$\limsup_{\lambda\to\infty} \frac{\ln\|\varepsilon_\tau(\lambda)\|}{\lambda} \leqslant -\tau.$$

This gives

$$A\biggl(\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t) + \varepsilon_\tau(\lambda)\biggr) = \lambda\biggl(\int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t) + \varepsilon_\tau(\lambda)\biggr) - \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)\cdot \mathrm{Id}\,.$$

Dividing this equation by λ and using partial integration we get

$$\begin{aligned} A\biggl(\frac{1}{\lambda}\mathrm{e}^{-\lambda\tau}S(\tau) + \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S^{[1]}(t) + \frac{\varepsilon_\tau(\lambda)}{\lambda}\biggr) \\ = \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t) + \varepsilon_\tau(\lambda) - \frac{1}{\lambda}\mathrm{e}^{-\lambda\tau}F(\tau)\cdot\mathrm{Id} + \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}F^{[1]}(t)\cdot\mathrm{Id}\,. \end{aligned}$$

Using the Phragmén–Doetsch inversion formula we see that $S^{[1]}(t)x \in D(A)$ and

$$AS^{[1]}(t)x = S(t)x - \int_0^t F(s)x\,\mathrm{d}s \quad \text{for all } x \text{ and } t \in [0,\tau].$$

It remains to show that the solution is unique. To this end we have

$$\int_0^\sigma \mathrm{e}^{-\lambda t}\,\mathrm{d}F(t)\cdot R(\lambda,A) = \int_0^\sigma \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t) + \varepsilon_\sigma(\lambda),$$

which implies

$$\mathrm{e}^{-\lambda\delta}R(\lambda,A) = \frac{\int_0^\sigma \mathrm{e}^{-\lambda t}\,\mathrm{d}S(t) + \varepsilon_\sigma(\lambda)}{\mathrm{e}^{\lambda\delta}}\biggl(\mathrm{e}^{-\lambda\sigma}F(\sigma) + \lambda\int_0^\sigma \mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t\biggr).$$

This gives

$$\lim_{\lambda\to\infty} \mathrm{e}^{-\lambda\delta}R(\lambda,A) = 0 \quad \text{for all } \delta > 0$$

and therefore

$$\limsup_{\lambda\to\infty} \frac{\ln\|R(\lambda,A)\|}{\lambda} \leqslant 0.$$

Uniqueness of the solution follows now from the Ljubich Uniqueness Theorem (Proposition 2.4). □

Corollary 3.4. *Let $A : D(A) \to X$ be a closed and densely defined linear operator on a Banach space X. Let $T > 0$, and for each $\tau \in [0,T)$, M_τ be a positive number. Furthermore, assume that E is a scalar-valued locally integrable function on $[0,T)$ with*

$$\liminf_{\lambda\to\infty} \frac{\ln|\hat{E}_\sigma(\lambda)|}{\lambda} \geqslant 0$$

for at least one (and then for all) $\sigma \in (0,T)$. Then the following statements are equivalent.

(1) *A generates a local E-convoluted semigroup $(U(t))_{0\leqslant t<T}$ on $[0,T)$ with*

$$\|U\|_{L_\infty[0,\tau]} \leqslant M_\tau \quad \text{for all } \tau \in [0,T).$$

(2) *There is $\omega \geqslant 0$ with $(\omega,\infty) \in \rho(A)$ and for all $\tau \in [0,T)$ the function,*

$$(\omega,\infty) \to \mathcal{L}(X),$$
$$\lambda \mapsto \int_0^\tau \mathrm{e}^{-\lambda t} E(t)\,\mathrm{d}t \cdot R(\lambda, A),$$

fulfils the local Widder conditions on $[0,\tau]$ with constant M_τ.

Proof. (1) $\Rightarrow$ (2). Let $F = E^{[1]}$. For every $x \in X$ the Cauchy problem,

$$u'(t) = Au(t) + F(t)x, \qquad 0 \leqslant t < T \quad \text{and} \quad u(0) = 0,$$

has a unique solution u_x, and so does the Cauchy problem,

$$v'(t) = Av(t) + F^{[1]}(t)x, \quad 0 \leqslant t < T \quad \text{and} \quad v(0) = 0.$$

The solution of the latter is

$$v(s) = \int_0^s u(t)\,\mathrm{d}t, \quad 0 \leqslant t < T.$$

Therefore, A generates a local F-convoluted semigroup $(S(t))_{0\leqslant t<T}$ on $[0,T)$, namely $S(t)x = u_x(t)$, and F fulfils

$$\liminf_{\lambda\to\infty} \frac{\ln|\hat{F}_\sigma(\lambda)|}{\lambda} \geqslant 0.$$

(Recall that the estimates (2.7) and (2.8) are equivalent.)

