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NILPOTENT SUBGROUPS OF GL(n, $\mathbb{Q}$)

B. A. F. WEHRFRITZ
School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, England
e-mail: b.a.f.wehrfritz@qmw.ac.uk

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Abstract. We compute the precise bounds, for every positive integer n, for the nilpotency class of nilpotent subgroups of GL(n, $\mathbb{Q}$) and GL(n, $\mathbb{Z}$).

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1. Introduction. Three times over the past few months I have been asked, directly and indirectly, about the maximal class of a nilpotent subgroup of GL(n, $\mathbb{Z}$). The bound $3n/2$ seems to be generally known. In fact the bound n holds and indeed for most n the true bound is less than n. In view of the recent interest in this question it seems worthwhile to publish a definitive result.

Theorem 1. *Let G be a nilpotent subgroup of* GL(n, $\mathbb{Z}$) *of class k. Then $k \leq n-1$ if n is not a power of* 2 *and $k \leq n$ if n is a power of* 2. *Moreover these bounds are attained for all n.*

Our approach is first to analyse the nilpotent subgroups of GL(n, $\mathbb{Q}$) and then to derive the integer case. Since the bound in the rational case is larger than that in the integer case, at first sight this might not seem sensible. However the analysis shows that the exceptional groups, that is those of class exceeding n, are relatively rare and have a sufficiently specific structure to be easily eliminated in the integer case. It also shows why the bound $3n/2$ can arise naturally This bound of $3n/2$ is in fact attained for some n in the rational case, namely for those $n > 1$ that are powers of 2. If n is not a power of 2 then the bound is less. Indeed for about one n in two it is less than n.

Theorem 2. *Let G be a nilpotent subgroup of* GL(n, $\mathbb{Q}$) *of class k.*
Define integers ρ and s by $n = 2^\rho + s$, where $0 \leq s < 2^\rho$. Then
$k \leq 1$ *if* $n = 1$; *otherwise*
$k \leq 3.2^{\rho-1}$ *if* $0 \leq s \leq 2^{\rho-1}$ *and*
$k \leq n-1$ *if* $2^{\rho-1} < s < 2^\rho$.
Moreover these bounds are attained for all n by subgroups of GL(n, $\mathbb{Z}[1/2]$).

One can carry out similar analyses for other ground fields, for example finite extensions of $\mathbb{Q}$. For the reals and complexes nothing interesting happens; 1-dimensional groups are abelian and GL(2, $\mathbb{R}$) contains copies of every dihedral group and hence contains nilpotent groups of every class.

The corresponding precise bounds for finite (every periodic subgroup of $GL(n, \mathbb{Q})$ is finite) nilpotent subgroups of $GL(n, \mathbb{Q})$ are generally smaller than those given in the Theorem 2. They can be read off from the analysis below and also from the structure of the Sylow subgroups of $GL(n, \mathbb{Q})$ given, for example, in [**1**]. Specifically, a finite p-subgroup (throughout this paper p denotes a positive integer prime) of $GL(n, \mathbb{Q})$ has class at most p^f, where $f = -1, 0, 1, 2, \ldots$ is defined by $p^f(p-1) \le n < p^{f+1}(p-1)$. If $p > 2$ and if $2^\rho \le n < 2^{\rho+1}$, then $p^f \le n/2 < 2^\rho$. Thus any finite nilpotent subgroup of $GL(n,\mathbb{Q})$ has class at most 2^ρ and in particular has class at most n with equality only if $n = 2^\rho$. Moreover these bounds are achieved by subgroups of $GL(n, \mathbb{Z})$, as can be seen from either [**1**] or the examples below.

Theorem 2 claims that there exist nilpotent subgroups of $GL(n, \mathbb{Q})$ with class exceeding n for about one n in two. Such groups are not common. The previous paragraph shows that such a group must contain elements of infinite order. They must also contain involutions (and some of the involutions and elements of infinite order must interact in a relatively specific way).

Proposition 1. *Let G be a nilpotent subgroup of* $GL(n, \mathbb{Q})$, *where $n > 1$. Then the n-th term $\gamma^n G$ of the lower central series of G is a finite 2-group. In particular, if G contains no involutions, then G is nilpotent of class less than n.*

Let p be a prime. We have to work with something more general than p-groups. A group G is *p-primary* if G modulo its centre Z is a p-group. If such a group G is also linear, then the periodic group G/Z is locally finite (use [**6**, 6.2 and 4.9]). Consequently the derived subgroup G' of G is also a locally finite p-group. The basic building blocks of nilpotent linear groups are the unipotent groups and the p-primary groups for various primes p. We need at least part of the following proposition to prove the theorems.

