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ODD TRIPERFECT NUMBERS ARE DIVISIBLE BY TWELVE DISTINCT PRIME FACTORS

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Abstract

We prove that an odd triperfect number has at least twelve distinct prime factors.

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1. Introduction

A positive integer N is called a *triperfect number* if $\sigma(N) = 3N$, where $\sigma(N)$ is the sum of the positive divisors of N. Although six even triperfect numbers are known, no odd triperfect (OT) numbers have been found.

McDaniel [8] and Cohen [4] proved that if N is OT, then $\omega(N) \geqslant 9$, where $\omega(N)$ is the number of distinct prime factors of N. The author [6] proved that $\omega(N) \geqslant 10$ and [7] that $\omega(N) \geqslant 11$; Bugulov [3] also proved that $\omega(N) \geqslant 11$. Beck and Najar [2] showed that $N > 10^{50}$, and Alexander [1] proved that $N > 10^{60}$. Cohen and Hagis [5] showed that the largest prime factor of N is at least 100129, that the second largest prime factor is at least 1009, that $\omega(N) \geqslant 11$, and that $N > 10^{70}$.

In this paper we prove the

THEOREM. *If N is an odd triperfect number, then* $\omega(N) \geqslant 12$.

REMARK. This theorem was independently proved by H. Reidlinger [10].

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2. Preliminaries

In the rest of this paper we let

$$N = \prod_{i=1}^{11} p_i^{a_i}$$

where the p_i's are odd primes, $p_1 < \cdots < p_{11}$, where the a_i's are positive integers, and where N is OT. We call $p_i^{a_i}$ a *component* of N and write $p_i^{a_i} \| N$ and $a_i = V_{p_i}(N)$.

DEFINITION. $S(N) = \sigma(N)/N$. We note that S is multiplicative, and extend the definition is include the case $a_i = \infty$ by setting

$$S(p_i^{\infty}) = p_i/(p_i - 1).$$

The next lemma is stated in [5].

LEMMA 1. $p_{10} \geqslant 1009$ *and* $p_{11} \geqslant 100129$.

LEMMA 2. $p_1 = 3,\ p_2 = 5,\ p_3 = 7,\ p_4 = 11,\ p_5 \leqslant 17,\ p_6 \leqslant 23,\ p_7 \leqslant 31$ *and* $p_8 \leqslant 79$.

PROOF. Since

$$S(3^{\infty}5^{\infty}7^{\infty}13^{\infty}17^{\infty}19^{\infty}23^{\infty}31^{\infty}1009^{\infty}100129^{\infty}) < 3,$$

we have $p_1 = 3$, $p_2 = 5$, $p_3 = 7$ and $p_4 = 11$. The proofs of other parts are similar.

The proof of the next lemma is easy.

LEMMA 3. (1) *The a_i's are even for all i.*
(2) *If q is a prime and $q \mid \sigma(p_i^{a_i})$ for some i, then $q = 3$ or $q = p_j$ for some j.*

The next three lemmas are stated in [9].

LEMMA 4. *Suppose p and q are odd primes and d is the order of p* mod q. *Then*

$$V_q(\sigma(p^a)) = \begin{cases} V_q(a+1) & \text{if } d = 1, \\ V_q(a+1) + V_q(p^d - 1) & \text{if } d > 1 \text{ and } d \mid a+1, \\ 0 & \text{otherwise}. \end{cases}$$

LEMMA 5. *If p is an odd prime, if $d > 1$, and if $d \mid a + 1$, then $\sigma(p^a)$ has a prime factor q such that the order of p is d* mod q *and* $q \equiv 1\ (d)$.

COROLLARY 5. *If* $d^b \mid a+1$, *then* $\sigma(p^a)$ *has* b *distinct prime factors congruent to* $1\ (d)$.

LEMMA 6. *Suppose* p *is a prime*, q *is a Fermat prime* $(3, 5, 17$ *etc.*), a *is even and* $q^b \mid \sigma(p^a)$. *Then* $p \equiv 1\ (q)$, *and* $\sigma(p^a)$ *has* b *distinct prime factors congruent to* $1\ (q)$.

LEMMA 7. *N has at most seven prime factors congruent to* 1 (3).

PROOF. $S(3^\infty 5^\infty 7^\infty 11^\infty 13^\infty 19^\infty 31^\infty 37^\infty 43^\infty 1009^\infty 100129^\infty) < 3$.