Furthermore, the estimate $\|S\|_{\mathrm{Lip}[0,\tau]} = \|S\|_{\mathrm{Lip}_0[0,\tau]} \leqslant \|U\|_{L_\infty[0,\tau]}$ follows from $S(t)x = \int_0^t u_x'(s)\,\mathrm{d}s = \int_0^t U(s)x\,\mathrm{d}s$ for $x \in X$ and $0 \leqslant t \leqslant \tau$. Now Theorem 3.2 gives the desired result.

(2) $\Rightarrow$ (1). Again let $F = E^{[1]}$. Then, by virtue of Theorem 3.2, the operator A generates a local F-convoluted semigroup $(S(t))_{0\leqslant t<T}$ on $[0,T)$ with $\|S\|_{\mathrm{Lip}_\omega[0,\tau]} \leqslant M_\tau$ for all $\tau \in [0,T)$. Consequently,

$$S(t)x = A\int_0^t S(s)x\,\mathrm{d}s + \int_0^t F(s)x\,\mathrm{d}s$$

for all $x \in X$ and all $t \in [0,T)$. For $x \in D(A)$ this becomes

$$S(t)x = \int_0^t S(s)Ax\,\mathrm{d}s + \int_0^t F(s)x\,\mathrm{d}s.$$

Defining $U(t)x = S'(t)x$ we get

$$U(t)x = S(t)Ax + F(t)x.$$

Because A commutes with every $S(t)$ we have

$$U(t)x = A\int_0^t U(s)x\,\mathrm{d}s + \int_0^t E(s)x\,\mathrm{d}s. \tag{3.4}$$

For every $\tau \in [0,T)$ the linear operator $U_\tau : D(A) \to C([0,\tau],X)$ defined by $U_\tau x(t) = U(t)x$ is bounded with $\|U_\tau\| \leqslant \mathrm{e}^{\omega\tau}\|S\|_{\mathrm{Lip}_\omega[0,\tau]}$ and therefore can be extended uniquely to the closure of $D(A)$, i.e. to X. We again use the notation U for the family $U : [0,T) \to \mathcal{L}(X)$ defined by $U(t)x = U_t(x)(t)$. It remains to show that the Cauchy problem,

$$u'(t) = Au(t) + F(t)x, \quad 0 \leqslant t < T \quad \text{and} \quad u(0) = 0,$$

has for every $x \in X$ the unique solution $u(t) = \int_0^t U(s)x\,\mathrm{d}s$. Because A is closed, equation (3.4) holds for $x \in X$, and uniqueness follows from the fact that A generates a local convoluted semigroup. □

From the local cases one gets a result for global convoluted semigroups. First we clarify the notation. By σ_c we denote the abscissa of convergence, i.e. σ_c is the infimum of all real λs such that $\lim_{R\to\infty}\int_0^R \mathrm{e}^{-\lambda t}E(t)\,\mathrm{d}t$ converges (see [**18**]). If $\sigma_\mathrm{c} \leqslant \eta$ with $\eta \geqslant 0$, then $\limsup_{t\to\infty}|E^{[1]}(t)|/t \leqslant \eta$ and $E^{[2]}$ is in $\mathrm{Lip}_{\eta+\varepsilon}$ for every $\varepsilon > 0$.

Definition 3.5. Let $A : D(A) \to X$ be a closed linear operator on a Banach space X and $E : [0,\infty) \to \boldsymbol{K}$ be a locally integrable function with abscissa of convergence $\sigma_\mathrm{c} < \infty$. We say that A generates an exponentially bounded E-convoluted semigroup $(S(t))_{t\geqslant 0}$, if there is a strongly continuous function $S : [0,\infty) \to \mathcal{L}(X)$ with $S(0) = 0$, constants $\omega \geqslant \sigma_\mathrm{c}$ and $M \geqslant 0$ with $\|S(t)\| \leqslant M\mathrm{e}^{\omega t}$ and $(\omega,\infty) \subset \rho(A)$ such that

$$\hat{E}(\lambda)\cdot R(\lambda,A) = \int_0^\infty \mathrm{e}^{-\lambda t}S(t)\,\mathrm{d}t \quad \text{for all } \lambda > \omega.$$

Hieber [**10**] did this for $E(t) = t^{\alpha-1}/\Gamma(\alpha)$ and $\alpha > 0$, which leads to α-times integrated semigroups, and Arendt [**1**] considered n-times integrated semigroups, i.e. exponentially bounded $(t^{n-1}/(n-1)!)$-convoluted semigroups.