Proposition 2. *Let n be a positive integer, p a positive prime and set* $n_p = \max\{1, n/(p-1)\}$. *Let G be an irreducible p-primary subgroup of* $GL(n, \mathbb{Q})$. *Then G is nilpotent of class k, where*

$k < n_p \le n$	*if n_p is not a power of p,*
$k \le n_p < \max\{n, 2\}$	*if $p > 2$ and if n_p is a power of p and*
$k \le 3n/2$	*if $p = 2$ and if $n = n_2$ is a power of 2.*

Over the rationals there are clearly differences between the 2-primary and the 2-group cases. However the p-primary and the p-group cases for odd p are not quite as similar as at first sight they might appear. For example, $GL(n, \mathbb{Q})$ contains an irreducible p-subgroup only if $n = p^f(p-1)$ for some integer f, apart from the trivial case $n = 1$, while $GL(n, \mathbb{Q})$ contains an irreducible p-primary subgroup for every p and n.

2. Examples.

2.1. The full unitriangular group $Tr_1(n, \mathbb{Z})$ over $\mathbb{Z}$ is nilpotent of class exactly $n - 1$ for all $n \ge 1$.

2.2. Let $n = 2^f$ for some $f \ge 0$. Then the wreath product $\{\pm 1\}$wr $Sym(n)$ embeds into $GL(n, \mathbb{Z})$ as generalized permutation matrices. If S_f is a Sylow 2-subgroup of $Sym(2^f)$, then S_{f+1} is isomorphic to $\{\pm 1\}$wr S_f, which embeds into $GL(n, \mathbb{Z})$. The nilpotency class of S_{f+1} is $2^f = n$; see [**1**] or [**4**, III.15.3e] for details.

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2.3. Let $n = (p-1)p^f$. Now GL($p-1, \mathbb{Z}$) contains the companion matrix of the polynomial $1 + X + X^2 + \ldots + X^{p-1}$. If here S_f denotes a Sylow p-subgroup of Sym(p^f), then GL($n, \mathbb{Z}$) contains via generalized permutation matrices a copy of S_{f+1}, and the latter is nilpotent of class $p^f = n/(p-1)$. Again, see **[1]** for details.

2.4. Let $r = 2^f$, for some $f \geq 0$, and set $n = 2r$. Then GL($n, \mathbb{Z}[1/2]$) contains a nilpotent 2-primary subgroup of class at least $3r = 3n/2$.

In fact the examples in 2.4 have class exactly $3n/2$. This can be computed directly, but it follows immediately once we have proved Theorem 2.

Proof. Let $K = \mathbb{Q}(i) \leq \mathbb{C}$, let γ denote complex conjugation, set $\Gamma =< \gamma >$ and let $< x >$ be a cyclic group of order $r = 2^f$. For $k = 1, 2, \ldots, r$ let $\alpha \longmapsto \alpha_k$ be an isomorphism of the split extension $\Gamma.K^*$ onto an isomorphic copy of itself, K^* here denoting the multiplicative group of K. Consider the wreath product

$$(\Gamma.K^*)\mathrm{wr} < x > = < x > .\prod\nolimits_{k=1}^{r} (\Gamma.K^*)_k,$$

where $< x >$ acts on the cartesian product by permuting the suffices cyclically in the natural way. Then $(\Gamma.K^*)\mathrm{wr} < x >$ embeds into GL(2r, $\mathbb{Q}$) via block permutation matrices of degree $2r$, where $\Gamma.K^*$ embeds into GL($2, \mathbb{Q}$) via

$$i \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\; \& \;\; \gamma \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the images of i, γ and $(1+i)^{-1}$ in GL($2, \mathbb{Q}$) all lie in GL($2, \mathbb{Z}[1/2]$). Consider the subgroup

$$G =< i_k, \gamma_k, y = x(1+i)_1 : k = 1, 2, \ldots, r >$$

of $(\Gamma.K^*)\mathrm{wr} < x >$. We claim the following.