COROLLARY 7. *If* $p^a \| N$, *then* $3^7 \nmid \sigma(p^a)$.

PROOF. Suppose $3^7 \mid \sigma(p^a)$. Then by Lemma 6, $p \equiv 1\ (3)$ and $\sigma(p^a)$ has seven more primes congruent to 1 (3), so we get a contradiction by Lemmas 3 and 7.

The proofs of the next two lemmas are similar.

LEMMA 8. *N has at most five prime factors equivalent to* 1 (5).

COROLLARY 8. *If* $p^a \| N$, *then* $5^5 \nmid \sigma(p^a)$.

LEMMA 9. *N has at most three prime factors equivalent to* 1 (17).

COROLLARY 9.1. *If* $p^a \| N$, *then* $17^3 \nmid \sigma(p^a)$.

COROLLARY 9.2. $17^8 \nmid N$.

PROOF. If $17^8 \mid N$, then by Corollary 9.1, 17 or $17^2 \mid \sigma(p_i^{a_i})$ for four distinct i, and $p_i \equiv 1\ (17)$ by Lemma 6, which contradicts Lemma 9.

LEMMA 10. *If* $13^a > (p_{10}^3 - 1)(p_{11}^3 - 1)$ *and if* $13^{a+2b} \mid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$, *then N has at least* b *distinct prime factors congruent to* 1 (13).

PROOF. Let $p^a = p_i^{a_i}$ and $13 \mid \sigma(p^a)$, where $i = 10$ or 11. If d is the order of p mod 13, then, by Lemma 4, $d = 1$ or $d > 1$, and $d \mid a+1$, in which case $d = 3$ because a is even and $d \mid 12$. Suppose $13^b \nmid a_{10} + 1$ and $13^b \nmid a_{11} + 1$. Then by the same lemma

$$\begin{aligned} a + 2b &\leqslant V_{13}\left(\sigma\left(p_{10}^{a_{10}} p_{11}^{a_{11}}\right)\right) = V_{13}\left(\sigma\left(p_{10}^{a_{10}}\right)\right) + V_{13}\left(\sigma\left(p_{11}^{a_{11}}\right)\right) \\ &\leqslant V_{13}(a_{10}+1) + V_{13}\left(p_{10}^3 - 1\right) + V_{13}(a_{11}+1) + V\left(p_{11}^3 - 1\right) \\ &< 2b + a, \end{aligned}$$

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which is a contradiction. Hence $13^b \mid a+1$, and, by Corollary 5, $\sigma(p^a)$ has b distinct prime factors $\equiv 1\ (13)$.

3. The case $17 \nmid N$

In this section we assume that $17 \nmid N$ and get a contradiction.

The proof of the next lemma is easy.

LEMMA 11. *If* $17 \nmid N$, *then* $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, $p_5 = 13$, $p_6 \leqslant 23$, $p_7 \leqslant 29$, $p_8 \leqslant 43$ *and* $p_9 \leqslant 167$.

LEMMA 12. *If* $17 \nmid N$, *if* $p^a \| N$, *and if* $43 \leqslant p \leqslant 167$, *then* $5 \nmid \sigma(p^a)$.

PROOF. Suppose that $5 \mid \sigma(p^a)$. Then by Lemma 6, $p \equiv 1\ (5)$ and $5 \mid a+1$, and so $p = 61, 71, 101, 131$ or 151, and $\sigma(p^4) \mid \sigma(p^a)$. Since

$$\begin{aligned}\sigma(61^4) &= 5 \cdot 131 \cdot 21491,\\ \sigma(71^4) &= 5 \cdot 11 \cdot 211 \cdot 2221,\\ \sigma(101^4) &= 5 \cdot 31 \cdot 491 \cdot 1381,\\ \sigma(131^4) &= 5 \cdot 61 \cdot 973001,\\ \sigma(151^4) &= 5 \cdot 104670301,\end{aligned}$$

we have $p \neq 61, 71, 101$ or 131 by Lemmas 1 and 11.