Before stating the announced theorem we need a lemma which shows the exceptional position of the resolvent with regard to Lemma 2.12 in [**11**].

Lemma 3.6. *Let $A : D(A) \to X$ be a closed linear operator on a Banach space X and $E : [0,\infty) \to \boldsymbol{K}$ a locally integrable function with abscissa of convergence $\sigma_\mathrm{c} < \infty$. Furthermore, assume that there is $\omega \geqslant \max\{0,\sigma_\mathrm{c}\}$ with $(\omega,\infty) \subset \rho(A)$ and*

$$\hat{E}(\lambda)\cdot R(\lambda,A) = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\phi(t) + \varepsilon(\lambda) \quad \text{for all } \lambda > \omega$$

with functions $\phi \in \mathrm{Lip}_\omega([0,\infty),\mathcal{L}(X))$ and ε satisfying

$$\lim_{\lambda\to\infty}\frac{\ln\|\varepsilon(\lambda)\|}{\lambda} = -\infty.$$

Then ε vanishes identically on (ω,∞).

Proof. Let $F = E^{[1]}$. Because of

$$\int_0^\infty \mathrm{e}^{-\lambda t} E(t)\,\mathrm{d}t \cdot R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\phi(t) + \varepsilon(\lambda),$$

we obtain, if $\lambda > \omega$,

$$A\biggl(\int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\phi^{[1]}(t) + \frac{\varepsilon(\lambda)}{\lambda}\biggr) = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\phi(t) + \varepsilon(\lambda) - \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}F^{[1]}(t) \cdot \mathrm{Id}\,.$$

Applying the Phragmén–Doetsch inversion formula gives

$$A\phi^{[1]}(t) = \phi(t) - F^{[1]}(t) \cdot \mathrm{Id}\,.$$

Thus we obtain

$$(\lambda - A) \int_0^\infty \mathrm{e}^{-\lambda t} \phi^{[1]}(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} F^{[1]}(t)\,\mathrm{d}t \cdot \mathrm{Id}$$

and after integrating by parts

$$\int_0^\infty \mathrm{e}^{-\lambda t} E(t)\,\mathrm{d}t \cdot R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\phi(t),$$

which gives the desired result. □

Theorem 3.7. *Let $A : D(A) \to X$ be a closed, densely defined linear operator on a Banach space X and $E : [0, \infty) \to \boldsymbol{K}$ a locally integrable function with abscissa of convergence $\sigma_{\mathrm{c}} < \infty$. Furthermore, assume that there is a $\lambda_0 \geqslant \eta := \max\{0, \sigma_{\mathrm{c}}\}$ with $\hat{E}(\lambda) \neq 0$ for all $\lambda > \lambda_0$.*

Then the following assertions are equivalent.

(1) *A generates an exponentially bounded E-convoluted semigroup.*

(2) *There are $\omega > \eta$ and $C > 0$ such that the abstract Cauchy problem,*

$$u(t) = A \int_0^t u(s)\,\mathrm{d}s + E^{[1]}(t)x,$$
$$u \in C([0, \infty), X) \text{ with } \|u(t)\| \leqslant C\mathrm{e}^{\omega t} \text{ and } u^{[1]}(t) \in D(A) \quad \text{for all } t \geqslant 0,$$

has a unique solution u_x for every $x \in X$.

(3) *There is $\omega > \eta$ with $(\omega, \infty) \subset \rho(A)$, and for every $\tau > 0$ the function*

$$(\omega, \infty) \to \mathcal{L}(X),$$
$$\lambda \mapsto \hat{E}(\lambda) \cdot R(\lambda, A)$$

fulfils the local Widder conditions on $[0, \tau]$ with constants M and ω (independent of τ).

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(4) *There is* $\omega > \eta$ *with* $(\omega, \infty) \subset \rho(A)$ *and*

$$\sup_{k \in \mathbb{N}_0} \sup_{\lambda > \omega} \left\| \frac{(\lambda - \omega)^{k+1}}{k!} (\hat{E}(\lambda) \cdot R(\lambda, A))^{(k)} \right\| \leqslant M$$

for some $M \geqslant 0$.

Remark 3.8.

(i) For n-times integrated semigroups the equivalence of the first two assertions was shown by Neubrander [**14**].

