ARTINIAN BAND SUMS OF RINGS

A. V. KELAREV

Abstract

Band sums of associative rings were introduced by Weissglass in 1973. The main theorem claims that the support of every Artinian band sum of rings is finite. This result is analogous to the well-known theorem on Artinian semigroup rings.

1991 *Mathematics subject classification (Amer. Math. Soc.)*: primary 16P20, 16W50; secondary 20M25.

Let B be a band, that is, a semigroup consisting of idempotents. An associative ring R is called a *band sum* of its subrings R_b, $b \in B$, if $R = \bigoplus_{b \in B} R_b$, the sum is direct for additive groups, and $R_a R_b \subseteq R_{ab}$ for every $a, b \in B$. Band sums of rings were introduced in [11]. A discussion of relations between band sums and other ring and semigroup constructions is contained in [5], where several examples of effective applications of band sums are presented. Many articles have been devoted to the investigations of various properties of band sums. In particular, a wide range of properties were considered in [1]. For a fairly complete list of relevant references the reader may turn to [5]. Here we mention only the more recent articles [6, 8].

The aim of this paper is to investigate Artinian band sums of rings. Looking for a way of research it seems natural to consider the well-known results on Artinian semigroup rings as a guide. It was proved in [12] that if a semigroup ring RS is Artinian then R is Artinian and S is finite. (The proof was simplified in [7, 9].) If R and S are commutative, a complete description of Artinian RS was obtained in [4]. We shall deal with an Artinian band sum $R = \bigoplus_{b \in B} R_b$ and deduce analogous results. Our main theorem asserts that the set $\{b \mid R_b \not\subseteq 0\}$ is finite. Clearly, B can be infinite since there may be infinitely many zero rings among the R_b.

Two examples will be given to show in general that each R_b being Artinian is neither a necessary nor sufficient condition for R to be Artinian. In the important case

where B is commutative a corollary to the main theorem will show that R is Artinian if all R_b are Artinian and the set $\{b \mid R_b \neq 0\}$ is finite. The converse implication is not valid. Now let us state the main result.

THEOREM. *If B is a band and $R = \bigoplus_{b \in B} R_b$ is an Artinian band sum of rings, then all but a finite number of the R_b are equal to zero.*

By Artinian we mean right Artinian. Some definitions are needed for the proof. A band is called a *semilattice (right zero band; left zero band; rectangular band)* if it satisfies the identity $xy = yx$ ($xy = y$; $xy = x$; $xyx = x$). For each band B there exist a semilattice S and rectangular bands H_s, $s \in S$, such that $B = \bigoplus_{s \in S} H_s$, $H_s H_t \subseteq H_{st}$ for every $s, t \in S$, and $H_s \cup H_t = \phi$ whenever $s \neq t$ (cf. [2]). In this case B is said to be a *semilattice S of rectangular bands H_s*. It is easily seen that each rectangular band H is isomorphic to a direct product of a left zero band and a right zero band. Note that every semilattice S is a partially ordered set under the natural order $\leq$ defined by the rule $s \leq t$ if and only if $st = s$ (cf. [2, §1.8]).

Let $R = \bigoplus_{b \in B} R_b$ be a band sum of rings. Each element r in R has a unique representation of the form $r = \sum_{b \in B} r_b$, where $r_b \in R_b$. The set

$$\text{supp}(r) = \{b \in B \mid r_b \neq 0\}$$

is called the *support* of r. For any $I \subseteq R$ the union $\bigcup_{r \in I} \text{supp}(r)$ will be denoted by supp(I) and called the *support* of I. An ideal I of R is said to be *homogeneous* if $I = \bigoplus_{b \in B} I \cap R_b$. Putting $I_b = I \cap R_b$ we see that the homgeneous ideal I is a band sum of the I_b. In proving the theorem, we shall several times use the easy fact that the quotient ring R/I is a band sum of R_b/I_b : $R/I = \bigoplus_{b \in B} R_b/I_b$ provided that I is homogeneous.