Suppose $p = 151$. Then it is easy to show that $p_8 = 29$, $p_9 = 151$, $p_{10} \leqslant 4243$ and $p_{11} = 104670301$. Since

$$\begin{aligned}\sigma(5^2) &= 31,\\ \sigma(5^4) &= 11 \cdot 71,\\ \sigma(5^6) &= 19531,\end{aligned}$$

we have $5^7 \mid N$. Furthermore, $5^2 \nmid \sigma(11^{a_4})$, $5^2 \nmid \sigma(151^{a_9})$ and $5 \nmid \sigma(104670301^{a_{11}})$, for otherwise N would have a prime factor $q > 100000$ and $q \neq 104670301$. Then $5^6 \mid \sigma(p_{10}^{a_{10}})$, contradicting Corollary 8. Hence $p \neq 151$.

LEMMA 13. *If* $17 \nmid N$, *if* $p^a \| N$, *and if* $p = 11, 31$ *or* 41, *then* $5^2 \nmid \sigma(p^a)$.

PROOF. Suppose $5^2 \mid \sigma(p^a)$. Then by Lemma 4, $5^2 \mid a+1$, and so $\sigma(p^{24}) \mid \sigma(p^a)$. Since

$$\begin{aligned}\sigma(11^{24}) &= 5^2 \cdot 3221 \cdot 24151 \cdot M_1,\\ \sigma(31^{24}) &= 5^2 \cdot 101 \cdot 4951 \cdot 17351 \cdot M_2,\\ \sigma(41^{24}) &= 5^2 \cdot 16651 \cdot 579281 \cdot M_3,\end{aligned}$$

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where the M_i's have no prime factors less than 100000, and where $579281 \nmid M_3$, we get a contradiction by Lemmas 1 and 11.

LEMMA 14. *If* $17 \nmid N$, *then* $5^3 \nmid \sigma(\prod_{i=1}^{9} p_i^{a_i})$.

PROOF. By Lemmas 11, 12 and 13, we have $5^4 \nmid \sigma(\prod_{i=1}^{9} p_i^{a_i})$. Suppose $5^3 \mid \sigma(\prod_{i=1}^{9} p_i^{a_i})$. Then by the same lemmas, $5 \mid \sigma(p_i^{a_i})$ where $p_i = 11$, 31, and 41. Then $3221 \cdot 17351 \cdot 579281 \mid N$ because

$$\sigma(11^4) = 5 \cdot 3221,$$
$$\sigma(31^4) = 5 \cdot 11 \cdot 17351,$$
$$\sigma(41^4) = 5 \cdot 579281,$$

which contradicts Lemma 11.

LEMMA 15. *If* $17 \nmid N$, *and if* $5^2 \mid \sigma(\prod_{i=1}^{9} p_i^{a_i})$, *then* $5 \nmid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$.

PROOF. By Lemmas 11, 12 and 13, $5 \mid \sigma(p^a)$ and $5 \mid \sigma(q^b)$, where $p^a \| N$, $q^b \| N$, and $p, q = 11$, 31 or 41. Then, as in the proof of Lemma 14, $p_{10} = 3221$ or 17351, and $p_{11} = 579281$. Since

$$\sigma(3221^4) = 5 \cdot 11 \cdot M_1,$$
$$\sigma(17351^4) = 5 \cdot 11 \cdot M_2,$$
$$\sigma(579281^4) = 5 \cdot 2131 \cdot M_3,$$

where the M_i's have no prime factors less than 100000, and where $579281 \nmid M_i$, we have $5 \nmid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$ by Lemma 11.

COROLLARY 15. *If* $17 \nmid N$, *and if* $5^4 \mid N$, *then* $5^2 \nmid \sigma(\prod_{i=1}^{9} p_i^{a})$.

LEMMA 16. *If* $17 \nmid N$, *then* $5^{10} \nmid N$.

PROOF. Suppose that $5^{10} \mid N$. Then by Corollary 15, $5^2 \nmid \sigma(\prod_{i=1}^{9} p_i^{a_i})$, and so $5^9 \mid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$. Hence $5^5 \mid \sigma(p_{10}^{a_{10}})$ or $5^5 \mid \sigma(p_{11}^{a_{11}})$, contradicting Corollary 8.

LEMMA 17. *If* $17 \nmid N$, *then* $5^6 \nmid N$.

PROOF. Suppose that $5^8 \| N$. Then $829 \mid N$ because $829 \mid \sigma(5^8)$, which contradicts Lemmas 1 and 11. Suppose $5^6 \| N$. Since $\sigma(5^6) = 19531$, it follows that $p_{10} = 19531$; moreover, $5 \nmid \sigma(p_{10}^{a_{10}})$ because $191 \mid \sigma(19531^4)$, and because $p_9 \leqslant 167$ by Lemma 11. By Corollary 15, $5^2 \nmid \sigma(\prod_{i=1}^{10} p_{11}^{a_{11}})$, contradicting Corollary 8. Lemma 17 now follows from Lemma 16.