(a) G embeds into GL($n, \mathbb{Z}[1/2]$).

(b) G is a nilpotent 2-primary group.

(c) The nilpotency class of G is at least $3r$.

(a) Since here $n = 2r$, this follows from the above.

(b) $I =< i_k : k = 1, 2, \ldots, r >$ is a normal subgroup of G. Set $J =< \gamma_k : k = 1, 2, \ldots, r >$. Then IJ is a 2-subgroup (of order 2^{3r}) of G. Now IJ is normal in G, for

$$\begin{aligned} \gamma_k^y &= (1+i)_1^{-1} x^{-1} \gamma_k x (1+i)_1 = (1+i)_1^{-1} \gamma_{k+1} (1+i)_1 = \gamma_{k+1}, \text{ if } k < r \text{ and} \\ &= (1+i)_1^{-1} \gamma_1 (1+i)_1 = \gamma_1 (1-i)_1^{-1} (1+i)_1 = \gamma_1 i_1 \in IJ, \text{ if } k = r. \end{aligned}$$

Also

$$y^r = (x(1+i)_1)^r = x^r (1+i)_r (1+i)_{r-1} \ldots (1+i)_2 (1+i)_1$$

and $(1+i)^4 = -4 \in \mathbb{Q}$. Thus y^{4r} lies in the centre Z of G and hence G/Z is a finite 2-group (of order dividing $2^{3r+2+\rho}$). In particular G is nilpotent and 2-primary.

(c) We prove this by showing that the multiple commutator $[\gamma_{1,2r-1} y, \gamma_{1,r-1} y]$ of length $3r - 1$ is non-trivial. If M is a $< y >$ module and if a lies in M, then

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$$[a, {}_h y] = a(y-1)^h = \sum_{k=0}^{h}(-1)^{h-k}\binom{h}{k}ay^k.$$

The proof of (b) above yields that $[\gamma_k, y] = \gamma_k^{-1}\gamma_{k+1}$ if $k < r$ and $[\gamma_k, y] = \gamma_r^{-1}\gamma_1 i_1$ if $k = r$. Hence

$$[\gamma_{1,r-1}y] = \prod_{k=0}^{r-1}\gamma_{1+k}{}^{\binom{r-1}{k}}$$

(the (-1)'s are irrelevant since $|\gamma| = 2$) and

$$[\gamma_1, {}_r y] = (\prod_{k=0}^{r-1}\gamma_{1+k}{}^{\binom{r}{k}})(\gamma_1 i_1) = i_1,$$

since the binomial coefficients here are even for $0 < k < r$. Then

$$[\gamma_{1,2r-1}y] = [i_1, {}_{r-1}\, y] = \prod_{k=0}^{r-1} i_{1+k}^{(-1)^{r-1-k}\binom{r-1}{k}},$$

and so $[\gamma_{1,2r-1}y, \gamma_1] = [i, \gamma]_1{}^{(-1)^{r-1}} = (-1)_1$.

Therefore

$$[\gamma_{1,2r-1}y, \gamma_{1,r-1}y] = \prod_{k=0}^{r-1}(-1)_{1+k}{}^{\binom{r-1}{k}} = \prod_{k=1}^{r}(-1)_k \neq 1.$$

The claim follows.

3. Special cases. In the main general arguments suffice. However there are a number of small cases that need to be handled separately. Fortunately most of these follow a common pattern. These we deal with in this section.

Say that a p-primary group G has ec-*height at most* n, an integer, if G has a central series (running from $< 1 >$ to G itself) of length n whose factors with at most one exception are elementary abelian p-groups and that exception, if it exists, has its maximal p-subgroup of exponent dividing p. In particular such a group is nilpotent of class at most n. The exceptional factor can always be taken to be the top factor of the series; for if

$$< 1 >= G_0 \leq G_1 \leq G_2 \leq \ldots \leq G_n = G$$

is such a series with G_1 the exceptional factor, then, since G' is a p-group, $[G_2, G] \leq T_1$, the maximal p-subgroup of G_1, and then

$$< 1 >= G_0 \leq T_1 \leq G_2 \leq \ldots \leq G_n = G$$

is also such a series, but with G_2/T_1 the exceptional factor. In this way the exceptional factor can be pushed to the top of the series. In practice we only need the concept of ec-height for the primes 2 and 3.