LEMMA 1. *Let $R = \bigoplus_{b \in B} R_b$ be a band sum, E an infinite set of ideals of R, $|\text{supp}(\sum_{I \in E} I)| = \infty$, and k a natural number such that $|\text{supp}(I)| \leq k$ for every $I \in E$. Then R is not Artinian.*

PROOF. We will construct an infinite sequence of ideals $I_1, I_2, \ldots$ in E such that supp(I_i) is not included in $\bigcup_{j > i} \text{supp}(I_j)$ for any $i \geq 1$.

Take a finite subset E_1 of E such that $|\bigcup_{i \in E_1} \text{supp}(I)| > k$. For any $J \in E_1$ let G_J denote the set of all $I \in E$ such that $\text{supp}(I) \not\supseteq \text{supp}(J)$. For each $I \in E$, supp(I) does not contain at least one of the supp(J), $J \in E_1$. Therefore $E = \cup_{J \in E_1} G_J$. Since supp($\sum_{I \in E} I$) $= \bigcup_{J \in E_1} \text{supp}(\sum_{I \in G_J} I)$, there exists I_1 in E_1 such that the set $G_1 = \{I \in E \mid \text{supp}(I) \not\supseteq \text{supp}(I_1)\}$ is infinite and, moreover, $|\text{supp}(\sum_{I \in G_1} I)| = \infty$.

In G_1 we choose a finite subset E_2 such that $|\bigcup_{I \in E_2} \text{supp}(I)| > k$. As above, there exists I_2 in E_2 such that the set $G_2 = \{I \in G_1 \mid \text{supp}(I) \not\supseteq \text{supp}(I_2)\}$ is infinite. Again we can pick I_3 such that $I_3 \in G_2$ and $G_3 = \{I \in G_2 \mid \text{supp}(I) \not\supseteq \text{supp}(I_3)\}$ is

infinite, and so on. For each $i > 1$ the ideals $I_{i+1}, I_{i+2}, \ldots$ are taken from G_i, whence supp (I_i) is not contained in supp (I_j) for every $j > i$. Setting $P_i = \sum_{j \geq i} I_j$ for $i \geq 1$, we get supp $(P_{i+1}) = \bigcup_{j > i}$ supp $(I_j) \not\supseteq$ supp (I_i) and so $P_i \neq P_{i+1}$. Therefore $P_1 \supset P_2 \supset P_3 \supset \ldots$, implying that R is not Artinian.

LEMMA 2. *Let* $R = \bigoplus_{b \in B} R_b$ *be a band sum,* I *a nilpotent homogeneous ideal of* R *such that* supp (I) *is infinite and only a finite number of the* R_b *are not contained in* I*. Then* R *is not Artinian.*

PROOF. Since I is nilpotent and supp (I) is infinite, there exists a natural number $n > 1$ such that the number of the R_b which do not lie in I^n is infinite. Assume that n is the least natural number of this sort. It is enough to show that R/I^n is not Artinian.

Evidently, I^n is homogeneous, and therefore we may pass to the quotient ring R/I^n and assume that $I^n = 0$.

By the choice of n only a finite number of the R_b are not contained in I^{n-1}. Hence supp (I^{n-1}) is infinite. Thus I^{n-1} satisfies all the conditions of Lemma 2. To simplify the notation we assume that from the very beginning $I = I^{n-1}$, that is, $n = 2$.

Let C denote the subsemigroup generated in B by all b such that R_b is not contained in I. It is well-known that every band is locally finite. Therefore C is finite. Set $m = |C|$, $D = \{b \in B \mid R_b \neq 0, b \notin C\}$. Clearly D is infinite and $R_d \subseteq I$ for every $d \in D$.