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DEFINITION. Suppose that p is a prime. Define

$$a(p) = \text{minimum}\{c \mid c \text{ is even and } p^{c+1} > 10^{11}\},$$

$$b_i = \begin{cases} a_i & \text{if } a_i < a(p_i), \\ a(p_i) & \text{if } a_i \geqslant a(p_i), \end{cases}$$

$$c_i = \begin{cases} a_i & \text{if } a_i < a(p_i), \\ \infty & \text{if } a_i \geqslant a(p_i). \end{cases}$$

LEMMA 18. *Let* $M = \prod_{i=1}^{11} p_i^{b_i}$. *Then* $0 \leqslant \log 3 - \log S(M) < 11 \cdot 10^{-11}$.

PROOF. Suppose that $p^a \| N$ and that $a \geqslant a(p)$. Then

$$\begin{aligned} 0 &\leqslant \log S(p^a) - \log S(p^{a(p)}) \\ &< \log p/(p-1) - \log(p^{a(p)+1} - 1)/p^{a(p)}(p-1) \\ &= \log p^{a(p)+1}/(p^{a(p)+1} - 1) = \log(1 + 1/(p^{a(p)+1} - 1)) \\ &< 1/(p^{a(p)+1} - 1) < 10^{-11}. \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leqslant \log 3 - \log S(M) = \log S(N) - \log S(M) \\ &\leqslant \sum_{i=1}^{11} \left(\log S(p_i^{a_i}) - \log S(p_i^{b_i})\right) < 11 \cdot 10^{-11}. \end{aligned}$$

LEMMA 19. *If* $17 \nmid N$, *then* $p_{10} < 3547$.

PROOF. Suppose that $p_{10} \geqslant 3547$. Then

$$\sum_{i=1}^{9} \log S(p_i^{a_i}) < \log 3$$

and

$$\sum_{i=1}^{9} \log S(p_i^{a_i}) + \log S(3547^\infty 100129^\infty) \geqslant \log 3.$$

Using a computer (Burroughs 6800 at East Carolina University), we searched for an $M = \prod_{i=1}^{9} p_i^{b_i}$ which satisfied Lemmas 3 ($p_i \leqslant 71$ and $a_i < a(p_i)$) 11, 17, and also

$$\log S(M) < \log 3$$

and

$$\log S(M) + 9 \cdot 10^{-11} + \log S(3547^\infty 100129^\infty) \geqslant \log 3.$$

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The results were

(1) $3^{24}5^{2}7^{14}11^{10}13^{10}19^{8}23^{8}31^{8}59^{6}$,

(2) $3^{4}5^{4}7^{14}11^{2}13^{10}19^{8}23^{8}31^{8}71^{6}$,

(3) $3^{4}5^{4}7^{14}11^{2}13^{10}19^{8}23^{8}31^{8}71^{2}$,

(4) $3^{4}5^{4}7^{2}11^{10}13^{10}19^{8}23^{8}29^{8}71^{6}$.

In (1), $3^2 \nmid \sigma(7^{a_3})$, $3 \nmid \sigma(13^{a_5})$, $3 \nmid \sigma(19^{a_6})$, $3 \nmid \sigma(31^{a_8})$ because $37 \mid \sigma(7^8)$, $61 \mid \sigma(13^2)$, $127 \mid \sigma(19^2)$, and $331 \mid \sigma(31^2)$. Hence, $3^{23} \mid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$. Then $3^{12} \mid \sigma(p_{10}^{a_{10}})$ or $3^{12} \mid \sigma(p_{11}^{a_{11}})$, contradicting Corollary 7.

In (2) and (3), $\sigma(3^4) = 11^2$ and $\sigma(5^4) = 11 \cdot 71$, so that $11^3 \mid N$, which is a contradiction.