(ii) Although there is no locally integrable function E on $[0, \infty)$ whose Laplace transform is identically equal to 1, the argument used in the proof of Theorem 3.7 applies to the case of the Dirac measure concentrated at 0 (whose Laplace transform is identically equal to 1). This is important for (iii) below.

(iii) To obtain Theorem 1.1, we take $\hat{E}(\lambda) \equiv 1$, that is, E is the Dirac delta function. Theorem 1.1 is then (1) $\Leftrightarrow$ (3). In this case, $E^{[1]}(t) \equiv 1$. The fact that well-posedness of the Cauchy problem,

$$u'(t) = Au(t) + x, \quad 0 \leqslant t \leqslant \tau, \qquad u(0) = 0 \quad \text{for some } \tau > 0,$$

is equivalent to A generating a C_0-semigroup was originally observed by van Casteren (see [**2**, Theorem 1.2].

Proof of the theorem. (1) $\Rightarrow$ (4). There are constants $\omega > \eta$ and $M \geqslant 0$ and a strongly continuous function $S : [0, \infty) \to \mathcal{L}(X)$ with $S(0) = 0$ and with $\|S(t)\| \leqslant M\mathrm{e}^{\omega t}$ such that

$$\hat{E}(\lambda) \cdot R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)x \,\mathrm{d}t \quad \text{for all } \lambda > \omega \text{ and all } x \in X.$$

The function $\phi_x(t) = \int_0^t S(s)x \,\mathrm{d}s$ is in $\mathrm{Lip}_\omega([0, \infty), X)$ with $\|\phi\|_{\mathrm{Lip}_\omega} \leqslant M\|x\|$; thus Corollary 2.13 of [**11**] yields (4).

(4) $\Rightarrow$ (3). This implication is a consequence of Lemma 2.12 and Corollary 2.13 of [**11**].

(3) $\Rightarrow$ (2). Applying Lemma 2.12 of [**11**] and Lemma 3.6 from this paper, we obtain a function $\phi \in \mathrm{Lip}_\omega([0, \infty), \mathcal{L}(X))$ with $\|\phi\|_{\mathrm{Lip}_\omega} \leqslant M$ such that

$$\hat{E}(\lambda) \cdot R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t} \,\mathrm{d}\phi(t) \quad \text{for all } \lambda > \omega.$$

This gives

$$A \int_0^\infty \mathrm{e}^{-\lambda t} \,\mathrm{d}\phi^{[1]}(t) = \int_0^\infty \mathrm{e}^{-\lambda t} \,\mathrm{d}\phi(t) - \int_0^\infty \mathrm{e}^{-\lambda t} \,\mathrm{d}E^{[2]}(t) \cdot \mathrm{Id},$$

and the Phragmén–Doetsch inversion formula yields

$$A\phi^{[1]}(t) = \phi(t) - E^{[2]}(t) \cdot \mathrm{Id}\,.$$

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Therefore, for every $x \in X$ the abstract Cauchy problem,

$$\left.\begin{aligned} v(t) &= \int_0^t v(s)\,\mathrm{d}s + E^{[2]}(t)x, \\ v \in C([0,\infty,X) &\text{ exponentially bounded and } v^{[1]}(t) \in D(A), \end{aligned}\right\} \tag{3.5}$$

has a solution $v_x(t) = \phi(t)x$. But this is the only one, since if v solves the above Cauchy problem for some x we have

$$(\lambda - A)\int_0^\infty \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} E^{[1]}(t)x\,\mathrm{d}t,$$

which gives

$$\int_0^\infty \mathrm{e}^{-\lambda t} v(t)\,\mathrm{d}t = \frac{1}{\lambda}\hat{E}(\lambda)\cdot R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t}\phi(t)x\,\mathrm{d}t.$$

Thus $v(t) = \phi(t)x$.