Take any d in D and consider the ideal I_d generated in R by R_d. We get

$$I_d = R_d + RR_d + R_dR + RR_dR$$
$$= R_d + \bigoplus_{b \in B} R_b R_d + \bigoplus_{c \in B} R_d R_c + \bigoplus_{b,c \in B} R_b R_d R_c.$$

If b and c are not in C, then $R_b \subseteq I$ and $R_c \subseteq I$, whence $R_b R_d + R_d R_c \subseteq I^2 = 0$, because $R_d \subseteq I$. Therefore

$$I_d = R_d + \bigoplus_{b \in C} R_b R_d + \bigoplus_{c \in C} R_d R_c + \bigoplus_{b,c \in C} R_b R_d R_c.$$

Hence supp (I_d) cannot contain more than $k = 1 + m + m + m^2$ elements, where $m = |C|$. So the set $E = \{I_d \mid d \in D\}$ satisfies the conditions of Lemma 1. Thus R is not Artinian.

PROOF OF THE THEOREM. Let B be a semilattice S of rectangular bands $H_s : B = \bigcup_{s \in S} H_s$. Then $R = \bigoplus_{s \in S} Q_s$, where $Q_s = \bigoplus_{b \in H_s} R_b$. Set $T = \{t \in S \mid Q_t \neq 0\}$. First we will prove that only a finite number of the Q_s are not equal to zero, that is T is finite. (Note that in the special case where every Q_s is not radical, the claim follows from [10, Proposition 1.3].)

Suppose the contrary: let T be infinite. If T contains a descending chain of elements $s_1 > s_2 > \ldots$, then we consider ideals D_k generated in R by $\sum_{i \geq k} S_{s_i}$, $k = 1, 2, \ldots$. Clearly $D_1 \supset D_2 \supset \ldots$, which gives a contradiction. Thus T satisfies the descending chain condition.

By transfinite induction we shall construct an infinite ascending chain of ideals $I_1 \subset I_2 \subset \ldots$ of S such that T is contained in $\bigcup_{k=1}^{\infty} I_k$. Since T satisfies the descending chain condition, there exists a minimal element t_1 in T. denote by I_1 the ideal generated in S by t_1. Suppose that for some ordinal number v all the ideals I_u, $u < v$ are defined. Set $U_v = \bigcup_{u<v} I_v$. If v is a limit ordinal, then put $I_v = U_v$. If v is not limit, that is, if there exists $v - 1$, then choose any minimal element t_v in $T \setminus I_{v-1}$ and denote by I_v the ideal generated in S by $I_{v-1} \cup \{t_v\}$. Having defined I_v, we check whether T is contained in $\bigcup_k I_k$, and if not, then we continue the process. Clearly, the chain of ideals terminates, and so for a certain ordinal number w we have $T \subseteq I_w$. Since T is infinite, w is infinite too. Easy induction shows that $I_v \cap T = \{t_u \mid u < v, u \text{ a non-limit ordeal}\}$ for each v.

Consider the chain of ideals $A_1 \subset A_2 \subset \ldots \subset A_w = R$, where $A_v = \bigoplus_{u \in I_v} Q_u$. It is well-known that every semisimple Artinian ring is Noetherian. Therefore in $R/J(R)$ the ascending chain of ideals

$$A_1 + J(R) \subseteq A_2 + J(R) \subseteq \ldots \subseteq A_w + J(R)$$

breaks. Hence there exists a positive integer n such that

$$A_n + J(R) = A_{n+1} + J(R) = \ldots = A_w + J(R) = R + J(R).$$

This implies that the quotient ring $F = R/A_n$ is radical. Since A_n is homogeneous, F is a semilattice sum of its subrings $F_s = Q_s/(A_n \cap Q_s)$, $s \in S$. Evidently, $A_v = \bigcup_{u<v} A_u$ for a limit $v \leq w$. If v is not limit, then $I_v \cap T = \{t_v\} \cup \{I_{v-1} \cap T\}$, and so $A_v = Q_{t_v} + A_{v-1}$, $A_v \neq A_{v-1}$. Therefore supp (F) is infinite. Obviously, F is a homogeneous ideal in F containing all F_s. Since F is radical, it is nilpotent. By Lemma 2 F is not Artinian, whence neither is R. This contradiction shows that T is finite.