3.1. *Let A be an abelian normal subgroup of prime index p in the p-primary group G. Suppose that the maximal p-subgroup T of A has exponent dividing p. Then G has* ec-*height at most p.*

Proof. Now $G =< g > A$, for some g in G, and $g^p \in A$. Also G' is a p-group. It is convenient to write A additively. Then $A(g-1) = [A, g] = G' \leq T$. Also

$$A(g-1)(1+g+g^2+\ldots+g^{p-1}) = A(g^p-1) =< 0 > .$$

But $A(g-1) \leq T$ and any t in T has order dividing p, by hypothesis, so that

$$t(1+g+g^2+\ldots+g^{p-1}) = t(g-1)^{p-1}.$$

Thus $A(g-1)^p =< 0 >$.

Consider the series

$$< 1 >\leq C_T(g) \leq [A,\ _{p-2}\,g].C_T(g) \leq \ldots \leq [A,\ _i\,g].C_T(g) \leq \ldots \leq [A, g].C_T(g) \leq G.$$

This is a central series of G of length p with each of its factors an elementary abelian p-group except possibly the final factor $H = G/K$ for $K = [A, g].C_T(g)$. If G/T has no non-trivial p-elements, then T/K is the maximal p-subgroup of H. If not we may choose g to be a p-element and then $g^p \in T$, so that $g^p \in C_T(g)$ and $T/K \times < gK >$ is the maximal p-subgroup of H. Either way the maximal p-subgroup of H is an elementary abelian p-group. The lemma follows.

3.2. *Let X be a group such that, for some prime p, every p-primary subgroup of X has* ec-*height at most r. Let C be a cyclic group of order p. Then every p-primary subgroup of the wreath product $Y = X\mathrm{wr}C$ has* ec-*height at most pr.*

Proof. Let G be a p-primary subgroup of Y. If B denotes the base group of Y, then $B \cap G$ has ec-height at most r and $B \cap G$ has a central series $< 1 >= B_0 \leq B_1 \leq \ldots B_r = B \cap G$ each factor of which has its maximal p-subgroup elementary abelian and each factor of which, with at most one exception, is a p-group. We can also choose each B_i characteristic in $B \cap G$ and hence normal in G.

By 3.1, the section G/B_{r-1} has ec-height at most p. Pick g in G with $G =< g > (B \cap G)$, so that $g^p \in B$. For $i \leq r$ apply 3.1 to the split extension of B_i/B_{i-1} by $< g > / < g^p >$. Then this split extension has ec-height at most p. Moreover, with at most one exception, each of these split extensions is a p-group. Therefore G has ec-height at most pr.

The same proof yields the following, which we only use for $p = 2$.

3.3. *Let X be a group such that every p-subgroup of X has* ec-*height at most r, for some prime p. Let C be a cyclic group of order p. Then every p-subgroup of the wreath product $Y = X\mathrm{wr}C$ has* ec-*height at most pr.*

3.4. *Let K denote the field $\mathbb{Q}(i) \leq \mathbb{C}$, let $\Gamma = \mathrm{Gal}(K/\mathbb{Q}) \cong C_2$, let $\Gamma.K^*$ be the split extension of the multiplicative group K^* of K by Γ and set $W = (\Gamma.K^*)\mathrm{wr}P$, the* *permutational wreath product, where P is a transitive 2-subgroup of* $\mathrm{Sym}(r)$. *Then any 2-primary subgroup of W has* ec*-height at most $3r$, while any 2-subgroup of W has* ec*-height at most $2r$.*

Proof. If $Q \geq P$ is a Sylow 2-subgroup of $\mathrm{Sym}(r)$, then Q too is transitive and W is a subgroup of $(\Gamma.K^*)\mathrm{wr}Q$. Hence we may assume that $P = Q$. Then $r = 2^\rho$, for some $\rho \geq 0$, and P is the wreath power of ρ copies of the cyclic group of order 2. See [**3**, §5.9] or [**4**, §III.15]. The result will then follow from the case $\rho = 0$, a simple induction on ρ and results 3.2 and 3.3. Hence we can assume that $\rho = 0$ and $W = \Gamma.K^*$.