In (4), $14591 \leqslant p_{10} \leqslant 17053$, and $p_{11} \leqslant 15613471$ because

$$S(3^{4}5^{4}7^{2}11^{\infty}13^{\infty}19^{\infty}23^{\infty}29^{\infty}71^{\infty}17077^{\infty}100129^{\infty}) < 3,$$
$$S(3^{4}5^{4}7^{2}11^{10}13^{10}19^{8}23^{8}29^{8}71^{6}14563^{2}) > 3,$$

and

$$S(3^{4}5^{4}7^{2}11^{\infty}13^{\infty}19^{\infty}23^{\infty}29^{\infty}71^{\infty}14591^{\infty}15613483^{\infty}) < 3.$$

If $10 \leqslant a_5 \leqslant 38$, then $\sigma(13^{a_5})$ has a prime factor $q < 100129$, and $q \neq p_i$ for $1 \leqslant i \leqslant 10$, or it has two prime factors greater than 17053. Hence $a_5 \geqslant 40$. For $1 \leqslant i \leqslant 9$ and $i \neq 8$, the order of p_i mod 13 is even, so that $13 \nmid \sigma(p_i^{a_i})$ by Lemmas 3 and 4. Also $13 \nmid \sigma(29^{a_8})$ because the order of 29 mod 13 is 3, and because $67 \mid \sigma(29^2)$. Hence $13^{a_5} \mid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$. Since $a_5 \geqslant 40$, and since

$$(p_{10}^3 - 1)(p_{11}^3 - 1) < (17053^3 - 1)(15613471^3 - 1) < 13^{31},$$

N would have four prime factors congruent to 1 (13) by Lemma 10, and this is a contradiction because $p_i \neq 1$ (13) for $1 \leqslant i \leqslant 9$.

LEMMA 20. *If* $17 \nmid N$, *then* N *is not* OT.

PROOF. By Lemma 19, $p_{10} < 3547$. Using the computer, we searched for an $M = \prod_{i=1}^{10} p_i^{b_i}$ which satisfied Lemmas 1, 3 ($p_i \leqslant 71$ and $a_i < a(p_i)$), 11, 17, and also

$$\log S(M) < \log 3$$

and

$$\log S(M) + 10 \cdot 10^{-11} + \log S(100129^{\infty}) > \log 3.$$

There were forty-eight such M's; however, in every case there were primes q and r such that $q < r$, such that no primes occurred between q and r, and such that

$$S\left(\prod_{i=1}^{10} p_i^{b_i}\right) S(q^2) > 3,$$
$$S\left(\prod_{i=1}^{10} p_i^{c_i}\right) S(r^{\infty}) < 3.$$

(In eight cases it was necessary to consider higher values for the b_i's.) Hence N cannot be OT.

4. The case $17 \mid N$

In this section we assume that $17 \mid N$.

LEMMA 21. $17^6 \nmid N$.

PROOF. By Corollary 9.2, $17^8 \nmid N$. Suppose that $17^6 \| N$. Since $\sigma(17^6) = 25646167$, it follows that $25646167 \mid N$. By Lemmas 2, 6, 9, and by Corollary 9.1, $17^2 \mid \sigma(p_i^{a_i})$ for $i = 9, 10, 11$ because 103 is the smallest prime congruent to 1 (17). Then we get a contradiction by Lemma 2 because $1123 \cdot q \cdot r \mid \sigma(2564167^{288})$, where q and r are distinct primes exceeding 100000 and different from 25646167.

LEMMA 22. *If* $17^4 \| N$, *then* $p_8 \leqslant 61$, $p_9 \geqslant 103$, $p_{10} = 88741$ *and* $5^6 \nmid N$.

PROOF. Since $\sigma(17^4) = 88741$, it follows that p_{10} (or p_9) $= 88741$. If $17^2 \mid \sigma(p^a)$ for some $p^a \| N$, then, by Lemma 6, N has three prime factors congruent to 1 (17); otherwise, $17 \mid \sigma(p^a)$ for four distinct components p^a, which contradicts Lemmas 6 and 9. Since 103 is the smallest prime congruent to 1 (17), we have $103 \leqslant p_9 \equiv p_{10} \equiv p_{11} \equiv 1$ (17). Then $p_8 \leqslant 61$ because

$$S(3^\infty 5^\infty 7^\infty 11^\infty 13^\infty 17^\infty 19^\infty 67^\infty 103^\infty 88741^\infty 100129^\infty) < 3.$$