Now suppose that there are infinitely many non-zero rings among the R_b, $b \in B$. We will show that this leads to a contradiction.

Clearly $J_B(R) = \bigoplus_{b \in B} J(R)R_b$ is a homogeneous ideal of R. Morover, it is nilpotent because $J_B(R) \subseteq J(R)$. If only a finite number of the R_b are not included in $J_B(R)$, then supp $(J_B(R))$ is infinite, and Lemma 2 yields that R is not Artinian. Hence an infinite number of the components R_b do not lie in $J_B(R)$. Therefore $R/J_B(R)$ is a band sum with an infinite number of non-zero components $R_b/(R_b \cap J_B(R))$. Thus we may assume that from the very beginning $J_B(R) = 0$.

For $s \in S$ set $T_s = \{b \in H_s \mid R_b \neq 0\}$. Since we have proved that T is finite, only a finite number of the T_s are not empty. So we can choose a maximal element t in S

such that T_t is infinite. It is routine to verify that the set $V = \bigcup\{H_s \mid s \in S, st \neq t\}$ is an ideal of S, $W = \bigoplus_{b \in V} R_b$ is a homogeneous ideal in R, and the quotient ring $F = R/W$ satisfies exactly the same conditions as R. Therefore we can replace R by F. In order to preserve simple notation we just assume that $W = 0$ instead of substituting F for R. Then t becomes a minimal element such that T_t is not empty. (Indeed, $H_s \subseteq V$ for every $s < t$. Since $W = 0$, then $Q_s = 0$ for $s < t$, and so $T_s \neq \emptyset$.) Moreover, we may delete all s such that $st \neq t$ from S and assume that t is the least element of S.

The rectangular band H_t is isomorphic to a direct product of a left zero band C and a right zero band D. Identifying H_t with this product, we have $H_t = C \times D$. Moreover we may assume that $C, D \subseteq H_t$. For $d \in D$ set $C_d = \bigoplus_{c \in C} R_{(c,d)}$. Take any $b \in B$. There exists $s \in S$ such that $b \in H_s$. By the assumption above $st = t$. Therefore $b(c, d) \in H_t$ for any $c \in C$, $d \in D$. Let $b(c, d) = (c_1, d_1) \in H_t$. Consequently, $b(c, d) = b(c, d)(c, d) = (c_1, d_1)(c, d) = (c_1, d)$ implying $R_b R_{(c,d)} \subseteq C_d$. Thus C_d is a left ideal in R. Similarly, $D_c = \bigoplus_{d \in D} R_{(c,d)}$ is a right ideal of R for every $c \in C$.

Given that R is right Artinian, only a finite number of the right ideals D_c are non-zero. Since $D_c \cap C_d = R_{(c,d)}$, it follows that each C_d is a sum of a finite number of non-zero $R_{(c,d)}$'s. As T_t is infinite, there exist an infinite number of $C_d \neq 0$, $d \in D$.

Define on the union $A = D \cup (\bigcup_{s \neq t} H_s)$ a multiplication $*$ by setting

$$a * b = \begin{cases} ab & \text{if } ab \neq H_t \\ d & \text{if } ab = (c, d) \in H_t. \end{cases}$$

It is easy to check that $*$ is associative. To illustrate this let us take any $x, y, z \in A$ and consider the case when $xyz \in H_t$, x, $yz \notin H_t$. Let $xyz = (c, d) \in H_t$. Then $x * (y * z) = x * (yz) = d$. On the other hand, if $xy \notin H_t$, we get $(x * y) * z = (xy) * z = d$. If $xy = (c_1, d_1) \in H_t$, then $(x * y) * z = d_1 * z = d$, because $d_1 = d_1(c_1, d_1)$, and $d_1 z = d_1 xyz = d_1(c, d) = (c_2, d)$ for some $c_2 \in C$. Thus $x * (y * z) = (x * y) * z$ in the case we are considering. Other cases are similar.