Let G be a 2-primary subgroup of W. Modulo $< i >$, the subgroup $G \cap K^*$ of G is torsion-free abelian of index at most 2. Also G' is a 2-group. Thus $G' \leq < i >$ and G modulo $< i >$ is abelian with its torsion subgroup of order 1 or 2. Also $|i| = 4$. Therefore G has ec-height at most 3.

A 2-subgroup H of $\Gamma.K^*$ has order dividing 8 and Γ does not centralize $< i >$, the torsion subgroup of K^*. Therefore H is not cyclic of order 8 and consequently H has ec-height at most 2.

3.5. *Let K be a subfield of $\mathbb{C}$, where $(K : \mathbb{Q}) = 2$ and K does not contain i, and let $\Gamma = \mathrm{Gal}(K/\mathbb{Q}) \cong C_2$. Otherwise, let P and W be as in 3.4. Then every 2-primary subgroup of W has* ec*-height at most 2r.*

Proof. As in the proof of 3.4 we may assume that $r = 1$ and $W = \Gamma.K^*$. Let G be a 2-primary subgroup of W. Then $G' \leq K^*$ and G' is a 2-subgroup. Therefore $G' =< -1 >$, which has order 2, and G modulo $< -1 >$ is abelian with its maximal 2-subgroup of exponent dividing 2. Consequently G has ec-height at most 2.

3.6. *Let K be a subfield of $\mathbb{C}$ with $(K : \mathbb{Q}) = 4$ and let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$. Otherwise let P and W be as in 3.4. Then every 2-primary subgroup of W has* ec*-height at most* $4r$.

Proof. As in the proof of 3.4 we may assume that $r = 1$ and $W = \Gamma.K^*$. Let G be a 2-primary subgroup of W. Then $G' \leq K^*$ and G' is a 2-subgroup. If K does not contain a primitive 8th root of 1, then $G' \leq < i >$ and G modulo $< i >$ is abelian with its maximal 2-subgroup of exponent dividing 4. Therefore G has ec-height at most 4 in this case. Now assume that $K = \mathbb{Q}(\omega) \leq \mathbb{C}$, where ω is a primitive 8th root of 1. Here $\Gamma = \mathrm{Gal}(K/\mathbb{Q}) \cong C_2 \times C_2$. Being a 2-subgroup, $G' \leq < w >$ and G modulo $< \omega >$ is abelian with its maximal 2-subgroup of exponent dividing 2. Therefore G again has ec-height at most 4.

3.7. *Let $K = \mathbb{Q}(\omega) \leq \mathbb{C}$, where w is a primitive* 9*th root of* 1 *and let Γ be the Sylow* 3*-subgroup (of order* 3*) of* $\mathrm{Gal}(K/\mathbb{Q})$. *Set $W = (\Gamma.K^*)\mathrm{wr}P$, where P is now a transitive* 3*-subgroup of* $\mathrm{Sym}(r)$. *Then any* 3*-primary subgroup of W has ec-height at most* $3r$.

Proof. As in the proof of 3.4 we may assume that $r = 1$ and $W = \Gamma.K^*$. Let G be a 3-primary subgroup of W. Then G' is a 3-subgroup of K^*, so that $G' \leq < \omega >$, which has order 9. Modulo $< \omega >$, the group G is abelian with its maximal 3-subgroup of order dividing 3. Thus G has ec-height at most 3.

4. General results. Suppose that G is a nilpotent irreducible subgroup of $GL(n, \mathbb{Q})$. Then G modulo its centre Z is finite [**6**, 3.13]. Assume that G is p-primary, for some prime p, and let A be a maximal abelian normal subgroup of G. Then $A \geq Z$; also A is completely reducible (Clifford's Theorem). Hence $R = \mathbb{Q}[A] \leq \mathbb{Q}^{n\times n}$ is semi-simple and $R = K_1 \times K_2 \times \ldots \times K_r$ for some fields K_i. Then if V denotes row n-space over $\mathbb{Q}$ regarded as a right R-module in the obvious way, $V = H_1 \oplus H_2 \oplus \ldots \oplus H_r$, where H_i is the homogeneous component of V corresponding to K_i.

Since G is irreducible, G permutes the H_i and the K_i transitively. Set $N = \cap_i N_G(H_i) = \cap_i N_G(K_i)$. Then G/N is isomorphic to a transitive p-subgroup P of Sym(r). In particular $r = p^\rho$ for some $\rho \geq 0$ and P is nilpotent of class at most r/p. See [**3**, §5.9] or [**4**, III.15.3]. It also follows that the K_i are all isomorphic, to K say.