Next we remark that A commutes with every $\phi(t)$. This follows the same idea as in the proof of Proposition 2.6. For $x \in D(A)$ define the function $p(s) = v_{Ax}(s) + E^{[1]}(s)x$, which gives

$$A\int_0^t p(s)\,\mathrm{d}s = A\int_0^t v_{Ax}(s)\,\mathrm{d}s + E^{[2]}(t)x = v_{Ax}(t).$$

So we have

$$p(t) = A\int_0^t p(s)\,\mathrm{d}s + E^{[1]}(t)x,$$

which gives

$$p^{[1]}(t) = A\int_0^t p^{[1]}(s)\,\mathrm{d}s + E^{[2]}(t)x.$$

Thus $p^{[1]} = v_x$ because $p^{[1]}$ solves the Cauchy problem (3.5), and $A\phi(t)x = Ap^{[1]}(t) = v_{Ax}(t) = \phi(t)Ax$. Therefore, we have

$$\phi'(t)x = \phi(t)Ax + E^{[1]}(t)x \quad \text{for } t \geqslant 0 \text{ and } x \in D(A). \tag{3.6}$$

Now define the Banach space $Y = \{f \in C([0,\infty,X) : \text{there is } M \geqslant 0 \text{ with } \|f(t)\| \leqslant M\mathrm{e}^{\omega t} \text{ for all } t \geqslant 0\}$ with norm $\|f\|_Y = \inf\{M \geqslant 0 : \|f(t)\| \leqslant M\mathrm{e}^{\omega t}\}$, and consider the linear operator

$$V : D(A) \to Y \quad \text{with } Vx(t) = \phi'(t)x.$$

V is bounded, since the function Vx is continuous because of (3.6); furthermore, we have, for $t > s \geqslant 0$,

$$\left\|\frac{1}{t-s}\int_s^t Vx(r)\,\mathrm{d}r\right\| = \left\|\frac{\phi(t)x - \phi(s)x}{t-s}\right\| \leqslant \|\phi\|_{\mathrm{Lip}_\omega}\frac{1}{t-s}\int_s^t \mathrm{e}^{\omega r}\,\mathrm{d}r\cdot\|x\|.$$

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For $s \uparrow t$ we obtain

$$\|Vx(t)\| \leqslant \|\phi\|_{\mathrm{Lip}_\omega} \cdot \mathrm{e}^{\omega t} \cdot \|x\|.$$

Consequently, $\|V\| \leqslant \|\phi\|_{\mathrm{Lip}_\omega}$. Because A is densely defined we extend V on X and will again call this operator V.

Because A is closed it follows from (3.6) that

$$Vx(t) = A\int_0^t Vx(s)\,\mathrm{d}s + E^{[1]}(t)x \quad \text{for } t \geqslant 0 \text{ and } x \in X.$$

If on the other hand u solves the Cauchy problem in (2), we have

$$u^{[1]}(t) = A\int_0^t u^{[1]}(s)\,\mathrm{d}s + E^{[2]}(t)x.$$

Consequently, $u^{[1]} = \phi(t)x$ because $u^{[1]}$ solves (3.5).

(2) $\Rightarrow$ (1). Define Y as above and consider the linear operator

$$T : X \to Y \quad \text{with } Tx = u_x.$$

T is closed and therefore bounded. Furthermore, consider the strongly continuous function

$$S : [0,\infty) \to \mathcal{L}(X) \quad \text{with } S(t)x = u_x(t).$$

Then $S(0) = 0$ and $\|S(t)\| \leqslant \|T\| \cdot \mathrm{e}^{\omega t}$.

We notice that A commutes with every $S(t)$, because with

$$p(s) = \int_0^s u_{Ax}(r)\,\mathrm{d}r + E^{[1]}(s)x$$

for $x \in D(A)$ we have $p \in Y$ and

$$A\int_0^t p(s)\,\mathrm{d}s = \int_0^t u_{Ax}(s) - E^{[1]}(s)Ax\,\mathrm{d}s + E^{[2]}(t)Ax = p(t) - E^{[1]}(t)x.$$

Thus $p = u_x$ and $S(t)Ax = AS(t)x$. Because u_x solves the given Cauchy problem, we obtain

$$(\lambda - A)\int_0^\infty \mathrm{e}^{-\lambda t}u_x(t)\,\mathrm{d}t = \hat{E}(\lambda)x \quad \text{for } \lambda > \omega.$$

For $\lambda > \max\{\omega, \lambda_0\}$ we define the linear operator $R \in \mathcal{L}(X)$ by

$$Rx = \frac{1}{\hat{E}(\lambda)}\int_0^\infty \mathrm{e}^{-\lambda t}S(t)x\,\mathrm{d}t.$$

Then we have

$$\|R\| \leqslant \frac{\|T\|}{(\lambda - \omega)\hat{E}(\lambda)}$$

and $(\lambda - A)Rx = x$.

Because A commutes with every $S(t)$ we also have $R(\lambda - A) = \mathrm{Id}_{D(A)}$. Therefore, A generates the exponentially bounded E-convoluted semigroup $(S(t))_{t\geqslant 0}$. □

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