For $a \in A$ set $P_a = R_a$, when $a \in B$, and set $P_a = C_a$ when $a \in D$. It is routine to verify that A is a band under $*$, and $R = \bigoplus_{a \in A} P_a$ is a band sum. It has been proved that an infinite number of P_a, $a \in A$, are not zero. Therefore we may consider A and P_a instead of B and R_b. To make further notation simpler assume that $A = B$, and so $H_t = D$, $C_d = R_d$.

Denote by J_t the set $\bigoplus_{d \in D} J(R) \cap R_d$. Because t is minimal with $Q_t \neq 0$, Q_t is an ideal in R. For any $b \in B$, $d \in D$ it is easily seen that $R_b(J(R) \cap R_d) \subseteq J(R) \cap R_{bd}$ and $(J(R) \cap R_d)R_b \subseteq R_{db}$. Therefore J_t is an ideal of R. Since $J(R)$ is nilpotent, it follows that J_t is a homogeneous nilpotent ideal of R. By Lemma 2 an infinite number of the R_b, $b \in B$, do not lie in J_t. Since t is maximal with infinite T_t, it follows that the number of R_b, $b \in H_t$, which are not contained in J_t, is infinite. Therefore we may pass to the quotient ring R/J_t and assume that $J_t = 0$.

The semisimple Artinian ring $R/J(R)$ is also left Noetherian. Hence its left ideal $(Q_t + J(R))/J(R)$ is finitely generated. Since every $R_d, d \in D$, is a left ideal, there exists a finite subset F of D such that

$$Q_t + J(R) = J(R) + \bigoplus_{f \in F} R_f.$$

For each $d \in T_t \backslash F$ choose any x in R_d and fix elements $r(d) \in \bigoplus_{f \in F} R_f, y(d) \in J(R)$ such that $x = r(d) + y(d)$. Since $y(d) = x - r(d)$, then supp $(y(d)) \subseteq F \cup \{d\}$, and so $|\text{supp}|(y(d)) \leq m$, where $m = |F| + 1$.

Let $I(d)$ denote the ideal generated in R by $y(d)$. Put $E = \{I(d) \mid d \in T_t \backslash F\}$. We claim that E satisfies all the conditions on the set of ideals in Lemma 1. Indeed, the union $\bigcup_{I \in E}$ supp (I) contains $T_t \backslash F$, and so it is infinite. It remains to evaluate $|\text{supp} (I(d))|$. To this end note that

$$I(d) = Zy(d) + Ry(d) + y(d)U + y(d)Q_t + Ry(d)U + Ry(d)Q_t,$$

where Z is the set of integers, and $U = \bigoplus_{b \in B \backslash D} R_b$. Evidently, supp $(Zy(d) + Ry(d)) \subseteq$ supp $(y(d))$ implying $|\text{supp} (Zy(d) + Ry(d))| \leq m$. Since only a finite number of $T_s, s \in S$, are not empty and T_t is the only infinite set among them, we obtain that $\bigcup_{s \neq t} T_s$ is finite. Therefore supp (U) contains a finite number of elements, say n. Then $|\text{supp} (y(d)U)| \leq mn$. Similarly, $|\text{supp} (Ry(d)U)| \leq |\text{supp} (y(d))\text{supp} (U)| \leq mn$. Further, for $b \in D$ we have $y(d)R_b \subseteq J(R) \cap R_b \subseteq J_t = 0$. This implies $Ry(d)Q_t = 0$, too. Thus $|\text{supp} (I(d))| \leq m + 2mn$ for every d. Lemma 1 completes the proof.