Let p^c be the largest order of a p-power root-of-unity ω in K. Assume G is not abelian. Then G' is a non-trivial p-group, $G' \cap Z \neq < 1 >$ and A contains an element of order p. Thus $c \geq 1$. Also $(\mathbb{Q}(\omega) : \mathbb{Q}) = \phi(p^c) = p^{c-1}(p-1)$ and this must divide $m = (K; \mathbb{Q})$. Let $m = p^e(p-1)h$, where $e \geq c-1$ and p does not divide h. Let d be the dimension of H_i over K_i (d does not depend upon i, note). Then $n = dmr = dp^e(p-1)hr$. Also $n > 1$ since we have assumed that G is non-abelian.

Regard P as a subgroup of $GL(n, \mathbb{Q})$ of block permutation matrices corresponding to the decomposition $V = \oplus_i H_i$, so that $H_i\sigma = H_{i\sigma}$ for all σ in P. Then $\dim_{\mathbb{Q}} H_i = dm$ and $< P, G >$ embeds into the wreath product GL(dm, $\mathbb{Q}$)wr$P$. Also $C_G(A) = A$, and so G/A embeds into Aut$R \cong \Gamma$wr Sym(r), for Γ the Galois group of K over $\mathbb{Q}$. Thus G/A embeds into Γwr$P$ and hence into one of its Sylow p-subgroups. Consequently G/A embeds into Πwr$P$ for Π a Sylow p-subgroup of Γ and $|\Pi| = p^e$. Therefore G/A is nilpotent of class at most $\max\{r/p, er\}$ by [**4**, III.15.2]. Also $[A, G]$ is a p-group of order dividing p^{cr}. Thus G is nilpotent of class k, where

$$k \leq \max\{r/p, er\} + 1 + cr.$$

Note also that if $r = 1$, then G/A embeds into Π and A is contained in K^*. Hence if also $e > 0$ then the group G', being a p-group, has order dividing p^{e-1+c} and $k \leq e + c \leq 2e + 1 < 2(e+1)r$. We have now to make case-by-case estimates of these bounds.

Case $e = 0$ *and* $r = 1$. Here $G = A$ and $k < 2 \leq n$.

Case $e = 0$ *and* $r > 1$. Here $c = 1$ and $k \leq r/p + 1 + r$. Hence $k \leq 2r$, with equality only if $p = r = 2$. Also $n = dmr = d(p-1)hr$. If $p > 2$, then $k < n$. Let $p = 2$. If $dh > 2$, then $k < n$. If $dh = 2$ and $r > 2$, then $k < 2r \leq n$. If $dh = r = 2$, then $k \leq 4 = n$. Suppose that $dh = 1$. Then $m = 1$, $K = \mathbb{Q}$, $\Gamma = \Pi =< 1 >$ and G embeds into $\mathbb{Q}^*$wr$P$. Now $\mathbb{Q}^* =< -1 > \times F$, where F is free abelian. Hence G embeds into the direct product of $< -1 >$ wr$P$ and Fwr$P$. Therefore G is nilpotent of class at most $\max\{r, 1 + r/2\} = r = n$. In particular $n = 2^\rho$, a power of 2.

Case $e > 0$. Here $k \leq er + 1 + cr \leq (2e+1)r + 1$ and $n = dp^e(p-1)hr$. Thus (even if $r = 1$, see above) we have $k < 2(e+1)r$ and so $k < n$ whenever $dp^e(p-1)h \geq 2^{e+1}$. Hence if $p > 2$ or if $dh > 1$, then $k < n$. Let $p = 2$ and $dh = 1$. Then $n = 2^e r = 2^{e+\rho}$, a power of 2. If $e \geq 3$ then $k < n$. There remains the two special cases $p = 2$, $dh = 1$, $e = 1$, $m = 2$, $n = 2r$ and $p = 2$, $dh = 1$, $e = 2$, $m = 4$, $n = 4r$.