Since $4451 \cdot 5441 \cdot 46558947881 \mid \sigma(88741^4)$, $3221 \mid \sigma(11^4)$ and $3221 \not\equiv 1$ (17), since $17351 \mid (31^4)$ and $17351 \not\equiv 1$ (17), since $579281 \mid \sigma(41^4)$ and $579281 \not\equiv 1$ (17), and since $21491 \mid \sigma(61^4)$ and $21491 \not\equiv 1$ (17), it follows that we have $5 \nmid \sigma(\prod_{i=1}^{8} p_i^{a_i})$ and $5 \nmid \sigma(p_{10}^{a_{10}})$. Suppose that $5^{10} \mid N$. Then $5^5 \mid \sigma(p_9^{a_9})$ or $5^5 \mid \sigma(p_{11}^{a_{11}})$, contradicting Corollary 8. Since $829 \mid \sigma(5^8)$ and $821 \not\equiv 1$ (17), and since $19531 \mid \sigma(5^6)$ and $19531 \not\equiv 1$ (17), we have $5^8 \nparallel N$ and $5^6 \nparallel N$.

LEMMA 23. *If* $17^2 \| N$, *then* $p_8 \leqslant 43$, $p_9 = 307$ *and* $5^4 \nmid N$.

PROOF. Since $\sigma(17^2) = 307$, we have $p_9 = 307$ by Lemmas 1 and 2. Then $p_8 \leqslant 43$ because

$$S(3^\infty 5^\infty 7^\infty 11^\infty 13^\infty 17^2 19^\infty 47^\infty 307^\infty 1009^\infty 100129^\infty) < 3.$$

As in the proof of Lemma 13, we have $5^2 \nmid \sigma(p^a)$ if $p^a \| N$ and if $p = 11$, 31 or 41. Then $5^4 \nmid \sigma(\prod_{i=1}^{9} p_i^{a_i})$ because $5 \nmid \sigma(p_9^{a_9})$ by Lemma 6. Suppose that $5^{10} \mid N$. Then $5^7 \mid \sigma(p_{10}^{a_{10}} p_{11}^{a_{11}})$, and so $5^4 \mid \sigma(p_{10}^{a_{10}})$ or $5^4 \mid \sigma(p_{11}^{a_{11}})$. Hence N has five prime

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factors $\equiv 1$ (5), which is a contradiction because

$$S(3^{\infty}5^{\infty}7^{\infty}11^{\infty}13^{\infty}17^{2}31^{\infty}41^{\infty}307^{\infty}1021^{\infty}100151^{\infty}) < 3.$$

Since $829 \mid \sigma(5^8)$, we have $5^8 \nparallel N$ by Lemma 1. Suppose that $5^6 \| N$. Then $p_{10} = 19531$ because $\sigma(5^6) = 19531$. As in the proof of Lemma 13, we have $5^2 \nmid \sigma(p^a)$ if $p^a \| N$ and if $p = 41$. Since $3221 \mid \sigma(11^4)$, $17351 \mid \sigma(31^4)$ and $191 \mid \sigma(19531^4)$, we have $5^2 \nmid \sigma(\prod_{i=1}^{10} p_i^{a_i})$. Then $5^5 \mid \sigma(p_{11}^{a_{11}})$, contradicting Corollary 8. Since $71 \mid \sigma(5^4)$, we have $5^4 \nparallel N$.

Lemma 24. *If* $17 \mid N$, *then* N *is not* OT.

Proof. As in the case $17 \nmid N$, we searched for an $M = \prod_{i=1}^{9} p_i^{b_i}$ which satisfied Lemmas 2, 3 ($p_i \leqslant 71$ and $a_i < a(p_i)$), 22, 23, and also

$$\log S(M) < \log 3$$

and

$$\log S(M) + 9 \cdot 10^{-11} + S(3547^{\infty}100129^{\infty}) \geqslant \log 3,$$

but there were none. (If $17^4 \| N$, we let $p_9 = 88741$, so that possibly $p_{10} < p_9 < p_{11}$.) Hence $p_{10} < 3547$. Then we searched for an $M = \prod_{i=1}^{10} p_i^{a_i}$ which satisfied Lemmas 1, 2, 3 ($p_i \leqslant 71$ and $a_i < a(p_i)$), 22, 23, and also

$$\log S(M) < \log 3$$

and

$$\log S(M) + 10 \cdot 10^{-11} + \log(100129^{\infty}) > \log 3,$$

but there were none. Hence N is not OT.

Lemmas 20 and 24 prove our theorem. The computer time for Lemmas 19, 20 and 24 was less than 10 minutes.

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