EXAMPLE. Let Q be the field of rational numbers, D a right zero band, $|D| > 1$. Then the semigroup ring $QD = \bigoplus_{d \in D} Qd$ is a band sum, where all the Qd are Artinian, since they are isomorphic to Q. However, QD is not Artinian. It has an infinite descending chain of right ideals $I_2 \supset I_3 \supset \ldots$, where $I_k = \{m(c - d)/n \mid m, n$ are integers: $2, \ldots, k$ do not divide $n\}$ for fixed $c, d \in D, c \neq d$.

EXAMPLE. Let $R = Q_n$ be the ring of $n \times n$ matrices over Q. Denote by e_{ij}, where $1 \leq i, j \leq n$, the standard matrix units. Put $R_{(i,j)} = Qe_{ij}$. On the set $B = \{(i, j) \mid 1 \leq i, j \leq n\}$ define a multiplication by $(i_1, j_1)(i_2, j_2) = (i_1, j_2)$. This makes B a rectangular band. It is easy to see that $R = \bigoplus_{b \in B} R_b$ is an Artinian band sum. However, $R_{(i,j)}$ is a ring with zero multiplication for $i \neq j$, and so it is not Artinian.

PROPOSITION. *Let B be a semilattice, $R = \bigoplus_{b \in B} R_b$ a semilattice sum such that all R_b are Artinian and the set $\{b \in B \mid R_b \neq 0\}$ is finite. Then R is Artinian.*

PROOF. If B has only two elements, say $a < b$, then R is an extension of R_a by R_b, and so R is Artinian. The general case follows by induction on $|B|$ as in the proof of [3, Theorem 1].

The following example appeared in [3, proof of Theorem 5]:

EXAMPLE. Let Q^0 be the ring with zero multiplication and the same additive group Q^+ as Q. The ring R defined by

$$R^+ = Q^+ \bigoplus Q^+$$
$$(p, q)(r, s) = (ps + qr, qs)$$

is commutative and is a semilattice sum of Q^0 and Q, with Q^0 an ideal of R. It is easy to check that R has only three ideals: 0, Q^0 and R. Therefore R is Artinian, although Q^0 is not.

References

[1] H. L. Chick and B. J. Gardner, 'The preservation of some ring properties by semilattice sums', *Comm. Algebra* **15** (1987), 1667–1677.
[2] A. Clifford and G. Preston, 'The algebraic theory of semigroups', *Math. Surveys Monographs* **7** (Amer. Math. Soc., Providence, 1964).
[3] B. J. Gardner, 'Radicals of supplementary semilattice sums of associative rings', *Pacific J. Math.* **58** (1975), 387–392.
[4] R. Gilmer, 'Chain conditions in commutative semigroup rings', *J. Algebra* **103** (1986), 592–599.
[5] A. V. Kelarev, 'When is the radical of a band sum of rings homogeneous', *Comm. Algebra* **18** (1990), 585–603.
[6] ———, 'Hereditary radicals and band sums of associative rings', *J. Austral. Math. Soc. (Ser. A)* **51** (1991), 62–72.
[7] I. B. Kozhukhov, *Semigroups and semigroup rings with conditions on congruence and ideals* (Dissertation, Moscow State University, 1978).
[8] W. D. Munn, 'A class of band-graded rings', *J. London Math. Soc.* **45** (1992), 1–16.
[9] J. Okninski, 'Finiteness conditions for semigroup rings', *Acta Univ. Carolin.-Math. Phys.* **25** (1984), 29–32.
[10] P. Wauters and E. Jespers, 'When is a semigroup ring of a commutative semigroup local or semilocal', *J. Algebra* **108** (1987), 188–194.
[11] J. Weissglass, 'Semigroup rings and semilattice sums of rings', *Proc. Amer. Math. Soc.* **39** (1973), 471–478.
[12] E. I. Zelmanov, 'Semigroup algebras with identities', *Sibirsk. Mat. Zh.* **18** (1977), 787–798 (in Russian).

Department of Mathematics
University of Tasmania
G.P.O. Box 252 C
Hobart, Tasmania 7001
Australia