If $e = 1$ then K is Galois over $\mathbb{Q}$. Suppose that $e = 2$ and K is not Galois over $\mathbb{Q}$. Then $|\Gamma| < m$ and $K > \mathbb{Q}(\omega)$. Thus $|\Gamma| = 2$ and $c - 1 < e = 2$; also

$$k \leq \max\{r/2, r\} + 1 + 2r = 3r + 1 \leq 4r = n.$$

Now assume K is Galois over $\mathbb{Q}$. Then $\mathbb{Q}^{m\times m}$ is isomorphic to the skew group ring of K by Γ and the normalizer of K in the latter is $\Gamma.K^*$. By the Skolem-Noether Theorem (e.g. [**2**, p. 262]) any two copies of K in $\mathbb{Q}^{m\times m}$ are conjugate. Hence G embeds into $(\Gamma.K^*)\mathrm{wr}P$. If $e = 1$, then $k \leq 2r = n$ or $K = \mathbb{Q}(i)$ and $k \leq 3r = 3n/2$ (even $k \leq 2r = n$ if G is actually a 2-group) by 3.4 and 3.5. If $e = 2$ then $k \leq 4r = n$ by 3.6.

4.1. Summarizing the above, if G is a nilpotent irreducible p-primary subgroup of $\mathrm{GL}(n, \mathbb{Q})$ of class k, then either n is not a power of 2 and $k < n$, or $p > 2$, $n > 1$ and $k < n$, or n is a power of 2 and either $k \leq n$ or $n > 1$, $p = 2$, the group G embeds into $(\Gamma.K^*)\mathrm{wr}P$, in the above notation with $K = \mathbb{Q}(i)$, and $k \leq 3r = 3n/2$. Moreover if G is periodic, then $k \leq n$ in all cases.

4.2. *The Proof of Theorem 2*. The bounds claimed in Theorem 2 are attained by the examples in §2. Let G be a nilpotent subgroup of $\mathrm{GL}(n, \mathbb{Q})$. Now $\mathbb{Q}$ is a perfect field, so that the Jordan decompositions of elements of $\mathrm{GL}(n, \mathbb{Q})$ take place in $\mathrm{GL}(n, \mathbb{Q})$. See for example [**5**, 3.1.6]. Hence $G \leq G_u \times G_d \leq \mathrm{GL}(n, \mathbb{Q})$, where G_u is unipotent, and therefore torsion-free nilpotent of class at most $n - 1$; also G_d is a nilpotent d-group [**5**, 3.1.7].

Assume $G = G_d$ is a d-group. Then G has trivial unipotent radical and so we may assume that G is completely reducible (over $\mathbb{Q}$). Hence the centre Z of G has finite index in G [**6**, 3.13]. Thus G is a central product (see [**5**, 3.2.2]) of p-primary groups, one for each prime p, each of which will also be completely reducible by Clifford's Theorem. Clearly, therefore, we may assume that G is p-primary for some prime p, so that G/Z is now a finite p-group; we may also assume that G is irreducible. Then by 4.1 above G is nilpotent of class k, where $k < n$ if n is not a power of 2, where $k \leq 1$ if $n = 1$ and where $k \leq 3n/2$ otherwise. Theorem 2 follows.

4.3. *The Proof of Theorem 1*. Let G be a nilpotent subgroup of $\mathrm{GL}(n, \mathbb{Z})$. As in the proof of Theorem 2 (see 4.2), we reduce to the case in which G is primary and rationally irreducible. We need to be slightly more careful to keep G in $\mathrm{GL}(n, \mathbb{Z})$. With V again denoting row n-space over the rationals, we have $V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$ as $\mathbb{Q}G$-module, where all the $\mathbb{Q}G$-composition factors of each V_i are isomorphic; see [**7**, 2.3]. The class k of G is bounded by the classes of the $G/C_G(V_i)$ and hence we may assume that all the $\mathbb{Q}G$-composition factors of V are isomorphic. If there are at least two of these, then the proof 4.2 yields that k is at most $\max\{n - 1, 3n/4\}$, so that $k \leq n - 1$.

Thus we may assume that G is rationally irreducible. Then G modulo its centre Z is finite and k is bounded by the maximum class of a primary subgroup H of G. Repeating the above reduction we may reduce to the case in which H is rationally irreducible. Therefore we may assume that $G = H$ is p-primary, for some prime p. Assuming G is also a counterexample to the theorem, it follows from 4.1 that $p = 2$, that n is a power of 2 and $G \leq (\Gamma.K^*)\mathrm{wr}P \leq \mathrm{GL}(n, \mathbb{Q})$, where $K = \mathbb{Q}(i)$, where $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$ has order 2 and where P is a transitive 2-subgroup of degree $r = n/2$.

We continue the notation of the proof of 4.1. The centre Z of G generates a subfield S of $R = \mathbb{Q}[A] = K_1 \times K_2 \times \ldots \times K_r$. Then S embeds into K and so is isomorphic to $\mathbb{Q}$ or K. If S is $\mathbb{Q}$, then Z consists of scalar rational matrices in GL($n, \mathbb{Z}$). Thus $Z \leq < \pm 1 >$, the group G is a 2-group and the conclusion $k \leq n$ follows from the finite case. (See 4.1 or the Introduction.) Now assume that $S \cong K$. Then the central field S projects onto each K_i and we may assume that we initially chose the isomorphisms between the K_i so that S is the diagonal copy of K in R. Also Γ does not centralize K. Consequently G in fact embeds into K^*wr$P$. Now $K^* =< i > \times F$ for some torsion-free abelian group F, G embeds into the direct product of $< i >$ wr P and Fwr P and [**4**, III.15.2] yields that $k \leq \max\{2r, 1 + r/2\} = 2r$. Finally $n = d2^e hr \geq 2r$ here and so $k \leq n$ yet again. The proof is complete.

4.4. *The Proof of Proposition 1*. Let G be a nilpotent subgroup of GL($n, \mathbb{Q}$), where $n > 1$. Consider the proof of Theorem 2 and let $\gamma^i G$ denote the i-th term of the lower central series of G. Now the unipotent component G_u of G is nilpotent of class at most $n - 1$, so that $\gamma^n G = \gamma^n G_d$. For odd primes p the p-primary subgroups of G_d are nilpotent of class at most $n - 1$ and the derived subgroup of a 2-primary group is a 2-group. Since n is at least 2, this shows that $\gamma^n G$ is a 2-group. Necessarily such a 2-subgroup of GL(n, $\mathbb{Q}$) is finite. The remainder of the proposition follows.

4.5. *The Proof of Proposition 2*. The case in which $p = 2$ follows at once from 4.1. Hence assume that $p > 2$. Over the complexes the group G is monomial and so if $n \leq p - 1$, then certainly G is abelian and $k \leq 1 = n_p < \max\{n, 2\}$. From now on assume that $n > p - 1$ (so that $n_p > 1$) and G is non-abelian. We continue the notation of the proof of 4.1. Thus A contains an element of order p, the integer $p - 1$ divides m, which divides n, the integer $k \leq \max\{r/p, er\} + 1 + cr$ and $n = dmr = dp^e(p - 1)hr$.

If $e \geq 2$, then $k \leq (2e + 1)r + 1 < p^e r \leq n_p$. If $e = 1$ and $p > 3$, then $k \leq 3r + 1 < 5r \leq pr \leq n_p$. If $e = 1$, $p = 3$ and $dh > 1$, then $k \leq 3r + 1 < 6r \leq n_3$. Suppose $e = 1$, $p = 3$ and $dh = 1$. If $c = 1$, then $k \leq 2r + 1 \leq 3r = n_3 = 3^{\rho+1}$. Suppose $c = 2$. Then $K = \mathbb{Q}(\omega)$ is Galois over $\mathbb{Q}$, $\mathbb{Q}^{6\times 6}$ is the skew group ring of Γ over K and $\Pi < \Gamma$ has order 3. Further G embeds into $(\Pi.K^*)$wr P. By 3.7 the group G has class at most $3r$ and here $n = 6r$, so that $k \leq n_3 = 3^{\rho+1}$. Finally suppose $e = 0$, so that here c must be 1. If also $r = 1$, then $G = A$ and $k \leq 1 < n_p$. Suppose $r > 1$. Then $k \leq r/p + 1 + r$ and $n_p = dhr$. Thus $k < n_p$ unless $dh = 1$ and $n_p = r = p^\rho$, a power of p. In this case G embeds into K^*wr$P$ and $K^* = C \times F$, where $|C| = p$ and F is p-free. Then $k \leq \max\{r, 1 + r/p\} \leq r = n_p$. The proof is complete.

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